7. \( f(x, y) = (x - y)(1 - xy) = x - y - x^2y + xy^2 \)
\[ f_x = 1 - 2xy + y^2, \quad f_y = -x^2 + 2xy, \quad f_{xx} = -2y, \quad f_{xy} = -2x + 2y, \quad f_{yy} = 2. \]
Thus \( f_x(0, 0) = 1 - 2(0)(0) + (0)^2 = 1 \neq 0 \) and \( f_y(0, 0) = -1 + 2(0)(0) = -1 \neq 0 \) which has no real solution. If \( y = x \) then substitution into \( f_x = 0 \) gives \( 2x - 2x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \), so the critical points are \((1, 1)\) and \((-1, -1)\). Now \( D(1, 1) = (-2)(2) - 0^2 = -4 < 0 \) and \( D(-1, -1) = (2)(-2) - 0^2 = -4 < 0 \), so \((1, 1)\) and \((-1, -1)\) are saddle points.

9. \( f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2 \Rightarrow f_x = 6xy - 12x, f_y = 3y^2 + 3x^2 - 12y, f_{xx} = 6y - 12, f_{xy} = 6x, f_{yy} = 6y - 12. \) Then \( f_x = 0 \) implies \( 6x(y - 2) = 0 \) so \( x = 0 \) or \( y = 2 \). If \( x = 0 \) then substitution into \( f_y = 0 \) gives \( 3y^2 - 12y = 0 \Rightarrow 3y(y - 4) = 0 \Rightarrow y = 0 \) or \( y = 4 \), so we have critical points \((0, 0)\) and \((0, 4)\). If \( y = 2 \), substitution into \( f_y = 0 \) gives \( 12 + 3x^2 - 24 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \), so we have critical points \((\pm 2, 0)\). \( D(0, 0) = (-12)(-12) - 0^2 = 144 > 0 \) and \( f_{xx}(0, 0) = -12 < 0 \), so \( f(0, 0) = 2 \) is a local maximum. \( D(0, 4) = (12)(12) - 0^2 = 144 > 0 \) and \( f_{xx}(0, 4) = 12 > 0 \), so \( f(0, 4) = -30 \) is a local minimum.

\( D(\pm 2, 0) = (0)(0) - (\pm 12)^2 = -144 < 0 \), so \((\pm 2, 2)\) are saddle points.

16. \( f(x, y) = e^y(y^2 - x^2) \Rightarrow f_x = -2xe^y, f_y = (2y + y^2 - x^2)e^y, f_{xx} = -2e^y, f_{xy} = -2xe^y, f_{yy} = (2 + 4y + y^2 - x^2)e^y. \) Then \( f_x = 0 \) implies \( x = 0 \) and substituting into \( f_y = 0 \) gives \( 2y + y^2)e^y = 0 \Rightarrow y(2 + y) = 0 \Rightarrow y = 0 \) or \( y = -2 \), so the critical points are \((0, 0)\) and \((0, -2). \) \( D(0, 0) = (-2)(-2) - (0)^2 = -4 < 0 \) so \((0, 0)\) is a saddle point. \( D(0, -2) = (2e^{-2})(-2e^{-2}) - (0)^2 = 4e^{-4} > 0 \) and \( f_{xx}(0, -2) = 4e^{-2} > 0 \), so \( f(0, -2) = 4e^{-2} \) is a local maximum.

20. \( f(x, y) = x^2 ye^{-x^2 - y^2} \Rightarrow f_x = x^2 ye^{-x^2 - y^2}(2x) + 2xye^{-x^2 - y^2} = 2xy(1 - x^2)e^{-x^2 - y^2}, f_y = x^2 ye^{-x^2 - y^2}(-2y) + x^2 e^{-x^2 - y^2} = x^2(1 - 2y^2)e^{-x^2 - y^2}, f_{xx} = 2y(2x^4 - 5x^2 + 1)e^{-x^2 - y^2}, f_{xy} = 2x(1 - x^2)(1 - 2y^2)e^{-x^2 - y^2}, f_{yy} = 2xy(2y^2 - 3)e^{-x^2 - y^2}. \) \( f_x = 0 \) implies \( x = 0 \) or \( y = 0 \), or \( x = \pm 1. \) If \( x = 0 \) then \( f_y = 0 \) for any \( y \)-value, so all points of the form \((0, y)\) are critical points. If \( y = 0 \) then \( f_y = 0 \Rightarrow x^2 e^{-x^2} = 0 \Rightarrow x = 0 \), so \((0, 0)\)(already included above) is a critical point. If \( x = \pm 1 \) then \((1 - 2y^2)e^{-1 - y^2} = 0 \Rightarrow y = \pm \frac{1}{\sqrt{2}}, \) so \((\pm 1, \frac{1}{\sqrt{2}})\) and \((\pm 1, -\frac{1}{\sqrt{2}})\) are critical points. Now
\[ D \left( \pm 1, \frac{1}{\sqrt{2}} \right) = 8e^{-3} > 0, f_{xx} \left( \pm 1, \frac{1}{\sqrt{2}} \right) = -2\sqrt{2}e^{-3/2} < 0 \) and \( D \left( \pm 1, -\frac{1}{\sqrt{2}} \right) = \)
\[ 8e^{-3} > 0, f_{xx} \left( \pm 1, -\frac{1}{\sqrt{2}} \right) = 2\sqrt{2}e^{-3/2} > 0, \] so \( f \left( \pm 1, \frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} e^{-3/2} \) are local maximum points while \( f \left( \pm 1, -\frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{2} e^{-3/2}} \) are local minimum points. At all critical points \((0, y)\) we have \( D(0, y) \), so the
Second Derivatives Test gives no information. However, if \( y > 0 \) then \( x^2 ye^{-x^2-y^2} \geq 0 \) with equality only when \( x = 0 \), so we have local minimum values \( f(0, y) = 0, y > 0 \). Similarly, if \( y < 0 \) then \( x^2 ye^{-x^2-y^2} \leq 0 \) with equality when \( x = 0 \) so \( f(0, y) = 0, y < 0 \) are local maximum values, and \((0,0)\) is a saddle point.

29. Since \( f \) is a polynomial it is continuous on \( D \), so an absolute maximum and minimum exist. Here \( f_x = 2x−2, f_y = 2y \), and setting \( f_x = f_y = 0 \) gives \((1,0)\) as the only critical point which is inside \( D \), where \( f(1,0) = −1 \). Along \( L_1 : x = 0 \) and \( f(0, y) = y^2 \) for \(-2 \leq y \leq 2 \), a quadratic function which attains its minimum at \( y = 0 \), where \( f(0,0) = 0 \), and its maximum at \( y = ±2 \), where \( f(0,±2) = 4 \). Along \( L_2 : y = x−2 \) for \( 0 \leq x \leq 2 \), and \( f(x, x−2) = 2x^2−6x+4 = 2 \left( x−\frac{3}{2} \right)^2−\frac{1}{2} \), a quadratic which attains its minimum at \( x = \frac{3}{2} \), where \( f \left( \frac{3}{2},−\frac{1}{2} \right) = -1 \), and its maximum at \( x = 0 \), where \( f(0,−2) = 4 \). Along \( L_3 : y = 2−x \) for \( 0 \leq x \leq 2 \), and \( f(x, 2−x) = 2x^2−6x+4 = 2 \left( x−\frac{3}{2} \right)^2−\frac{1}{2} \), a quadratic which attains its minimum at \( x = \frac{3}{2} \), where \( f \left( \frac{3}{2},\frac{1}{2} \right) = 1 \), and its maximum at \( x = 0 \), where \( f(0,2) = 4 \). Thus the absolute maximum of \( f \) on \( D \) is \( f(0,±2) = 4 \) and the absolute minimum is \( f(1,0) = -1 \).

34. \( f_x = y^2 \) and \( f_y = 2xy \), and since \( f_x = 0 \Leftrightarrow y = 0 \), there are no critical points in the interior of \( D \). Along \( L_1 : y = 0 \) and \( f(x, 0) = 0 \).

Along \( L_2 : x = 0 \) and \( f(0, y) = 0 \). Along \( L_3 : y = \sqrt{3−x^2} \), so let \( g(x) = f \left( x, \sqrt{3−x^2} \right) = 3x−x^3 \) for \( 0 \leq x \leq \sqrt{3} \). Then \( g'(x) = 3−3x^2 = 0 \Leftrightarrow x = 1 \). The maximum value is \( f(1, \sqrt{2}) = 2 \) and the minimum occurs both at \( x = 0 \) and \( x = \sqrt{3} \) where \( f(0, \sqrt{3}) = f(\sqrt{3},0) = 0 \). Thus the absolute maximum of \( f \) on \( D \) is \( f(1, \sqrt{2}) = 2 \), and the absolute minimum is \( 0 \) which occurs at all points along \( L_1 \) and \( L_2 \).

42. The distance from the origin to a point \((x, y, z)\) on the surface is \( d = \sqrt{x^2 + y^2 + z^2} \) where \( y^2 = 9 + xz \), so we minimize \( d^2 = x^2 + 9 + xz + z^2 = f(x, z) \). Then \( f_x = 2x+z, f_z = x+2z, \) and \( f_x = 0, f_z = 0 \Rightarrow x = 0, z = 0 \), so the only critical point is \((0,0)\). \( D(0,0) = (2)(2)−1 = 3 > 0 \) with \( f_{xx}(0,0) = 2 > 0 \), so this is a minimum. Thus \( y^2 = 9 + 0 \Rightarrow y = ±3 \) and the points on the surface closest to the origin are \((0,±3,0)\).

45. Center the sphere at the origin so that its equation is \( x^2 + y^2 + z^2 = r^2 \), and orient the inscribed rectangular box so that its edges are parallel to the coordinate axes. Any vertex of the box satisfies \( x^2 + y^2 + z^2 = r^2 \), so take \((x, y, z)\) to be the vertex in the first octant. Then the box has length \( 2x \), width \( 2y \), and height \( 2z = 2\sqrt{r^2−x^2−y^2} \) with volume given by \( V(x, y) = (2x)(2y) \left( 2\sqrt{r^2−x^2−y^2} \right) = 8xy\sqrt{r^2−x^2−y^2} \) for \( 0 < x < r, 0 < y < r \). Then
53. Let $x, y, z$ be the dimensions of the rectangular box. Then the volume of
the box is $xyz$ and $L = \sqrt{x^2 + y^2 + z^2} \Rightarrow L^2 = x^2 + y^2 + z^2$
$\Rightarrow z = \sqrt{L^2 - x^2 - y^2}$.

Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}(x, y > 0)$,

$V_x = xy\frac{1}{2}(L^2-x^2-y^2)^{-1/2}(-2x) + \sqrt{r^2-x^2-y^2}8y = \frac{8y(r^2-2x^2-y^2)}{\sqrt{r^2-x^2-y^2}}$

and $V_y = \frac{8x(r^2-x^2-2y^2)}{\sqrt{r^2-x^2-y^2}}$.

Setting $V_x = 0$ gives $y = 0$ or $2x^2 + y^2 = r^2$, but $y > 0$ so only the latter
solution applies. Similarly, $V_y = 0$ with $x > 0$ implies $x^2 + 2y^2 = r^2$.

Substituting, we have $2x^2 + y^2 = x^2 + 2y^2 \Rightarrow x^2 = y^2 \Rightarrow x = y$.

Then $x^2 + 2y^2 = r^2 \Rightarrow 3x^2 = r^2 \Rightarrow x = \sqrt{r^2/3} = r/\sqrt{3} = y$. Thus
the only critical point is $(r/\sqrt{3}, r/\sqrt{3})$. There must be a maximum
volume and here it must occur at a critical point, so the maximum volume
occurs when $x = y = r/\sqrt{3}$ and the maximum volume is $V \left( \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}} \right) = 8 \left( \frac{r}{\sqrt{3}} \right) \left( \frac{r}{\sqrt{3}} \right) \sqrt{r^2 - \left( \frac{r}{\sqrt{3}} \right)^2} = \frac{8}{3\sqrt{3}} r^3$.

55. Note that the variables are $m$ and $b$, and $f(m, b) = \sum_{i=1}^{n} [y_i - (mx_i + b)]^2$.

Then $f_m = \sum_{i=1}^{n} -2x_i [y_i - (mx_i + b)] = 0$

implies $\sum_{i=1}^{n} (x_i y_i - mx_i^2 - bx_i) = 0$ or $\sum_{i=1}^{n} x_i y_i = m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i$

and $f_b = \sum_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies
\[
\sum_{i=1}^{n} y_i = m \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} b = m \left( \sum_{i=1}^{n} x_i \right) + nb. \text{ Thus we have the two desired equations.}
\]

Now \( f_{mm} = \sum_{i=1}^{n} 2x_i^2, f_{bb} = \sum_{i=1}^{n} 2 = 2n \) and \( f_{mb} = \sum_{i=1}^{n} 2x_i \). And \( f_{mm}(m, b) > 0 \) always and

\[
D(m, b) = 4n \left( \sum_{i=1}^{n} x_i^2 \right) - 4 \left( \sum_{i=1}^{n} x_i \right)^2 = 4 \left[ n \left( \sum_{i=1}^{n} x_i^2 \right) - \left( \sum_{i=1}^{n} x_i \right)^2 \right] > 0
\]
always so the solutions of these two equations do indeed minimize \( \sum_{i=1}^{n} d_i^2 \).