**1012微乙01-05班期中考解答和評分標準**

1. (15%) 求過曲面 $z = e^{x^2y-1}$ 上點 $(1,1,1)$ 的切平面方程式。

**Solution:**
Let $f = e^{x^2y-1} - z$
Then

$$\frac{\partial f}{\partial x}\Big|_{(1,1,1)} = e^{x^2y-1}\cdot 2xy\Big|_{(1,1,1)} = 2 \qquad \text{(5 pt)}$$
$$\frac{\partial f}{\partial y}\Big|_{(1,1,1)} = e^{x^2y-1}\cdot x^2\Big|_{(1,1,1)} = 1 \qquad \text{(5 pt)}$$
$$\frac{\partial f}{\partial z}\Big|_{(1,1,1)} = -1$$

Therefore, tangent plane: $\nabla f(1,1,1)\cdot(x-1, y-1, z-1) = 0$
That is,

$$2(x-1) + 1(y-1) - 1(z-1) = 0 \Rightarrow z = 2x + y - 2 \qquad \text{(5 pt)}$$

2. 求 $f(x,y) = y^4 + 2xy^3 + x^2y^2$
   (a) (5%) 在點 $(0,1)$ 的梯度;
   (b) (5%) 在點 $(0,1)$ 沿著 $(1,2)$ 方向的方向導數。

**Solution:**

(a) $\nabla f(x,y) = (2y^3 + 2xy^2, 4y^3 + 6xy^2 + 2x^2y)$ (4 points)
$\nabla f(0,1) = (2,4)$ (1 point)

(b) let $u = (1,2)$, $\sqrt{1^2+2^2} = \sqrt{5}$ (2 points)

$$\frac{\partial f}{\partial u}(0,1) = \nabla f(0,1)\cdot\frac{u}{\sqrt{5}} = (2,4)\cdot(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) \text{ (2 points)}$$

$$= 2\sqrt{5} \text{ (1 point)}$$

3. (12%) 求函數 $f(x,y) = xy - x^2y - xy^2$ 的 (a)極值候選點 (b)並討論其極值性質(含鞍點)。

**Solution:**
滿足$\nabla f(x,y) = (0,0)$的點，就是極值候選點:
$\frac{\partial f}{\partial x} = y - 2xy - y^2 = y(1-2x-y) = 0$ ...(1)
$\frac{\partial f}{\partial y} = x - x^2 - 2xy = x(1-x-2y) = 0$ ...(2)
由(1): $y = 0$ or $1 - 2x - y = 0$
(i) $y = 0$ 代入(2): $x(1-x) = 0$ , $x = 0$ or $x = 1$
得到(0,0) , (1,0)是候選點。
(ii)$1 - 2x - y = 0$ 即$y = 1 - 2x$ 代入(2): $x(3x-1) = 0$ , $x = 0$ or $x = \frac{1}{3}$
得到(0,1) , $(\frac{1}{3}, \frac{1}{3})$ 也是候選點。
由二階測試:

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\partial x}\right)^2\Bigg|_{(x,y)} = (-2y)(-2x) - (1-2x-2y)^2$$

$D(0,0) = -1 < 0$,...鞍點
$D(0,1) = -1 < 0$,...鞍點
$D(1,0) = -1 < 0$,...鞍點

$D(\frac{1}{3},\frac{1}{3})=\frac{4}{9}>0$,且$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(\frac{1}{3},\frac{1}{3})}=\frac{-2}{3}<0$ , ...極大值

評分標準:
(a)計算$\bigtriangledown f(x,y)$ 正確:2分，解極值候選點:4分。
(b)計算$D(x,y)$ 正確:2分，檢驗4個點為何種類型:4分。

4. (12%) 使用 Lagrange 乘子法求從點 $(0,0)$ 到曲線 $y=x^2-\frac{5}{4}$ 的最短距離。

**Solution:**
Let $f(x,y)=x^2-y-\frac{5}{4}$, $g(x,y)=x^2+y^2$. Notice that $\{f(x,y)=0\}$ is the given curve, and $g(x,y)$ is the "square" of distance from the origin.
By Lagrange multiplier method, there exists $\lambda\in\mathbf{R}$ such that

$$\nabla f(x,y)=\lambda\nabla g(x,y).$$

Hence we need to solve

$$\begin{cases} 2x &= \lambda\, 2x \\ -1 &= \lambda\, 2y \\ y &= x^2-\frac{5}{4}. \end{cases}$$

(5 pts if you complete the settings above)

By the first colume, we get $\lambda=1$ or $x=0$.

**1.** If $\lambda=1$, then plug into 2nd colume get $y=-\frac{1}{2}$. By the 3rd colume, $x=\frac{\sqrt{3}}{2}$ or $x=-\frac{\sqrt{3}}{2}$.

**2.** If $x=0$, then by the 3rd colume, $y=-\frac{5}{4}$.

Therefore, the critical points are $(\frac{\sqrt{3}}{2},-\frac{1}{2})$, $(-\frac{\sqrt{3}}{2},-\frac{1}{2})$, and $(0,-\frac{5}{4})$.(2 pts for each) Then plug into $g(x,y)$ to find the minimal distance is $\sqrt{1}=1$.(1 pt)

5. (15%) 求 $\iint_\Omega \frac{1}{(1+x+y)^2}\,dA$，其中 $\Omega=[0,2]\times[0,3]$。

**Solution:**

$$\begin{aligned} &\iint_\Omega \frac{1}{(1+x+y)^2}\,dA \quad \text{where } \Omega=[0,2]\times[0,3] \\ &=\int_0^3\int_0^2 \frac{1}{(1+x+y)^2}\,dxdy=\int_0^3 \left.\frac{-1}{1+x+y}\right|_{x=0}^{2}\,dy \\ &=\int_0^3\left(\frac{-1}{3+y}+\frac{1}{1+y}\right)\,dy \quad \text{(做到此項得 5 分)} \\ &=-\ln 6+\ln 3+\ln 4-\ln 1 \\ &=\ln\frac{4\times 3}{6\times 1}=\ln 2 \quad \text{(做到此項得滿分 15分)} \end{aligned}$$

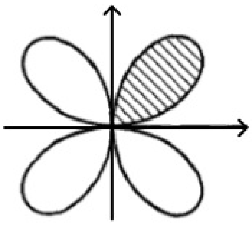
註：若是在計算過程中些微計算錯誤得 10分

6. (12%) 求 $\int_0^1\int_{x^{\frac{1}{4}}}^1 \frac{1}{1+y^5}\,dydx$ 之值。

**Solution:**

$$
\begin{aligned}
\int_0^1 \int_{x^{\frac{1}{4}}}^1 \frac{1}{1+y^5}\, dydx &= \int_0^1 \int_0^{y^4} \frac{1}{1+y^5}\, dxdy \text{......(6 points)} \\
&= \int_0^1 \frac{y^4}{1+y^5}\, dy \\
&= \frac{1}{5}\int_1^2 \frac{du}{u} \text{......(4 points)} \\
&= \frac{1}{5}\ln 2 \text{......(2 points)}
\end{aligned}
$$

7. (12%) $r = \sin 2\theta$ 之圖如下，求 $\iint_\Omega xy\; dA$，其中 $\Omega$ 為第一象限之一葉。



(提示：$\cos\theta\sin\theta = \dfrac{\sin 2\theta}{2}$)

**Solution:**

$r = \sin 2\theta$ and calculate the following area

$$\Omega = \begin{cases} 0 < \theta < \dfrac{\pi}{2}, \\ 0 < r < \sin 2\theta. \end{cases} \quad \text{(2 pts)}$$

$$
\begin{aligned}
\iint_\Omega xy\, dx\, dy &= \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r\cos\theta r\sin\theta r\, drd\theta && \text{(2 pts)} \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} \frac{1}{2} r^3 \sin 2\theta\, drd\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta \cdot \frac{1}{4} r^4 \Big|_0^{\sin 2\theta} d\theta \\
&= \frac{1}{8}\int_0^{\frac{\pi}{2}} \sin^5 2\theta\, d\theta, \text{ let } u = \cos 2\theta, du = -2\sin 2\theta\, d\theta && \text{(2 pts)} \\
&= -\frac{1}{16}\int_1^{-1} (1-u^2)^2\, du \\
&= \frac{1}{16}\int_{-1}^{1} (1-2u^2+u^4)\, du \\
&= \frac{1}{16}(u - \frac{2}{3}u^3 + \frac{1}{5}u^5)\Big|_{-1}^{1} \\
&= \frac{1}{15} && \text{(6 pts)}
\end{aligned}
$$

8. (12%) 求 $\iint_\Omega (3x+y)^6\; dA$，其中 $\Omega$ 為 $x+y=\pm 1$ 及 $3x+y=\pm 1$ 所包圍的平行四邊形。

**Solution:**

**Step 1. Change of variables (3 points)**

$$\begin{cases} u = 3x+y, & -1 \le u \le 1 \\ v = \ \ x+y, & -1 \le v \le 1 \end{cases}$$

**Step 2. Compute the Jacobian (4 points)**

$$\begin{cases} x = \dfrac{u-v}{2} \\ y = \dfrac{3v-u}{2} \end{cases}$$

$$\implies J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{vmatrix} = \frac{1}{2}$$

**Setp 3. Integration (5 points)**

$$\begin{aligned} & \iint_{\Omega} (3x+y)^6 dA \\ = & \int_{-1}^{1}\int_{-1}^{1} u^6 \cdot \frac{1}{2}\, dudv \\ = & \frac{1}{2}\left(\int_{-1}^{1} u^6 du\right)\left(\int_{-1}^{1} 1dv\right) \\ = & \frac{1}{2}\cdot\frac{2}{7}\cdot 2 \\ = & \frac{2}{7} \end{aligned}$$

**Remark.** If you have something wrong with the Jacobian in Setp 3, for example forgetting it or +/-, you will lose 2 points.