1012微乙01-05班期末考解答和評分標準

1. (10%) 求 $y(t)$ 滿足 $(t^2+1)y' = \frac{1}{y}$ 且 $y(0)=2$。

Solution:

separable: $(t^2+1)\frac{dy}{dt} = \frac{1}{y}$

$\Rightarrow ydy = \frac{1}{t^2+1}dt$

$\Rightarrow \int ydy = \int \frac{1}{t^2+1}$

$\Rightarrow \frac{1}{2}y^2 = tan^{-1}t + c$

$\Rightarrow y^2 = 2tan^{-1}t + c$

since y(0)=2>0

$\Rightarrow y = \sqrt{2tan^{-1}t + c}$

and $y(0) = \sqrt{0+c} = 2$

$\Rightarrow$ c=4

$\Rightarrow y = \sqrt{2tan^{-1}t + 4}$

Grading:

1. separable ...3%

2.$tan^{-1}t$ ...2%

3.y is nonnegative ...3%

4.c=4 ...2%

2. (10%) 求微分方程 $y' - \frac{2}{t}y = t$, $y(1) = -1$ 之解。

Solution:

The integration factor is

$$I(t) = e^{\int \frac{-2}{t}dt} = e^{-2\ln t} = \frac{1}{t^2} \text{...} (5pts)$$

So

$$\begin{aligned} y' - \frac{2}{t}y &= t \\ I(t)(y' - \frac{2}{t}y) &= I(t)t \\ (\frac{1}{t^2}y)' &= \frac{1}{t} \text{...} (3pts) \\ \frac{1}{t^2}y &= \int \frac{1}{t}dt \\ y &= t^2(\ln|t| + C) \end{aligned}$$

Since $y(1) = -1$, we have $-1 = \ln 1 + C \Rightarrow C = -1$. So the solution is

$$y = t^2(\ln|t| - 1) \text{..} (2pts)$$

3. (10%) 令 $V_2 = k$, $k \geq 2$, 表示在連續進行白努利試驗時，到第 k 次試驗才出現第 2 次 "+" 事件的隨機變數，求 $P(V_2 = 10)$。(設白努利試驗出現 "+" 機率為 p，出現 "−" 機率為 q，$p + q = 1$，$0 < p, q < 1$)

Solution:

$C_1^9 p^2 q^8$

Grading:
$C_1^{k-1} p^2 q^{k-1}$ ...9%
$C_2^{10} p^2 q^8$ ...4%
$C_1^9 p^2 (1-q)^8$ ...4%
others ...0%

4. (10%) 計算 $\int_{-\infty}^{\infty} xe^{-(x-1)^2}dx$。$\left(已知 \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}\right)$

Solution:

$$\begin{aligned}
&\int_{-\infty}^{\infty} xe^{-(x-1)^2}\,dx \quad \text{let } x-1=u,\ dx=du \\
&= \int_{-\infty}^{\infty} (u+1)e^{-u^2}\,du \quad \boxed{\text{1 point}} \\
&= \int_{-\infty}^{\infty} ue^{-u^2}\,du + \int_{-\infty}^{\infty} e^{-u^2}\,du \\
&= \int_{-\infty}^{\infty} ue^{-u^2}\,du + \sqrt{\pi}. \quad \boxed{\text{5 points}}
\end{aligned} \tag{1}$$

Now we check the improper integral $\int_{0}^{\infty} ue^{-u^2}\,du$ exists. By directly calculating,

$$\begin{aligned}
\int_{0}^{\infty} ue^{-u^2}\,du &= \lim_{b\to\infty}\int_{0}^{b} ue^{-u^2}\,du \\
&= \lim_{b\to\infty}\left(-\frac{1}{2}e^{-u^2}\Big|_{0}^{b}\right) \\
&= \lim_{b\to\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-b^2}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Similarly, the improper integral $\int_{-\infty}^{0} ue^{-u^2}\,du$ also exists. Since ue^{-u^2} is an odd function, we have

$$\int_{-\infty}^{\infty} ue^{-u^2}\,du = \int_{-\infty}^{0} ue^{-u^2}\,du + \int_{0}^{\infty} ue^{-u^2}\,du = 0 \quad \text{exists.}$$

Therefore, (1) becomes

$$\int_{-\infty}^{\infty} ue^{-u^2}\,du + \sqrt{\pi} = \sqrt{\pi}. \quad \boxed{\text{4 points}}$$

5. (15%) $f_X(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

求 $E(X)$ 及 $Var(X)$。

Solution:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} t f_X(t)\,dt \\ &= \int_0^{\infty} \lambda t e^{-\lambda t}\,dt \quad \boxed{\text{4 points}} \\ &= \frac{1}{\lambda}\int_0^{\infty} u e^{-u}\,du \\ &= \frac{1}{\lambda}\Big(-ue^{-u}\Big|_0^{\infty} + \int_0^{\infty} e^{-u}\,du\Big) \\ &= -\frac{1}{\lambda}e^{-u}\Big|_0^{\infty} \\ &= \frac{1}{\lambda}. \quad \boxed{\text{1 point}} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} t^2 f_X(t)\,dt \\ &= \int_0^{\infty} \lambda t^2 e^{-\lambda t}\,dt \quad \boxed{\text{4 points}} \\ &= \frac{1}{\lambda^2}\int_0^{\infty} u^2 e^{-u}\,du \\ &= \frac{1}{\lambda^2}\Big(-ue^{-u}\Big|_0^{\infty} + 2\int_0^{\infty} ue^{-u}\,du\Big) \\ &= \frac{2}{\lambda^2}. \quad \boxed{\text{1 point}} \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - \big(E(X)\big)^2 \quad \boxed{\text{4 points}} \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2}. \quad \boxed{\text{1 point}} \end{aligned}$$

6. 根據統計，在 15 歲到 35 歲之間酒駕死亡的機率密度為

$$f(x) = \frac{c}{x^2}, \qquad 15 \leq x \leq 35$$

(a) (8%) 求 c 之值。

(b) (7%) 求年齡在 21 到 25 歲間酒駕死亡的機率為何?

Solution:

1 Because $f(x) = \frac{c}{x^2}$ is the density function, $\int_{15}^{35} f(x)dx = 1$.
Then

$$\begin{aligned} 1 &= \int_{15}^{35} \frac{c}{x^2}dx \\ &= -\frac{c}{x} |_{15}^{35} \\ &= \frac{c}{15} - \frac{c}{35} \\ &= \frac{4c}{105} \\ \Rightarrow c &= \frac{105}{4} \end{aligned}$$

2

$$\begin{aligned} \int_{15}^{25} \frac{c}{x^2}dx &= \int_{21}^{25} \frac{105}{4x^2}dx \\ &= -\frac{105}{4x} |_{21}^{25} \\ &= \frac{105}{4 \times 21} - \frac{105}{4 \times 25} \\ &= \frac{1}{5} \end{aligned}$$

7. 某十字路口每年平均發生 36 次交通事故(註: 一年以 360 天計), 假設事故之發生遵守 Poisson 分配。

(a) (8%) 求在 15 天之內恰有一件交通事故的機率。

(b) (7%) 求在 15 天之內有二件或二件以上交通事故的機率。

Solution:

a. $P(k,m) = \frac{m^k}{k!}e^{-m}$ (3 points)

$m = \lambda T = \frac{36}{360} \times 15 = 1.5$ (3 points)

$P(1,1.5) = \frac{1.5^1}{1!}e^{-1.5} = 1.5e^{-1.5}$ (2 points)

b. $P(k,m) = \frac{m^k}{k!}e^{-m}$ (1 point)

$m = \lambda T = \frac{36}{360} \times 15 = 1.5$ (2 points)

$$\sum_{k=2}^{\infty} P(k,1.5) = 1 - P(0,1.5) - P(1,1.5) \text{ (2 points)}$$

$$= 1 - \frac{1.5^0}{0!}e^{-1.5} - \frac{1.5^1}{1!}e^{-1.5}$$

$$= 1 - e^{-1.5} - 1.5e^{-1.5}$$

$$= 1 - 2.5e^{-1.5} \text{ (2 points)}$$

8. (15%) 設 X, Y 為隨機變數取值 1 或 2。已知

$$P(X=1,Y=1)=\frac{1}{14}, P(X=1,Y=2)=\frac{5}{14}, P(X=2,Y=1)=\frac{6}{14}, P(X=2,Y=2)=\frac{2}{14}$$

求 $P(X=1)$，$P(X=2)$ 及 $E(X)$。(各 5%)

Solution:

$P(X=1)=P(X=1,Y=1)+P(X=1,Y=2)=\frac{3}{7}$,

$P(X=2)=P(X=2,Y=1)+P(X=2,Y=2)=\frac{4}{7}$,

$E(X)=\frac{3}{7}+2(\frac{4}{7})=\frac{11}{7}$.