1. (12 points) Test the following two series $\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$, where $p = 1$ and $p = 3/2$, for convergence.

Solution:

(a) method(1)

for $p = 1$

$$\sum_{k=1}^{\infty} \frac{lnk}{k}$$

let $u = lnx$ and $du = \frac{1}{x}dx$

$$\int_1^{\infty} \frac{lnk}{k}dk = \lim_{b\to\infty}\int_0^{lnb} \frac{lnu}{u}du = \lim_{b\to\infty}\frac{1}{2}(lnb)^2 \to \infty(4points)$$

By Integral test(1 points)

$$\sum_{k=1}^{\infty} \frac{lnk}{k}$$

diverges(1 points if the above process is correcet)

method(2)
for p=1

$$\sum_{k=1}^{\infty} \frac{lnk}{k}$$

after $k = 3$ the series with nonnegative terms and

$$\sum_{k=3}^{\infty} \frac{lnk}{k} > \sum_{k=3}^{\infty} \frac{1}{k}$$

and $\sum_{k=3}^{\infty} \frac{1}{k}$diverges (p-series with $p = 1$)(4 points)
By Basic comprasion test (1 point)

$$\sum_{k=1}^{\infty} \frac{lnk}{k}$$

diverges(1 points if the above process is correcet)

(b) method(1)

for $p = \frac{3}{2}$

$$\sum_{k=1}^{\infty} \frac{lnk}{k^{\frac{3}{2}}}$$

let $u = lnx$ and $du = \frac{1}{x}dx$
$dv = \frac{dx}{x^{\frac{3}{2}}}$ and $v = -2x^{\frac{-1}{2}}$

$$\int_1^{\infty} \frac{lnx}{x^{\frac{3}{2}}} = \lim_{b\to\infty}[(\frac{-2lnx}{\sqrt{x}})|_1^b + 2\int_1^b \frac{dx}{x^{\frac{3}{2}}}] = 0 + \lim_{b\to\infty} 2(-2)x^{\frac{-1}{2}}|_1^b = 4 < \infty(4points)$$

By Integral test(1 point)

$$\sum_{k=1}^{\infty} \frac{lnk}{k^{\frac{3}{2}}}$$

converges(1 points if the above process is correcet)

method(2)

for $p = \dfrac{3}{2}$

$$\sum_{k=1}^{\infty} \frac{lnk}{k^{\frac{3}{2}}}$$

$$\lim_{b\to\infty} \frac{\frac{lnk}{k^{\frac{3}{2}}}}{k^{\frac{5}{4}}} = \lim_{k\to\infty} \frac{lnk}{k^{\frac{1}{4}}} = 0(L'HospitalRule)$$

because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\frac{5}{4}}}$ converge(a p-seies with $p = \dfrac{5}{4} > 1$) (4 points)
By limit of comparison theorem(1 point)

$$\sum_{k=1}^{\infty} \frac{lnk}{k^{\frac{3}{2}}}$$

converges (1 point if process is correcet)

2. (12 points) Determine whether the series converge or diverge.

(a) $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k} - \sqrt{k-1})^{2k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(5k)!}$.

Solution:

2-(a) Let $a_k = (\sqrt{k} - \sqrt{k-1})^{2k}$

$$\lim_{k\to\infty} (a_k)^{\frac{1}{k}}$$
$$= \lim_{k\to\infty} (\sqrt{k} - \sqrt{k-1})^2$$
$$= \lim_{k\to\infty} \left(\frac{1}{\sqrt{k} + \sqrt{k-1}}\right)^2$$
$$= 0 < 1$$
$$\therefore \sum_{k=1}^{\infty} a_k \text{ converges by root test.}$$

2-(b) Let $a_k = \displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(5k)!}$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$
$$= \lim_{k\to\infty} \frac{\frac{((k+1)!)^2}{(5(k+1))!}}{\frac{(k!)^2}{(5k)!}}$$
$$= \lim_{k\to\infty} \frac{((k+1)!)^2}{(5k+5)!} \cdot \frac{(5k)!}{(k!)^2}$$
$$= \lim_{k\to\infty} \left(\frac{(k+1)!)}{(k!)}\right)^2 \cdot \frac{(5k)!}{(5k+5)!}$$
$$= \lim_{k\to\infty} (k+1)^2 \cdot \frac{1}{(5k+5)(5k+4)(5k+3)(5k+2)(5k+1)}$$
$$= 0 < 1$$
$$\therefore \sum_{k=1}^{\infty} a_k \text{ converges by ratio test.}$$

test使用正確: 2 points

極限算對: 2 points
註明收斂條件後寫出正確答案: 2 points

3. (12 points) Find the interval of convergence of the series $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) x^k$.

Solution:
$a_k = \ln(\frac{k+1}{k})$
$\frac{1}{\rho} = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$.(you should use L'H thm)
and then you must check the endpoint.
$x = 1$, $\Sigma_{k=1}^{\infty} \ln(\frac{k+1}{k}) = \Sigma_{k=1}^{\infty} \ln(k+1) - \ln(k) = \lim_{k\to\infty} \ln(k+1) = \infty$.
$x = -1, \Sigma_{k=1}^{\infty} \ln(\frac{k+1}{k})(-1)^k$
$\lim_{k\to\infty} a_k = 0$
$\frac{da_k}{dk} < 0$, and $a_1 > 0 \Rightarrow a_k$ decrease.
$\Longrightarrow [-1, 1)$ is convergence interval.

4. (12 points)

(a) Find the Taylor series for $f(x) = (x^2 + x + 1)\sqrt{x+1}$ at $x = 0$ up to the third power of x.

(b) Let $f(x) = \ln\sqrt{\frac{1+x^2}{1-x^2}}$. Find $f^{(10)}(0)$.

Solution:
(a) Find the Taylor series for $f(x) = (x^2 + x + 1)\sqrt{x+1}$ at $x = 0$ up to third power of x
Solution:
since $\sqrt{(1+x)} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - ...$
so $f(x) = (x^2 + x + 1)(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - ...) = 1 + \frac{3}{2}x + \frac{11}{8}x^2 + \frac{7}{16}x^3 + ...$
(each coefficient of power of x , 2 points)

(b) Let $f(x) = ln\sqrt{\frac{1+x^2}{1-x^2}}$ Find $f^{(10)}(0)$
Solution:
since $f(x) = \frac{1}{2}(ln(1+x^2) - ln(1-x^2))$(2 points)
also $ln(1+x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \frac{1}{5}x^{10}$...(1 points)
and $ln(1-x^2) = -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4}x^8 - \frac{1}{5}x^{10}$...(1 points)
compare the coefficient of power x^{10} we have $f^{(10)}(0) = \frac{10!}{5}$
(2 points for exactly right answer and right expension)

5. (10 points) Find the curvature $\kappa(t)$ of the curve $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k}$.

Solution:
$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{s'(t)} = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} \quad (5\%)$$
$$\mathbf{r}'(t) = (e^t,\ -e^{-t},\ \sqrt{2}), \quad ||\mathbf{r}'(t)|| = \sqrt{e^{2t} + e^{-2t} + 2} = e^t + e^{-t} \quad (2\%)$$
$$\mathbf{T}(t) = \frac{1}{e^t + e^{-t}}(e^t,\ -e^{-t},\ \sqrt{2}) = \left(\frac{1}{1+e^{-2t}},\ \frac{-1}{1+e^{2t}},\ \frac{\sqrt{2}}{e^t + e^{-t}}\right)$$

$$\mathbf{T}'(t) = \left(\frac{2e^{-2t}}{(1+e^{-2t})^2}, \frac{2e^{2t}}{(1+e^{2t})^2}, \frac{\sqrt{2}(e^{-t}-e^t)}{(e^t+e^{-t})^2} \right) = \left(\frac{2}{(e^t+e^{-t})^2}, \frac{2}{(e^t+e^{-t})^2}, \frac{\sqrt{2}(e^{-t}-e^t)}{(e^t+e^{-t})^2} \right)$$

$$||\mathbf{T}'(t)|| = \frac{\sqrt{4+4+2(e^{2t}+e^{-2t}-2)}}{(e^t+e^{-t})^2} = \frac{\sqrt{2(e^{2t}+e^{-2t}+2)}}{(e^t+e^{-t})^2} = \frac{\sqrt{2}}{(e^t+e^{-t})} \quad (2\%)$$

$$\kappa(t) = \frac{\sqrt{2}}{(e^t+e^{-t})^2} \quad (1\%)$$

sol:

$$\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^{2/3}} \quad (5\%)$$

$$\mathbf{r}'(t) = (e^t,\ -e^{-t},\ \sqrt{2}), \quad \mathbf{r}''(t) = (e^t,\ e^{-t},\ 0) \quad (2\%)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = (-\sqrt{2}e^{-t},\ \sqrt{2}e^t,\ 2), \quad ||\mathbf{r}'(t) \times \mathbf{r}''(t)|| = \sqrt{2}(e^t+e^{-t}) \quad (2\%)$$

$$\kappa(t) = \frac{\sqrt{2}}{(e^t+e^{-t})^2} \quad (1\%)$$

6. (15 points) Let $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(a) Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Is f continuous at $(0,0)$?

(b) Find the partial derivative $\dfrac{\partial f}{\partial x}$ at $(x,y) = (0,0)$ and at $(x,y) \neq (0,0)$.

(c) Is $\dfrac{\partial f}{\partial x}$ continuous at $(0,0)$?

Solution:

(a) By polar coordinate,(let $x = r\cos\theta,\ y = r\sin\theta$)

$$|g(r,\theta)| = |f(x,y)| = |f(r\cos\theta, r\sin\theta)| = |\frac{r^3(\cos^2\theta\sin^2\theta}{r^2}| = |r(\cos^2\theta\sin^2\theta)| \leq r.$$

Hence,

$$|\lim_{(x,y)\to(0,0)} f(x,y)| = |\lim_{\substack{r\to0,\\ \theta:\text{any angle}}} g(r,\theta)| \leq \lim_{r\to0} r = 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \ldots\ldots\ldots(5pts)$$

So $f(x,y)$ is continuous at $(0,0)$.

Note: If you didn't not emphasis that the angle θ is arbitrary, you only get the credits at most 4 points.

(b)

$$f_x(0,0) = \lim_{h\to0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to0} \frac{0-0}{h} = 0 \ldots\ldots\ldots(3pts)$$

For$(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{y(2xy^3)}{(x^2+y^2)^2} \ldots\ldots\ldots(2pts)$$

(c) Since

$$\lim_{y=x,x\to0} f_x(x,y) = \lim_{y=x,x\to0} \frac{y(2xy^3)}{(x^2+y^2)^2} = \lim_{x\to0} \frac{2x^4}{4x^4} = \frac{1}{2} \neq 0 = f_x(0,0),$$

the function $f_x(x,y)$ is not continuous at $(0,0)$.........(5pts)

7. (12 points) Let $u = u(x,y)$ be a function of rectangular coordinates x, y. Then u can be expressed in polar coordinates r, θ with $x = r\cos\theta$, $y = r\sin\theta$. Express $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in terms of r, θ, $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$.

Solution:

Sol. (I)

$$r(x,y) = \sqrt{x^2+y^2} \qquad (1\%)$$
$$\theta(x,y) = \tan^{-1}(\frac{y}{x}) \qquad (1\%)$$

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \cos\theta \\
\frac{\partial \theta}{\partial x} &= \frac{1}{1+(\frac{y}{x})^2}(-\frac{y}{x^2}) = \frac{-\sin\theta}{r} \\
\frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \sin\theta \\
\frac{\partial \theta}{\partial y} &= \frac{1}{1+(\frac{y}{x})^2}(\frac{1}{x}) = \frac{\cos\theta}{r} \qquad (1.5\% \text{ each})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad (\text{chain rule: } 2\%) \\
&= \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \qquad (\text{chain rule: } 2\%) \\
&= \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}
\end{aligned}$$

Sol. (II)

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial y}{\partial r} = \sin\theta$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta \qquad (1\% \text{ each})$$

$$\begin{cases}
\dfrac{\partial u}{\partial r} &= \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} \\[2ex]
\dfrac{\partial u}{\partial \theta} &= \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \theta} \quad (\text{chain rule: } 2\% \text{ each})
\end{cases}$$

$$\Rightarrow \begin{cases}
\dfrac{\partial u}{\partial r} &= \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta \\[2ex]
\dfrac{\partial u}{\partial \theta} &= \dfrac{\partial u}{\partial x}(-r\sin\theta) + \dfrac{\partial u}{\partial y}(r\cos\theta)
\end{cases}$$

Solve the system of equations for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ (say, by substitution or by Cramer's rule), we obtain

$$\begin{cases}
\dfrac{\partial u}{\partial x} &= \dfrac{\partial u}{\partial r}\cos\theta - \dfrac{\partial u}{\partial \theta}\dfrac{\sin\theta}{r} \\[2ex]
\dfrac{\partial u}{\partial y} &= \dfrac{\partial u}{\partial r}\sin\theta + \dfrac{\partial u}{\partial \theta}\dfrac{\cos\theta}{r} \quad (4\%)
\end{cases}$$

(Note: if you write the chain rule as in Sol. (I), you only get half the score for calculating the partial derivatives listed in Sol. (II), and vice versa.)

8. (15 points) Let $f(x,y) = xe^y + \cos(xy)$.

(a) Find the direction (a unit vector $\mathbf{u}$) in which $f(x,y)$ increases most rapidly at $(2,0)$ (that is, $f'_{\mathbf{u}}(2,0)$ is maximal).

(b) Find the direction in which $f(x,y)$ decreases most rapidly at $(2,0)$.

(c) What are the directions of zero change in f at $(2,0)$.

Solution:

(a) The direction which $f(x,y)$ increases most rapidly at $(2,0)$ is $\nabla f(x,y) = (e^y - y\sin(xy), xe^y - x\sin(xy))$ where $x = 2, y = 0$. That is $\dfrac{\nabla f(2,0)}{\|\nabla f(2,0)\|} = \dfrac{(1,2)}{\sqrt{5}}$.

(b) decreasing most rapidly is the inverse direction of $\nabla f(x,y)$. That is $\dfrac{-\nabla f(2,0)}{\|\nabla f(2,0)\|} = \dfrac{-(1,2)}{\sqrt{5}}$.

(c) The direction of zero change in $f(x,y)$ is those vector v satisfying $\langle v, \nabla f(x,y)\rangle = 0$. That is $\dfrac{(2,-1)}{\sqrt{5}}$ and $-\dfrac{(2,1)}{\sqrt{5}}$.

Grading:

The computation of $\nabla f(x,y)$ has 3 points. The rest of each question has 4 points. Those who write the correct concept but calculation is wrong or insert wrong parameter will gain only 2 point in each questions.