1. (10%) Find the volume of the solid bounded below by the cone $z^2 = 4(x^2+y^2)$ and above by the ellipsoid $4(x^2+y^2)+z^2 = 8$.

Solution:

Method 1 Use cylindrical coordinates:

Note that the region is **bounded** *below* by the cone and *above* by the ellipsoid, so it only consist of the part above the xy-plane!

$$\begin{aligned} V &= \iint\limits_{x^2+y^2\le 1} (\sqrt{8-4(x^2+y^2)} - \sqrt{4(x^2+y^2)})\, dxdy \\ &= \int_0^{2\pi}\int_0^1 2(\sqrt{2-r^2} - 2r)\, rdrd\theta \\ &= \frac{8\pi}{3}(\sqrt{2}-1) \end{aligned}$$

Scoring to parts of this method:

1. Integral domain $x^2+y^2 \le 1$: 2 pts
2. Upper bound of z , i.e. $\sqrt{8-4(x^2+y^2)}$: get 2 pts
3. Lower bound of z , i.e. $\sqrt{4(x^2+y^2)}$: get 2 pts
4. Jacobian of polar coordinates: get 2 pts
5. Your result fits the correct answer: get 2 pts, if you make a slight mistake, get 1 pt.

Method 2 Use spherical coordinates:

Let $(x,y,z) = (\sqrt{2}r\sin\phi\cos\theta, \sqrt{2}r\sin\phi\sin\theta, 2\sqrt{2}r\cos\phi)$. The Jacobian is

$$\begin{aligned} |J(u,v)| = |\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}| &= |\begin{vmatrix} \sqrt{2}\sin\phi\cos\theta & \sqrt{2}r\sin\phi\sin\theta & 2\sqrt{2}\cos\phi \\ -\sqrt{2}r\sin\phi\sin\theta & \sqrt{2}r\sin\phi\cos\theta & 0 \\ \sqrt{2}r\cos\phi\cos\theta & \sqrt{2}r\cos\phi\sin\theta & -2\sqrt{2}r\sin\phi \end{vmatrix}| \\ &= 4\sqrt{2}r^2\sin\phi \end{aligned}$$

Then the corresponding domain is

$$\begin{cases} 0 \le r \le 1 \\ 0 \le \phi \le \frac{\pi}{4} \\ 0 \le \theta \le 2\pi \end{cases}$$

$$V = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 4\sqrt{2}r^2\sin\phi drd\phi d\theta$$

Scoring to parts of this method:

1. Integral domain for r: 2 pts
2. Integral domain for ϕ: get 2 pts
3. Jacobian of spherical coordinate: get 4 pts, and if you miss the multiple before $r^2\sin\phi$, get 2 pts.
4. Your result fits the correct answer: get 2 pts, if you make a slight mistake, get 1 pt.

2. (12%) Evaluate $\iint\limits_{\Omega} e^{-4x^2-9y^2}\,dxdy$, where Ω is the region satisfying $2x \leq 3y$ and $x \geq 0$.

Solution:

Let

$$\begin{cases} x = \dfrac{r}{2}\cos\theta \\ y = \dfrac{r}{3}\sin\theta \end{cases} \quad (3\%)$$

then

$$J = \frac{r}{6} \quad (3\%)$$

Therefore

$$\begin{aligned} & \int\int_{\Omega} e^{-4x^2-9y^2}\,dxdy \\ = & \int_{\pi/4}^{\pi/2}\int_0^{\infty} e^{-r^2}|J|\,dr\,d\theta \quad (3\%) \\ = & \frac{\pi}{48} \quad (3\%) \end{aligned}$$

3. (12%) Evaluate the surface integral $\displaystyle\iint_S (x^2+y^2)z\,d\sigma$, where S is the part of the plane $z = 4+x+y$ that lies inside the cylinder $x^2+y^2=4$.

Solution:

To parametrize this surface: $\mathbf{r} = (x, y, 4+x+y)$, where $x^2+y^2 \leq 4$.

So $d\sigma = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \cdot dxdy = \sqrt{3}dxdy$

$$\iint_S (x^2+y^2)zd\sigma$$
$$= \iint_{x^2+y^2\leq 4} (x^2+y^2)(4+x+y)\sqrt{3}dxdy$$
$$= \int_0^{2\pi}\int_0^2 (r^2)(4+r\cos\theta+r\sin\theta)\sqrt{3}rdrd\theta$$
$$= \int_0^{2\pi}\int_0^2 (r^2)(4)\sqrt{3}rdrd\theta \text{ (by symmetry)}$$
$$= \int_0^{2\pi}\int_0^2 4\sqrt{3}r^3drd\theta$$
$$= \int_0^{2\pi} 16\sqrt{3}d\theta$$
$$= 32\sqrt{3}\pi$$

評分標準如下:

將曲面參數化: 2分
算出曲面的Jacobian: 2分
將面積分換成可處理的積分式，並且寫出完整的積分區域: 4分
計算積分: 4分

其餘錯誤酌量扣分

將此積分視為三重積分者一律不給分!!!!

4. (12%) Find the line integral

$$\int_{\mathcal{C}} (2x\sin(\pi y) - e^z)\,dx + (\pi x^2\cos(\pi y) - 3e^z)\,dy - xe^z\,dz$$

along the curve $\mathcal{C} = \left\{(x, y, z) | z = \ln\sqrt{1+x^2},\ y = x,\ 0 \le x \le 1\right\}$.

Solution:

Since $F = (2x\sin(\pi y) - e^z)\mathbf{i} + (\pi x^2\cos(\pi y) - 3e^z)\mathbf{j} - xe^z\mathbf{k}$ has

$F + 3e^z\mathbf{j} = \nabla(x^2\sin(\pi y) - xe^z)$ (4 points), so

$\int_C F \cdot dr = -\sqrt{2} - \int_0^1 3e^z dy$ (3 points)

$= -\sqrt{2} - 3\int_0^1 \sqrt{1+x^2}dx$ (3 points)

$= -(5\sqrt{2} + 3\ln(\sqrt{2}+1))/2$ (2 points)

5. (10%) Evaluate $\displaystyle\oint_{r=1-\cos\theta} (x^2y+y)dx-(xy^2-x)dy$ with the curve oriented counterclockwise.

Solution:

By Green's theorem,

$$\begin{aligned}\oint_{r=1-\cos\theta}(x^2y+y)dx-(xy^2-x)dy &= \iint_\Omega -(x^2+y^2)dA\\ &= \int_0^{2\pi}\int_0^{1-\cos\theta} -r^2\cdot r dr d\theta\\ &= -\frac{1}{4}\int_0^{2\pi}(1-\cos\theta)^4 d\theta\\ &= -\frac{35}{16}\pi.\end{aligned}$$

Green's thm : 3pts , region : 3pts ,Jacobian : 2pts , computation : 2pts.

6. (12%) Let $\mathbf{V} = (2x-y)\mathbf{i}+(2y+z)\mathbf{j}+x^2y^2z^2\mathbf{k}$ and let S be the upper half of the ellipsoid $\dfrac{x^2}{4}+\dfrac{y^2}{9}+\dfrac{z^2}{25}=1$. Find the flux of curl$\mathbf{V}$ in the direction of the upper unit normal $\mathbf{n}$ (pointing away from the origin.).

Solution:

Solution 1

By using Stokes' Theorem,

$\iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{n}\ ds = \oint_C \mathbf{V}(\mathbf{r})dr$ (4 points)

$C : r(\theta) = 2\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j}$, $\theta \in [0, 2\pi]$

$\oint_C \mathbf{V}(\mathbf{r})dr = \int_0^{2\pi} (10\sin\theta\cos\theta + 6\sin^2\theta)\ d\theta$ (4 points)

$= [\frac{5}{2}(-\cos 2\theta) + 3(\theta - \frac{1}{2}\sin 2\theta)]\ |_0^{2\pi}$

$= 6\pi.$ (4 points)

Solution 2

$\nabla \times \mathbf{V} = (2yx^2z^2 - 1)\mathbf{i} + (-2xy^2z^2)\mathbf{j} + \mathbf{k}.$ (3points)

$\because\ \nabla\cdot(\nabla\times\mathbf{V}) = 0$ (2points)

By using divergence theorem,

$\therefore \iint (\nabla \times \mathbf{V})\cdot\mathbf{n}\ ds = \iiint \nabla\cdot(\nabla\times\mathbf{V})dv = 0$ (3points)

In the suface of the buttom, $\mathbf{n} = -\mathbf{k}$. (1points)

We can find that the solution is $0 - [-(2\cdot 3\pi)] = 6\pi$. (3points)

ps. Using Stokes' Theorem twice is permitted.

ps2. If students observe the unit normal of the bottom surface, and they only calculate the k-component of curl $\mathbf{V}$. They do NOT pay for without calculating other components.

7. (12%) Evaluate the flux of

$$\mathbf{V}(x,y,z) = (z^2x + y^2z)\mathbf{i} + \left(\frac{1}{3}y^3 + z\tan x\right)\mathbf{j} + (x^2z + 2y^2 + 1)\mathbf{k}$$

across S: the upper half sphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with normal pointing away from the origin.

Solution:

$$S : \begin{cases} x^2 + y^2 + z^2 = 1 \\ 0 \leq z \end{cases} \quad \text{upward.}$$

$$S_1 : \begin{cases} z = 0 \\ x^2 + y^2 \leq 1 \end{cases} \quad \text{downward.}$$

$$\Omega : \begin{cases} x^2 + y^2 + z^2 \leq 1 \\ 0 \leq z \end{cases}$$

Let

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} .$$

Then

$$\Omega : \begin{cases} \rho^2 \leq 1 \\ 0 \leq \rho\cos\phi \end{cases} \Rightarrow \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \pi/2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$S : \begin{cases} \rho^2 = 1 \\ 0 \leq \rho\cos\phi \end{cases} \Rightarrow \begin{cases} \rho = 1 \\ 0 \leq \phi \leq \pi/2 \\ 0 \leq \theta \leq 2\pi \end{cases} .$$

Let

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} .$$

Then

$$S_1 : \begin{cases} z = 0 \\ r^2 \leq 1 \end{cases} \Rightarrow \begin{cases} z = 0 \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases} \quad \mathbf{d}\,\text{area} = -\mathbf{k}\, r\,dr\,d\theta$$

By divergent theorem,

$$\iiint_\Omega \nabla\cdot\mathbf{V}\,d\,\text{volumn} = \iint_{\partial\Omega = S+S_1} \mathbf{V}\cdot\mathbf{d}\,\text{area} = \iint_S \mathbf{V}\cdot\mathbf{d}\,\text{area} + \iint_{S_1} \mathbf{V}\cdot\mathbf{d}\,\text{area} \quad \text{(2 pts).}$$

$$\begin{aligned}\iiint_\Omega \nabla\cdot\mathbf{V}\,d\,\text{volumn} &= \iiint_\Omega z^2 + y^2 + x^2\,d\,\text{volumn} = \iiint_\Omega \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1} \rho^4\sin\phi\,d\rho\,d\phi\,d\theta \quad \text{(3 pts)} \\ &= \frac{\rho^5}{5}\bigg|_0^1 \cdot (-\cos\phi)\bigg|_0^{\pi/2} \cdot 2\pi = \frac{2}{5}\pi \quad \text{(1 pt)}\end{aligned}$$

$$\iint_{S_1} \mathbf{V} \cdot \mathbf{d}\,\text{area} = \iint_{S_1} -(x^2 z + y^2 + 1)\ r\,dr\,d\theta$$

$$= -\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \big((r \sin\theta)^2 + 1\big)\ r\,dr\,d\theta \quad \text{(3 pts)}$$

$$= -2 \cdot \frac{r^4}{4}\Big|_0^1 \cdot \pi - \pi = -\frac{3\pi}{2} \quad \text{(1 pt)}$$

$$\Rightarrow \iint_{S} \mathbf{V} \cdot \mathbf{d}\,\text{area} = \iiint_{\Omega} \nabla \cdot \mathbf{V}\, d\,\text{volumn} - \iint_{S_1} \mathbf{V} \cdot \mathbf{d}\,\text{area}$$

$$= \frac{19\pi}{10} \quad \text{(2 pts)}$$

8. (10%) Find stationary points of $f = 3xy - x^3 - y^3 + 2$. Determine which are local maximum, local minimum or a saddle point.

Solution:

$\nabla f = (3y - 3x^2, 3x - 3y^2) = (0, 0) \Rightarrow y = x^2$ and $x = y^2$

$\Rightarrow (x, y) = (0, 0)$ and $(1, 1)$ (stationary points)

$D = f_{xx} f_{yy} - f_{xy}^2 = 36xy - 9$

$D(0, 0) = -9 < 0 \Rightarrow$ (0,0) is a saddle point.

$D(1, 1) = 27 > 0$ and $f_{xx}(1, 1) = -6 < 0 \Rightarrow$ (1,1) is a local maximum.

2 pts if stationary points are all wrong.

5 pts if D is wrong or you just get one stationary point.

$6 - 8$ pts if you have some mistakes.

9. (10%) Use Lagrange multiplier to find the maximum and the minimum of $f(x,y) = 3x^2 - 2y^2$ for x, y on the curve $2x^2 - 2xy + y^2 = 1$. (You don't have to give the locations of these extrema.)

Solution:

by Lagrange multiplier, we have

$6x = \lambda(4x - 2y)$ and $-4y = \lambda(-2x + 2y)$ and $2x^2 - 2xy + y^2 = 1$

if $\lambda = 0$ then $x = y = 0$ it is not satisfies $2x^2 - 2xy + y^2 = 1$

so, $\lambda \neq 0$ we get

$\dfrac{6x}{\lambda} - 4x = -2y$ and $\dfrac{-4y}{\lambda} - 2y = -2x$ and $2x^2 - 2xy + y^2 = 1$

$\Rightarrow \dfrac{(\frac{6}{\lambda} - 4)}{-2} = \dfrac{-2}{\frac{-4}{\lambda} - 2}$

$\Rightarrow \lambda = 2$ or -3

if $\lambda = 2$ then $x = 2y$ and $y = \dfrac{x}{2}$

from $2x^2 - 2xy + y^2 = 1$ we get $8y^2 - 4y^2 + y^2 = 1 \Rightarrow y^2 = \dfrac{1}{5}$

and $2x^2 - x^2 + \dfrac{x^2}{4} = 1 \Rightarrow x^2 = \dfrac{4}{5}$

so, $3x^2 - 2y^2 = 2$ (maximum)

if $\lambda = -3$ then $3x = y$ and $x = \dfrac{y}{3}$

from $2x^2 - 2xy + y^2 = 1$ we get $2x^2 - 6x^2 + 9x^2 = 1 \Rightarrow x^2 = \dfrac{1}{5}$

and $\dfrac{2}{9}y^2 - \dfrac{2}{3}y^2 + y^2 = 1 \Rightarrow y^2 = \dfrac{9}{5}$

so, $3x^2 - 2y^2 = -3$ (minimum)

no point if you don't use Lagrange multiplier.

$2 - 3$ pts if you don't get any extreme values.

5 pts if both extreme values are wrong.

$7 - 8$ pts if one of the extreme values is wrong.