1011微乙01-05班期中考解答和評分標準

1. (10%) 求 $\lim_{x\to 2}\dfrac{\sqrt[3]{x^3+x-8}-\sqrt[3]{x}}{x-2}$.

Solution:

$$\begin{aligned}\frac{\sqrt[3]{x^3+x-8}-\sqrt[3]{x}}{x-2} &= \frac{\left(\sqrt[3]{x^3+x-8}-\sqrt[3]{x}\right)\left(\left(\sqrt[3]{x^3+x-8}\right)^2+\sqrt[3]{x^3+x-8}\sqrt[3]{x}+(\sqrt[3]{x})^2\right)}{(x-2)\left(\left(\sqrt[3]{x^3+x-8}\right)^2+\sqrt[3]{x^3+x-8}\sqrt[3]{x}+(\sqrt[3]{x})^2\right)}\\ &= \frac{x^3+x-8-x}{(x-2)\left(\left(\sqrt[3]{x^3+x-8}\right)^2+\sqrt[3]{x^3+x-8}\sqrt[3]{x}+(\sqrt[3]{x})^2\right)}\\ &= \frac{x^2+2x+4}{\left(\sqrt[3]{x^3+x-8}\right)^2+\sqrt[3]{x^3+x-8}\sqrt[3]{x}+(\sqrt[3]{x})^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt[3]{x^3+x-8}-\sqrt[3]{x}}{x-2} &= \lim_{x\to 2}\frac{x^2+2x+4}{\left(\sqrt[3]{x^3+x-8}\right)^2+\sqrt[3]{x^3+x-8}\sqrt[3]{x}+(\sqrt[3]{x})^2}\\ &= \frac{4+4+4}{\left(\sqrt[3]{2}\right)^2+\left(\sqrt[3]{2}\right)^2+\left(\sqrt[3]{2}\right)^2}\\ &= 2^{4/3}\end{aligned}$$

2. (10%) 用均值定理說明 $|\tan x-\tan y|\le 2|x-y|$ 對 $x,y\in[0,\pi/4]$ 均成立。

Solution:

If $x=y$, then $|tanx-\tan y|=2|x-y|=0$. In general case, we may assume that $x>y$, and x, $y\in[0,\pi/4]$. First, we take the differential

$$\frac{d}{dx}\tan x=\sec^2 x \quad \text{(2 points)}.$$

By the mean value theorem, we have

> there exists ξ with $y<\xi<x$ such that
>
> $$\tan x-\tan y=\sec^2\xi(x-y)$$
>
> for all x,$y\in[0,\pi/4]$.

(5 points)

In the interval $[0,\pi/4]$, we have

$$\sec^2\xi\le\sec^2\frac{\pi}{4}=2 \quad \text{(2 points)}$$

so

$$|\tan x-\tan y|=|\sec^2\xi||x-y|\le 2|x-y| \quad \text{(1 point)}$$

for all x,$y\in[0,\pi/4]$. It completes the proof.

3. (10%) 給定一個方程式 $xy^2+x^2y-2=0$，求在點$(x,y)=(1,1)$的切線方程式。

Solution:

Use implicit differentiation.

$$\frac{d}{dx}(xy^2+xy^2-2)=\frac{d}{dx}(0)$$

$$\Rightarrow y^2+x(2y)y'+2xy+x^2y'=0 \text{..}(8pts)$$

$$(x,y)=(1,1)\Rightarrow \quad 1+2y'+2+y'=0\Rightarrow y'=1.$$

So the tangent line at $(x, y) = (1, 1)$ is

$$\frac{y-1}{x-1} = -1 (\text{or} \;\; x + y = 2) \ldots\ldots (2pts)$$

4. 令 $f(x) = x^7 - 7x^6 + 5x^4$
   (a) (10%) 求 $f(x)$ 在 $x = 1$ 處之線性逼近。
   (b) (5%) 以之求 $f(0.92)$ 之近似值。

Solution:
(a)

$$\begin{aligned} f'(x) &= 7x^6 - 42x^5 + 20x^3 \\ f'(1) &= -15, f(1) = -1 \\ f(x) &\approx -15(x-1) - 1 \end{aligned}$$

(b)

$$f(0.92 \approx -15(0.92 - 1) - 1 = 0.2$$

5. (15%) (每小題5%)計算下列函數之導函數

$$(1)\ \sin(3^x) \quad (2)\ e^{\tan^{-1} x} \quad (3)\ \frac{\ln x}{x}$$

Solution:

(1) $(\sin(3^x))' = [2\%] \cos(3^x) \cdot (3^x)' = [2\%] \cos(3^x) \cdot 3^x \cdot \ln 3.$
The rest 1% depends on the detail of your answer.

(2) $(e^{\tan^{-1} x}) = [2\%] e^{\tan^{-1} x} \cdot (\tan^{-1} x)' = [2\%] e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}.$
The rest 1% depends on the detail of your answer.
Another solution: Apply the formula on p45, $(f(x)^{g(x)})' = [2\%] g(x) \cdot f(x)^{g(x)-1} \cdot f'(x) + [2\%] \ln f(x) \cdot f(x)^{g(x)} \cdot g'(x)$. (Here $f(x) = e, g(x) = \tan^{-1} x$)

(3) Use the quotient rule for derivatives: $\left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = [2\%] \frac{1}{x^2} - [2\%] \frac{\ln x}{x^2}$
The rest 1% depends on the detail of your answer.

6. (15%) 設計一圓柱形無蓋的杯子 (厚度忽略不計)，假設在容積固定為1000立方公分的條件下，試問使得杯子表面積最小的杯底半徑為何? (12%) 需使用極值測試法。(3%)

Solution:
let r be the radius of the bottom of the cylinder
and let h be the height of the cylinder

we have fixed volume $\pi r^2 h = 1000$ (2 pts)
and the surface area $A(r) = \pi r^2 + 2\pi r h$ (2 pts)

$$\begin{aligned} &= \pi r^2 + 2\pi r \frac{1000}{\pi r^2} \\ &= \pi r^2 + \frac{2000}{r} \text{ (2 pts)} \end{aligned}$$

$\Rightarrow A'(r) = 2\pi r - \dfrac{2000}{r^2} = \dfrac{2\pi r^3 - 2000}{r^2}$ (3 pts)

$A'(r) = 0$ iff $2\pi r^3 - 2000 = 0$ iff $r = \dfrac{10}{\sqrt[3]{\pi}}$ (3 pts)

if $r > \dfrac{10}{\sqrt[3]{\pi}}$, since $2\pi r^3 - 2000 > 0$ and $r^2 > 0$

we have $A'(r) = \dfrac{2\pi r^3 - 2000}{r^2} > 0$ for all $r > \dfrac{10}{\sqrt[3]{\pi}}$

$\Rightarrow A(r)$ is increasing for all $r > \dfrac{10}{\sqrt[3]{\pi}}$

if $0 < r < \dfrac{10}{\sqrt[3]{\pi}}$, since $2\pi r^3 - 2000 < 0$ and $r^2 > 0$

we have $A'(r) = \dfrac{2\pi r^3 - 2000}{r^2} < 0$ for all $0 < r < \dfrac{10}{\sqrt[3]{\pi}}$

$\Rightarrow A(r)$ is decreasing for all $0 < r < \dfrac{10}{\sqrt[3]{\pi}}$

we thus conclude that the cylinder has minimum surface area as $r = \dfrac{10}{\sqrt[3]{\pi}}$ (3 pts)

(if you just say that r is a local minima, you get only 1 pt)

(另解)
本題批改方式為分段給分，其中「設定區」佔6分，「計算區」佔6分，「極值測試」佔3分，。每一區詳細配分標註於解法中。從計算錯誤的地方開始之後都不給分，例如一開始就設定錯誤就不給分。

根據考試規則，只寫出結果者不給分，常見於「極值測試」中，直接寫出一階微分或二階微分正負號而沒有計算過程，或是只畫個簡圖或表格，都無法判斷是因為題目指定要最小值而把結果直接寫上去，或是自己算出來的，這樣的狀況都不給分。

設定區: 設底面半徑為 r, 高為 h，則:
體積 $V(r) = 1000 = \pi r^2 h$ (2 pts)
表面積 $A(r) = 2\pi rh + \pi r^2$ (注意題目為「無蓋」) (2 pts)
將 $h = \dfrac{1000}{\pi r^2}$ 帶入 $A(r)$ 得 $A(r) = \pi r^2 + \dfrac{2000}{r}$ (2 pts)

計算區: 求 $A(r)$ 的極值，即求滿足 $A'(r) = 0$ 的點：

$$A'(r) = 2\pi r - \frac{2000}{r^2} \quad \text{(3 pts)}$$
$$A'(r) = 0 \quad \Rightarrow \quad -2000 + 2\pi r^3 = 0$$
$$\Rightarrow \quad r = \frac{10}{\pi^{1/3}} \quad \text{(3 pts)}$$

極值測試: 一階測試：(3 pts)

$$A'(r) > 0 \quad \Leftrightarrow \quad -2000 + 2\pi r^3 > 0 \quad \Leftrightarrow \quad r > \frac{10}{\pi^{1/3}}$$
$$A'(r) < 0 \quad \Leftrightarrow \quad -2000 + 2\pi r^3 > 0 \quad \Leftrightarrow \quad r < \frac{10}{\pi^{1/3}}$$

故 $A(r)$ 在 r 大於 $\dfrac{10}{\pi^{1/3}}$ 遞增，在 r 小於 $\dfrac{10}{\pi^{1/3}}$ 時遞減，所以在 $r = \dfrac{10}{\pi^{1/3}}$ 發生最小值。

註1: 若只在 $r = \dfrac{10}{\pi^{1/3}}$ 這個點上用二階測試，只能確定他產生「局部極小值」，無法確定他是「最小值」，這種狀況給1分。

註2: 若在「計算區」中用算幾不等式者，當作沒有做極值測試，若都寫對的話給12分。

7. (25%) 令 $y = f(x) = \dfrac{x(x-2)+2}{x-1}$. 回答以下各小題 (若不存在的話，須註明不存在):

(a) $y = f(x)$ 的遞增區間 $(-\infty, 0] \cup [2, \infty)$， $y = f(x)$ 的遞減區間 $[0, 1) \cup (1, 2]$

(b) $y = f(x)$ 之極大值(座標) $(0, -2)$， $y = f(x)$ 之極小值(座標) $(2, 2)$

(c) $y = f(x)$ 的凹向上區間 $(1, \infty)$， $y = f(x)$ 的凹向下區間 $(-\infty, 1)$

(d) $y = f(x)$ 之反曲點(座標) f has no inflection point on $\mathbb{R}$

(e) $y = f(x)$ 所有的漸近線為 $x = 1$、$y = x - 1$

(f) 畫出 $y = f(x)$ 的圖形。

Solution:
For f to be well-defined, we need $x \neq 1$. Then we can simplify f as

$$\begin{aligned} f(x) &= (x-1) + \frac{1}{(x-1)} \\ \Longrightarrow f'(x) &= 1 - \frac{1}{(x-1)^2} \\ \Longrightarrow f''(x) &= \frac{2}{(x-1)^3} \end{aligned}$$

Hence we observe that:

(a.1) $f'(x) \geq 0$ for $x \in (-\infty, 0] \cup [2, \infty) \Longrightarrow f$ is increasing on $(-\infty, 0] \cup [2, \infty)$

(a.2) $f'(x) \leq 0$ for $x \in [0, 1) \cup (1, 2] \Longrightarrow f$ is decreasing on $[0, 1) \cup (1, 2]$

Furthermore, $f'(x) = 0$ for $x = 0$ or 2, and from the two observations above, we have

(b.1) f has a local maximum at $(0, -2)$

(b.2) f has a local minimum at $(2, 2)$

And to observe the convexity and concavity.

(c.1) $f''(x) \geq 0$ for $x \geq 1 \Longrightarrow f$ is convex on $(1, \infty)$

(c.1) $f''(x) \leq 0$ for $x \leq 1 \Longrightarrow f$ is concave on $(-\infty, 1)$

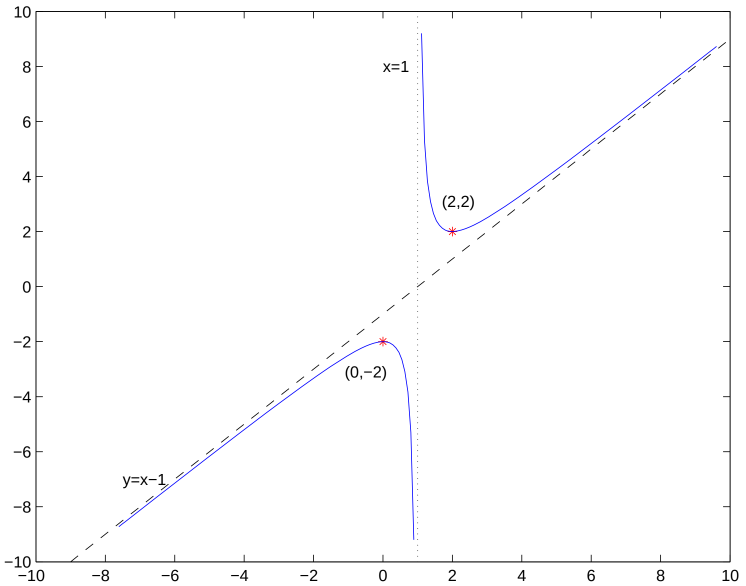
Now we observe the inflection point and asymptote.

(d) $f''(x) \neq 0$ for all $x \in \mathbb{R} \Longrightarrow f$ has no inflection point on $\mathbb{R}$

(e) $f(x)$ tends to infinite when x tends to $1 \Longrightarrow x = 1$ is an asymptote of f and
$f(x) - (x-1) = \dfrac{1}{(x-1)} \to 0$ as $x \to \infty \Longrightarrow y = x - 1$ is an asymptote of f

Finally, we can figure the graph by the above observations.

(f)



Grading evaluation:

(a.1) If your answer is "$(-\infty, 0), (2, \infty)$", "$(-\infty, 0], (2, \infty)$", "$(-\infty, 0), [2, \infty)$", or "$(-\infty, 0], [2, \infty)$", you will get 2 points.
If your answer only holds one of the two intervals, you will get 1 point.

(a.2) If your answer is "$(0, 1), (1, 2)$", "$[0, 1), (1, 2)$", "$(0, 1), (1, 2]$", or "$[0, 1), (1, 2]$", , you will gent 2 points.
Moreover, if your answer contains $\{1\}$, we also give you 2 points.
If your answer only holds one of the two intervals, you will get 1 point.

(b.1) If your answer is "$(0, -2)$" or "-2", you will get 2 points.

(b.2) If your answer is "$(2, 2)$" or "2", you will get 2 points.

(c.1) If your answer is "$(1, \infty)$" or "$[1, \infty)$", you will get 2 points.

(c.2) If your answer is "$(\infty, 1)$" or "$(\infty, 1]$", you will get 2 points.

(d) If your answer is "$x = 1$" or "nowhere", you will get 2 points.

(e) If your answer is "$x = 1, y = x - 1$", you will get 4 points.
If your answer holds one of the two asymptotes, you will get 2 points.

(f) If your graph of f is wrong, you have no point.
If your graph of f is nearly the true graph, you will get 1 point. And if you figure the two asymptotes, you will get 1 more point; if you mark the local maximum point and local minimum point (you also need to write the coordinates), you will get 1 more point.