1. (12 points) Evaluate the following limits.

(a) $\lim\limits_{x\to 2^-} \dfrac{x-\sqrt{2x}}{|x^2-4|}$.

(b) $\lim\limits_{x\to 0} \dfrac{1-\sin^2(ax)-\cos(ax)}{1+\sin^2(bx)-\cos(bx)}$, where a and b are non-zero real numbers.

Solution:

(1)

$$\begin{aligned}
\lim_{x\to 2^-}\frac{x-\sqrt{2x}}{|x^2-4|} &= \lim_{x\to 2^-}\frac{x-\sqrt{2x}}{4-x^2} && \text{(1.pt)}\\
&= \lim_{x\to 2^-}\frac{x^2-2x}{(4-x^2)(x+\sqrt{2x})} && \text{(2.pts)}\\
&= \lim_{x\to 2^-}\frac{-x}{(2+x)(x+\sqrt{2x})} && \text{(1pt.)}\\
&= \frac{-1}{8} && \text{(1.pt)}
\end{aligned}$$

(2)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\sin^2(ax)-\cos(ax)}{1+\sin^2(bx)-\cos(bx)} &= \lim_{x\to 0}\frac{2\sin^2\frac{a}{2}x-\sin^2(ax)}{2\sin^2\frac{b}{2}x+\sin^2(bx)} && \text{(2.pts)}\\
&= \lim_{x\to 0}\frac{2\cdot\frac{a^2x^2}{4}\cdot(\frac{\sin\frac{a}{2}x}{\frac{a}{2}x})^2-(a^2x^2)\cdot(\frac{\sin(ax)}{ax})^2}{2\cdot\frac{b^2x^2}{4}\cdot(\frac{\sin\frac{b}{2}x}{\frac{b}{2}x})^2+(b^2x^2)\cdot(\frac{\sin(bx)}{bx})^2} && \text{(2.pts)}\\
&= \lim_{x\to 0}\frac{a^2}{b^2}\frac{\cdot\frac{1}{2}\cdot(\frac{\sin\frac{a}{2}x}{\frac{a}{2}x})^2-(\frac{\sin(ax)}{ax})^2}{\frac{1}{2}\cdot(\frac{\sin\frac{b}{2}x}{\frac{b}{2}x})^2+(\frac{\sin(bx)}{bx})^2} && \text{(1.pt)}\\
&= \frac{a^2}{b^2}\frac{\frac{1}{2}-1}{\frac{1}{2}+1} = \frac{-a^2}{3b^2} \quad (\text{Since } \lim_{x\to 0}\frac{\sin x}{x}=1) && \text{(2.pts)}
\end{aligned}$$

One who uses **L'Höpital rule** without proof or giving a wrong proof will only get at most (3.pts) for each problem. The following is the answer of using L'Höpital rule to calculate the problems without proof and the credit distribution:

(1)

$$\begin{aligned}
\lim_{x\to 2^-}\frac{x-\sqrt{2x}}{|x^2-4|} &= \lim_{x\to 2^-}\frac{x-\sqrt{2x}}{4-x^2}\\
&= \lim_{x\to 2^-}\frac{1-\frac{1}{\sqrt{2x}}}{-2x} = -\lim_{x\to 2^-}\frac{\sqrt{2x}-1}{(2x)^{\frac{3}{2}}} && \text{(2.pts)}\\
&= -\frac{1}{8} && \text{(1.pt)}
\end{aligned}$$

(2)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\sin^2(ax)-\cos ax}{1+\sin^2 bx-\cos bx} &= \lim_{x\to 0}\frac{-2a\sin(ax)\cos(ax)+a\sin(ax)}{2b\sin(ax)\cos(bx)+b\sin(bx)} && \text{(1.pt)}\\
&= \lim_{x\to 0}\frac{-2a^2\left(\cos^2(ax)-\sin^2(ax)\right)+a^2\cos(ax)}{2b^2\left(\cos^2(ax)-\sin^2(bx)\right)+b^2\cos(bx)} && \text{(1.pt)}\\
&= \frac{-2a^2+a^2}{2b^2+b^2} = \frac{-a^2}{3b^2} \quad \text{(1.pt)}
\end{aligned}$$

2. (12 points) Suppose that $f(x)$ is differentiable on $(-1,1)$ with $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=L$, where L is a constant.

Define $g(x) = \begin{cases} f(x)\sin(\dfrac{1}{x}), \text{ for } 0 < |x| < 1; \\ A, \text{ for } x = 0. \end{cases}$

(a) Find $f(0)$ and $f'(0)$.
(b) If g is continuous at $x = 0$, find the value of A and compute $g'(0)$.
(c) Write down a formula of $g'(x)$ in terms of $f(x)$ and $f'(x)$ for $0 < |x| < 1$.
(d) Suppose that $f'(x)$ and $g'(x)$ are both continuous at 0. Find the value of L.

Solution:

(a) Since f is differentiable, f is continuous. Hence

$$\begin{aligned} f(0) &= \lim_{x\to 0} f(x) && (1) \\ &= \lim_{x\to 0} \frac{f(x)}{x^2}\cdot x^2 \\ &= \lim_{x\to 0} \frac{f(x)}{x^2}\cdot \lim_{x\to 0} x^2 \quad \text{since both limits exist} \\ &= L\cdot 0 = 0. \quad (1\%) && (2) \end{aligned}$$

By definition of derivative,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h-0} \quad (1\%) \\ &= \lim_{h\to 0} \frac{f(h)}{h} \quad \text{by (2)} && (3) \\ &= \lim_{h\to 0} \frac{f(h)}{h^2}\cdot h \\ &= \lim_{h\to 0} \frac{f(h)}{h^2}\cdot \lim_{h\to 0} h \quad \text{since both limits exist} \\ &= L\cdot 0 = 0. \quad (1\%) && (4) \end{aligned}$$

(b) Since g is continuous at $x = 0$,

$$\begin{aligned} A &= g(0) = \lim_{x\to 0} g(x) \\ &= \lim_{x\to 0} f(x)\sin(\frac{1}{x}) \quad \text{since } x \neq 0. \end{aligned}$$

Since $-1 \leq \sin(1/x) \leq 1$, we have

$$-|f(x)| \leq f(x)\sin(\frac{1}{x}) \leq |f(x)|.$$

Then by Pinching Theorem, we have by (1),

$$0 = \lim_{x\to 0} -|f(x)| \leq \lim_{x\to 0} f(x)\sin(\frac{1}{x}) \leq \lim_{x\to 0} |f(x)| = 0, \quad (1\%)$$

which means

$$A = g(0) = 0. \qquad (5)$$

By definition of derivative,

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h)-g(0)}{h-0} \quad (1\%) \\ &= \lim_{h\to 0} \frac{g(h)}{h} \quad \text{by (5)} \\ &= \lim_{h\to 0} \frac{f(h)}{h}\sin\frac{1}{x} \\ &= 0 \quad \text{by (3) and Pinching Theorem.} \quad (1\%) && (6) \end{aligned}$$

(c) For $x \neq 0$, by product rule and chain rule,

$$g'(x) = \frac{d}{dx} f(x)\sin(1/x) = f'(x)\sin(1/x) + f(x)\cos(1/x)(-x^{-2}). \quad (3\%) \qquad (7)$$

(d) Since $g'(x)$ is continuous at $x = 0$, by (7),

$$\begin{aligned} g'(0) &= \lim_{x\to 0} g'(x) \\ &= \lim_{x\to 0} \{f'(x)\sin(1/x) + f(x)\cos(1/x)(-x^{-2})\} \quad (1\%). \end{aligned} \tag{8}$$

Since $f'(x)$ is continuous at $x = 0$, by (4),

$$\lim_{x\to 0} f'(x) = f'(0) = 0.$$

Hence by Pinching Theorem,

$$\lim_{x\to 0} f'(x)\sin(1/x) = 0. \quad (1\%)$$

Therefore, by (6), (8) becomes

$$0 \;=\; g'(0) = \lim_{x\to 0} -\frac{f(x)}{x^2}\cos(1/x). \tag{9}$$

If $L \neq 0$,

$$\lim_{x\to 0} -\frac{f(x)}{x^2}\cos(1/x) = -L\lim_{x\to 0}\cos(1/x),$$

which doesn't exist and contradict to (9). (1%)

If $L(=\lim_{x\to 0}\frac{f(x)}{x^2}) = 0$, by Pinching Theorem,

$$\lim_{x\to 0}\frac{f(x)}{x^2}\cos(1/x) = 0,$$

which coincide with (9). Hence

$$L = 0. \tag{10}$$

3. (12 points) Let $a < b$. A function f in said to be a <u>*contraction*</u> on $[a, b]$ if there exists K, $0 < K < 1$, such that for all $x_1, x_2 \in [a, b]$ we have $|f(x_1) - f(x_2)| \leq K\,|x_1 - x_2|$.

   (a) Show by the $\epsilon - \delta$ definition that if f is a contraction on $[a, b]$, then f is continuous on $[a, b]$.

   (b) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| \leq q$, $0 < q < 1$, for all $x \in (a, b)$. Show that f is a contraction on $[a, b]$ and has at most one fixed point on $[a, b]$. (A point $c \in [a, b]$ is called a fixed point of f if $f(c) = c$.)

Solution:

(a)Take $c \in [a, b]$. Let $\epsilon > 0$ be given, take $\delta = \frac{\epsilon}{K}$. Then for all $x \in [a, b]$ and $|x - c| < \delta$, we have $|f(x) - f(c)| \leq K|x - c| < K \cdot \frac{\epsilon}{K} = \epsilon$. So f is continuous at c. As c is arbitrary, f is continuous on $[a, b]$.(4points)

(b)(1)By Mean-value theorem, for any $x_1 < x_2$ and $x_1, x_2 \in [a, b]$, we have $f^{'}(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, where $x_3 \in [x_1, x_2]$. Hence we have $|f(x_2) - f(x_1)| = |f^{'}(x_3)||x_2 - x_1| \leq q|x_2 - x_1|$. So we have $|f(x_2) - f(x_1)| \leq q|x_2 - x_1|$, where $0 < q < 1$. Then we can conclude f is a contraction.(Note that if $x_1 = x_2$, the result follows directly.(4 points)

(2)If not, $\exists x_4 < x_5$ and $f(x_4) = x_4, f(x_5) = x_5$. By mean-value theorem, we have $f^{'}(x_6) = \frac{f(x_5) - f(x_4)}{x_5 - x_4} = \frac{x_5 - x_4}{x_5 - x_4} = 1$,. $f^{'}(x_6) = 1$ contradicates the assumption, $f^{'}(x) < 1$, for all $x \in [a, b]$.Hence f has at most one fixed point on $[a, b]$.(4points)

4. (12 points) Suppose ABC is a triangle with vertices $A = (-5, 0), B = (0, 10)$ and $C = (5, 0)$. Let P be a point on the line segment that join B to the origin. Find the position of P that minimizes the sum of distances between P and the three vertices of the triangle ABC.

Solution:

Let $P = (0, y)$
and set

$$S(y) = (10 - y) + 2\sqrt{25 + y^2} \quad \text{(2.pts)}$$

One who wrote the wrong function will not get any point !

Then, we solve $S'(y) = 0$, we have

$$S'(y) = -1 + \frac{2y}{\sqrt{y^2 + 25}} = 0 \quad \Rightarrow \quad y = \frac{5}{\sqrt{3}}$$

[(2.pts) for Calculate $S'(y)$, (2.pts) for solving $y = \dfrac{5}{\sqrt{3}}$ and (1.pt) for choosing the right sign of y]

Next, we exam whether $S(y)$ takes minimum at $y = \dfrac{5}{\sqrt{3}}$, by Second derivative test, we have

$$S''(y)\Big|_{y=\frac{5}{\sqrt{3}}} = \frac{50 - y^2}{(y^2 + 25)^{\frac{3}{2}}}\Big|_{y=\frac{5}{\sqrt{3}}} > 0$$

[((3.pts) for calculate $S''(y)$), (1.pt) for $S''(\dfrac{5}{\sqrt{3}}) > 0$)]

One who didn't check the minimality will not get this (4.pts), other methods to check are 1. First derivative test, 2. using the Extreme value theorem to check critical point and end points.

Hence $S(y)$ has a minimum at $y = \dfrac{5}{3}$, and

$$P = (0, \frac{5}{\sqrt{3}}) \quad \text{(1.pt)}$$

5. (12 points) Suppose that $f(x)$ is continuous and increasing on $[-1, 2]$ with $f(x) > 0$. Let $F(x) = \displaystyle\int_0^x \left(t \int_1^t f(u)du \right) dt$.

(a) Classify all critical points of $F(x)$ in $(-1, 2)$.

(b) Show that $F''(x)$ is increasing on $(0, 1)$ and there is a point of inflection of $F(x)$ on $(\frac{1}{2}, 1)$.

Solution:

(a)

$F(x) = \displaystyle\int_0^x \left(t \int_1^t f(u)du \right) dt$

$\Rightarrow F'(x) = x \displaystyle\int_1^x f(u)du$ **(1 point)**

$\Rightarrow F''(x) = \displaystyle\int_1^x f(u)du + x \cdot f(x)$

$F'(x) = 0$ when $x = 0$ or $\displaystyle\int_1^x f(u)du = 0$

i.e. $x = 0$ (**1 point**) or $x = 1$ (**1 point**)

Observe that

$F'(x) > 0$ when $x \in (-1, 0) \cup (1, 2)$, (**1 point**)

$F'(x) < 0$ when $x \in (0, 1)$. (**1 point**)

(Or, $F''(0) = \displaystyle\int_1^0 f(u)du < 0$ and $F''(1) = f(1) > 0$)

Thus $x = 0$ is a local maximum, (**1 point**)

$x = 1$ is a local minimum. (**1 point**)

(b)(1)

$F''(x) = \displaystyle\int_1^x f(u)du + x \cdot f(x)$ (**1 point**)

For $x > 0$,
$\because \dfrac{d}{dx}\left(\int_1^x f(u)du\right) = f(x) > 0$
$\therefore \int_1^x f(u)du$ is increasing.
Since $x, f(x), \int_1^x f(u)du$ are all increasing, $F''(x)$ is increasing on $(0,1)$. (**1 point**)

(2)
$\because F''(1) = f(1) > 0$, (**1 point**)

$$\begin{aligned} F''(\tfrac{1}{2}) &= \int_1^{\frac{1}{2}} f(x)dx + \frac{1}{2}\cdot f(\frac{1}{2}) \\ &= -\int_{\frac{1}{2}}^1 f(x)dx + \int_{\frac{1}{2}}^1 f(\frac{1}{2})dx \\ &= \int_{\frac{1}{2}}^1 \left[f(\frac{1}{2}) - f(x)\right] dx \\ &< 0, \qquad \left(\because f(x) \text{ is increasing on } [\frac{1}{2},1] \Rightarrow f(\frac{1}{2}) < f(x),\ \forall x \in (\frac{1}{2},1]\right) \end{aligned}$$

(**1 point**)

And also, $F''(x)$ is continuous on $[\frac{1}{2},1]$
$\therefore$ By Intermediate Value Theorem, $F''(c) = 0$ for some $c \in (\frac{1}{2},1)$. (**1 point**)

Since $F''(x)$ is increasing on $(0,1)$,
$\Rightarrow F''(x) < 0$ for $x \in (\frac{1}{2},c)$ and $F''(x) > 0$ for $x \in (c,1)$
$\Rightarrow x = c$ is a point of inflection.

6. (12 points) From the equation $\sqrt{1+y} - \int_0^{x^2-1} \dfrac{dt}{1+t^2} + \tan(xy) = 1$ a differentiable function $y = y(x)$ can be determined around $(x,y) = (1,0)$.
   (a) Evaluate y' at $(x,y) = (1,0)$.
   (b) Evaluate y'' at $(x,y) = (1,0)$ and determine the concavity of $y = y(x)$ around this point.

Solution:
(i) Apply $\dfrac{\mathrm{d}}{\mathrm{d}x}$ on both sides of the equation.
(Calculation of derivatives 5%)

$$\frac{1}{2\sqrt{1+y}}y' - \frac{1}{1+(x^2-1)^2}(2x) + (y+xy')\sec^2(xy) = 0$$

1st term: 1%
2nd term: FTC 1%, chain rule 1%
3rd term: $\dfrac{d\tan x}{dx}$ 1%, chain rule 1%
Put in $(x,y) = (1,0)$
(Evaluation 1%. Your answer should be matched.)

$$\frac{1}{2}y'|_{(1,0)} - 2 + y'|_{(1,0)} = 0$$
$$y'|_{(1,0)} = \frac{4}{3}$$

(ii) Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ on both sides of the above equation again.

$$\frac{y''}{2\sqrt{1+y}} - \frac{y'}{4\sqrt{(1+y)^3}}y' - \frac{2(1+(x^2-1)^2) - 2x\cdot 2(x^2-1)2x}{(1+(x^2-1)^2)^2}$$
$$+ (y'+y'+xy'')\sec^2(xy) + (y+xy')2\sec(xy)[\sec(xy)\tan(xy)](y+xy') = 0$$

(If the calculation of derivatives in (i) is correct, there is total 4% in this part; otherwise, there is total 3% in this part. You get -1% for one mistake with respect to your result in (i) until you get 0% in this part.)
Put in $(x, y) = (1, 0)$
(Evaluation 1%. Your answer should be matched.)

$$\frac{y''|_{(1,0)}}{2} - \frac{1}{4}(\frac{4}{3})^2 - 2 + (2\cdot\frac{4}{3} + y''|_{(1,0)}) + 0 = 0$$
$$y''|_{(1,0)} = -\frac{4}{27}$$

(Determine your concavity with respect to your evaluation. 1%)
Since $y'' < 0$ at $(1, 0)$, the graph is concave down around $(1, 0)$.

7. (12 points) Express the following limit as a definite integral of certain function and then evaluate the integral:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{n+\sqrt{nk}}}.$$

Solution:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{n+\sqrt{nk}}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\sqrt{\frac{k}{n}}}} \quad \text{(3 points)}$$
$$= \int_0^1 \frac{1}{\sqrt{1+\sqrt{x}}}dx \quad \text{(3 points)} \tag{11}$$

Using the following change of variable

$$u = 1+\sqrt{x}$$
$$x = (u-1)^2$$
$$dx = 2(u-1)du \quad \text{(1 point)}$$

the integral (11) is transformed into

$$\int_1^2 \frac{2(u-1)}{\sqrt{u}}du = \int_1^2 (2u^{\frac{1}{2}} - 2u^{-\frac{1}{2}})du \quad \text{(2 points)}$$
$$= \frac{4}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\Big|_1^2 = (\frac{4}{3}2\sqrt{2} - 4\sqrt{2}) - (\frac{4}{3} - 4) = \frac{8}{3} - \frac{4}{3}\sqrt{2} \quad \text{(3 points)}$$

8. (16 points) Let $f(x) = 3\frac{x^{\frac{2}{3}}}{x-1}$.
   (a) Find all critical points.
   (b) Find the intervals of increasing and intervals of decreasing.
   (c) Find the intervals on which f is concave up and intervals on which f is concave down.
   (d) Find the points of inflection.
   (e) Determine whether f has any vertical tangent or vertical cusps.
   (f) Find all vertical or horizontal asymptotes.
   (g) Draw the graph of $f(x)$.

Solution:

(a) (3 pt.)
$f'(x) = \dfrac{-(x+2)}{x^{\frac{1}{3}}(x-1)^2}$ Critical Points $x = -2$ and $x = 0$
$f(x)$ does not exist at $x = 1$

(b) (2 pt.)
increasing $[-2, 0]$
decreasing $(-\infty, -2]$ $[0, 1), (1, \infty)$

(c) (3 pt.)
$f''(x) = \dfrac{(-\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}x^{\frac{-4}{3}})(x-1) + 2(x^{\frac{2}{3}} + 2x^{\frac{-1}{3}})}{(x-1)^3}$
Concave up $[\dfrac{-4-3\sqrt{2}}{2}, 0), (0, \dfrac{-4+3\sqrt{2}}{2},]\ (1, \infty)$
Concave down $(-\infty, \dfrac{-4-3\sqrt{2}}{2}],\ [\dfrac{-4+3\sqrt{2}}{2}, 1)$

(d) (2 pt.)
$x = \dfrac{-4 \pm 3\sqrt{2}}{2}$

(e) (2 pt.)
cusp at $x = 0$
no vertical tangent
$f(x)$ is not continuous at $x = 1$

(f) (2 pt.)
vertical asymptotes: x = 1
horizontal asymptotes: y = 0

(g) (2 pt.)