1. (15%) Sketch the region bounded by the curves $x^{1/3} + y^{1/3} = 1$ and $x + y = 1$. Locate the centroid of the region and find the volume generated by revolving the region about each of the coordinate axes.

Solution:
(a) Sketch of the picture (3 points).
(b)

$$\overline{X} = \overline{Y} = \frac{\int_0^1 x\big[(1-x) - (1-x^{1/3})^3\big]dx}{\text{area of the region}}, \tag{1}$$

where

$$\text{area of the region} = \int_0^1 (1-x) - (1-x^{1/3})^3 dx \tag{2}$$

$$= \int_0^1 1 - x - \big[1 - 3x^{1/3} + 3x^{2/3} - x\big] = 9/20 \tag{3}$$

and

$$\int_0^1 x\Big[(1-x) - (1-x^{1/3})^3\Big]dx = 9/56. \tag{4}$$

So $\overline{X} = \overline{Y} = 5/14$. (5 points)
(c) (7 points)

The volumn generated by rotating the region with respect to x-axis
$=2\pi \cdot \overline{Y} \cdot$ area of the region $= 9\pi/28$;
The volumn generated by rotating the region with respect to y-axis
$=2\pi \cdot \overline{X} \cdot$ area of the region $= 9\pi/28$.

2. (15%)

(a) Evaluate $\displaystyle\int \ln x \, dx$.

(b) Show that the function $f(x) = \ln x$ is increasing in $x > 0$.

(c) Consider the definite integrals of $f(x) = \ln x$ on $[1, n]$ and $[1, n+1]$. By comparing the upper sum and the lower sum for $f(x) = \ln x$ with regular partition of length $\triangle x = 1$, derive the inequalities

$$\int_1^n \ln x \, dx < \ln 1 + \ln 2 + \cdots + \ln n < \int_1^{n+1} \ln x \, dx.$$

Sketch a graph if necessary.

(d) Prove that $\left(\frac{n}{e}\right)^n < \frac{n!}{e} < \left(\frac{n+1}{e}\right)^{n+1}$.

Solution:

2.

(a) We use the integration by part to get

$$\int \ln x \, dx = x \ln x - x + C \quad \boxed{\text{2 points}}$$

for some constant $C \in \mathbb{R}$.

(b) $\boxed{\text{1 point}}$ There are two methods you can choose

(1) If $x_2 > x_1 > 0$, $\frac{x_2}{x_1} > 1$, then we have

$$\ln x_2 - \ln x_1 = \begin{cases} \ln\left(\frac{x_2}{x_1}\right) > 0 \quad \text{or} \\ \displaystyle\int_{x_1}^{x_2} \frac{dt}{t} > 0 \end{cases}.$$

(2) For $x > 0$, we have

$$\frac{d}{dx} \ln x = \frac{1}{x} > 0.$$

Then we can derive the result by applying the Mean Value Thm.

(c) From part (b), we observe that $\ln 1 + \ln 2 + \cdots + \ln n =$ the upper sum on $[1, n]$. Thus, we have

$$\ln 1 + \ln 2 + \cdots + \ln n > \int_1^n \ln x \, dx \quad \boxed{\text{3 points}}.$$

Similarly, $\ln 1 + \ln 2 + \cdots + \ln n =$ the lower sum on $[1, n+1]$. Thus, we have

$$\ln 1 + \ln 2 + \cdots + \ln n < \int_1^{n+1} \ln x \, dx \quad \boxed{\text{3 points}}.$$

Therefore, we conclude that

$$\int_1^n \ln x < \ln 1 + \ln 2 + \cdots + \ln n < \int_1^{n+1} \ln x \, dx.$$

(d) From part (a), we evaluate

$$\begin{aligned}
\int_1^n \ln x\,dx &= x\ln x - x\Big|_1^n \\
&= n\ln n - n + 1 \\
&= \ln\left(\frac{n^n}{e^{n-1}}\right), \quad \boxed{\text{2 points}}
\end{aligned}$$

$$\begin{aligned}
\int_1^{n+1} \ln x\,dx &= x\ln x - x\Big|_1^{n+1} \\
&= (n+1)\ln(n+1) - n \\
&= \ln\left(\frac{(n+1)^{n+1}}{e^n}\right), \quad \boxed{\text{2 points}}
\end{aligned}$$

and

$$\ln 1 + \ln 2 + \cdots + \ln n = \ln n! \quad \boxed{\text{2 points}}.$$

Then, from part (b) and the inequality of part (c), we have

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

So it implies

$$\left(\frac{n}{e}\right)^n < \frac{n!}{e} < \left(\frac{n+1}{e}\right)^{n+1}.$$

3. (15%) Calculate $\int \frac{x^4+2x^3+1}{(x-1)(x^2+1)^2}dx$.

Solution:

$$\int \frac{x^4+2x^3+1}{(x-1)(x^2+1)^2}\,\mathrm{d}x = \ln|x-1| + \arctan x - \frac{x}{x^2+1} + C$$

Write $\frac{x^4+2x^3+1}{(x-1)(x^2+1)^2}$ into partial fraction by letting

$$\frac{x^4+2x^3+1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

then $x^4+2x^3+1 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1)$

. Put $x=i$, we get $2-2i = (Di+E)(i-1) = (-D-E)+(D-E)i$, then $D=0$, $E=-2$

. Put $x=1$, we get $4=4A$, then $A=1$

. Put $x=0$, we get $1 = A-C-E = 3-C$, then $C=2$

. Put $x=-1$, we get $0 = 4A-4(B+C)-2(-D+E) = -4B$, then $B=0$

So,

$$\frac{x^4+2x^3+1}{(x-1)(x^2+1)^2} = \frac{1}{x-1} + \frac{2}{x^2+1} + \frac{-2}{(x^2+1)^2}$$

Integrate each term,

$$\int \frac{1}{x-1}\,\mathrm{d}x = \ln|x-1| + C$$

$$\int \frac{2}{x^2+1}\,\mathrm{d}x = 2\arctan x + C$$

For $\int \frac{-2}{(x^2+1)^2}\,\mathrm{d}x$, substitute $x=\tan u$, $\mathrm{d}x = \sec^2 u\mathrm{d}u$, we have

$$\int \frac{-2}{(x^2+1)^2}\,\mathrm{d}x$$

$$= -2\int \frac{1}{(\tan u^2+1)^2}\,\sec^2 u\mathrm{d}u$$

$$= -2\int \frac{1}{(\sec^2 u)^2}\,\sec^2 u\mathrm{d}u$$

$$= -2\int \frac{1}{\sec^2 u}\,\mathrm{d}u$$

$$= -2\int \cos^2 u\,\mathrm{d}u = -2\int \frac{1}{2}(1+\cos 2u)\,\mathrm{d}u$$

$$= -2(\frac{u}{2} + \frac{\sin 2u}{4}) + C = -u - \sin u\cos u + C$$

$$= -\arctan x - \frac{1}{\sqrt{x^2+1}}\frac{x}{\sqrt{x^2+1}} + C = -\arctan x - \frac{x}{x^2+1} + C$$

So,

$$\int \frac{x^4+2x^3+1}{(x-1)(x^2+1)^2}\,\mathrm{d}x = \ln|x-1| + \arctan x - \frac{x}{x^2+1} + C$$

評分標準：
寫出部分分式　得五分

三個分式的積分分別是得兩分、得三分、得五分

第三項的積分最後沒有做變數代換換回 x 或者沒有寫下三角函數與反三角函數合成的明確結果不算正確

少寫積分常數、粗心導致的細微錯誤等等扣一分

4. (15%) Solve the initial-value problem

$$\begin{cases} (\cos^2 x)y' - (\sec x + \sin^2 x)y^2 = 2\sec x - 2\cos^2 x + 2, \\ y(0) = \sqrt{2}. \end{cases}$$

Solution:

$$\begin{aligned}(\cos^2 x)y' - (\sec x + \sin^2 x)y^2 &= 2\sec x + 2(1 - \cos^2 x) \\ (\cos^2 x)y' &= (y^2 + 2)(\sec x + \sin^2 x)\end{aligned}$$

Around $x = 0$, $\cos x \neq 0$. So

$$\frac{1}{y^2+2}\frac{dy}{dx} = \sec^3 x + \tan^2 x \quad \text{...(6 } pts\text{)}$$

$$\begin{aligned}
\frac{1}{y^2+2}dy &= (\sec^3 x + \tan^2 x)dx \\
\int \frac{1}{y^2+2}dy &= \int (\sec^3 x + \tan^2 x)dx \\
\frac{1}{\sqrt{2}}\tan^{-1}(\frac{y}{\sqrt{2}}) &= \int (\sec^3 x + \tan^2 x)dx \quad \text{...(3 } pts\text{)} \\
\frac{1}{\sqrt{2}}\tan^{-1}(\frac{y}{\sqrt{2}}) &= \int \sec^3 x dx + \int (\sec^2 x - 1)dx \\
\frac{1}{\sqrt{2}}\tan^{-1}(\frac{y}{\sqrt{2}}) &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + \int (\sec^2 x - 1)dx \quad \text{.........(2 } pts\text{)} \\
\frac{1}{\sqrt{2}}\tan^{-1}(\frac{y}{\sqrt{2}}) &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + \tan x - x + C \quad \text{.............(2 } pts\text{)}
\end{aligned}$$

$$y(0) = \sqrt{2} \Rightarrow \quad \frac{1}{\sqrt{2}}\tan^{-1}(\frac{\sqrt{2}}{\sqrt{2}}) = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0 + \tan 0 - 0 + C \Rightarrow \quad C = \frac{\pi}{4\sqrt{2}} \quad \text{......................(2 } pts\text{)}$$

The solution is

$$\frac{1}{\sqrt{2}}\tan^{-1}(\frac{y}{\sqrt{2}}) = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + \tan x - x + \frac{\pi}{4\sqrt{2}}.$$

5. (15%) Consider the curve given by the polar equation $r = a + \cos\theta$ where $0 < a < 1$.

(a) Find the tangent lines of this curve at the origin.

Find the constant $a = a_0$ such that the tangent lines at the origin are $y = \frac{\sqrt{3}}{2}x$ and $y = -\frac{\sqrt{3}}{2}x$.

(b) Draw the curve $r = a_0 + \cos\theta$.

(c) Compute the area of the region bounded by the inner loop of $r = a_0 + \cos\theta$.

(d) Find the area of the surface given by revolving the inner loop about the x-axis.

Solution:

(a) 4 points tangent line of this curve at the origin

$$\frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta + \cos 2\theta}{-a\cos\theta - 2\cos\theta\sin\theta}|_{(\theta = \cos^{-1}(-a))} = \pm\frac{a^2-1}{a\sqrt{1-a^2}}$$

$$\pm\frac{\sqrt{3}}{2} = \pm\frac{a^2-1}{a\sqrt{1-a^2}}$$

$a = \sqrt{\frac{4}{7}}$

(b) 3 points follow the page 488 (Figure 10.3.9) the fourth picture of the textbook.

(c) 4 points the area of the region bounded by the inner loop

$$-2\int_{\cos^{-1}(-a)}^{\pi} \frac{1}{2}r^2 d\theta$$

$= a\sqrt{1-a^2}(\frac{3}{2}) - (a^2 + \frac{1}{2})(\pi - \cos^{-1} -a), a = \sqrt{\frac{4}{7}}$

(d) 4 points

the area of the surface given by revolving the inner loop about the x-axis.

$$S = \int_{\cos^{-1}(-a)}^{\pi} 2\pi y\sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}d\theta$$

$$S = \int_{\cos^{-1}(-a)}^{\pi} 2\pi(a + \cos\theta)\sin\theta\sqrt{1 + a^2 + 2a\cos\theta}d\theta$$

$$= 2\pi(\int_{-a^2}^{-a} \sqrt{1+a^2+2t}(-dt) + \int_{-a^2}^{-a} \frac{t}{a^2}\sqrt{1+a^2+2t}(-dt))(t = a\cos\theta)$$

$$= -2\pi\frac{1}{3}(1+a^2+2t)^{\frac{3}{2}}|_{-a^2}^{-a} - 2\pi/a^2[(\frac{t}{3})(1+a^2+2t)^{\frac{3}{2}})|_{-a^2}^{-a} - \int_{-a^2}^{-a}(1+a^2+2t)^{\frac{3}{2}}dt]$$

$$= 2\pi/3a(1-a)^3 - 2\pi/3(1-a)^3 - 2\pi/a^2(\frac{1}{5}(1-a)^{\frac{5}{2}} - \frac{1}{5}(1-a)^5)(a = \sqrt{\frac{4}{7}})$$

6. (15%) Compute the following limits

(a) $\lim_{x \to 0^+} \left(x^{-1} + x^{-2}\right)^{\frac{1}{\ln x}}$

(b) $\lim_{n \to \infty} \dfrac{\left[\ln\left(1 - \frac{1}{n}\right)\right]^3}{\tan\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)}$

(c) $\lim_{n \to \infty} \left(1 + \dfrac{1}{\ln n}\right)^{n^\alpha}$, where $\alpha > 0$

Solution:

(a)

Let $y = (x^{-1} + x^{-2})^{\frac{1}{\ln x}}$.

Then we have $\ln y = \dfrac{1}{\ln x}(\ln(x^{-1} + x^{-2})) = \dfrac{\ln(x^{-1} + x^{-2})}{\ln x}$ (2 %)

Since that as $x \to 0^+$, $\ln x \to -\infty$, and $\ln(x^{-1} + x^{-2}) \to \infty$

By L'Hopital's rule , we get

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(x^{-1} + x^{-2})}{\ln x}$$

$$= \lim_{x \to 0^+} \frac{\frac{-x^{-2} - 2x^{-3}}{x^{-1} + x^{-2}}}{\frac{1}{x}} \quad (3\%)$$

$$= \lim_{x \to 0^+} \frac{-x^{-1} - 2x^{-2}}{x^{-1} + x^{-2}} = \lim_{x \to 0^+} \frac{-x - 2}{x + 1} = -2 \quad (4\%)$$

so,

$$\lim_{x \to 0^+} (x^{-1} + x^{-2})^{\frac{1}{\ln x}} = \lim_{x \to 0^+} y = e^{-2} \quad (5\%)$$

(b)

Let $x = \dfrac{1}{n}$.Then as $n \to \infty$, we have $x \to 0^+$

So,

$$\lim_{n \to \infty} \frac{[\ln(1 - \frac{1}{n})]^3}{\tan\frac{1}{n} - \sin\frac{1}{n}} = \lim_{x \to 0^+} \frac{[\ln(1 - x)]^3}{\tan x - \sin x}$$

Since as $x \to 0^+$, we have $\tan x - \sin x \to 0$, and $[\ln(1 - x)]^3 \to 0$

By L'Hopital's rule , we get

$$\lim_{x \to 0^+} \frac{[\ln(1 - x)]^3}{\tan x - \sin x} = \lim_{x \to 0^+} \frac{3[\ln(1 - x)]^2}{\sec^2 x - \cos x} \frac{-1}{1 - x} \quad (2\%)$$

Since

$$\lim_{x \to 0^+} \frac{-1}{1 - x} = -1 \quad and \quad \sec^2 x - \cos x = \frac{1 - \cos^3 x}{\cos^2 x}$$

We have

$$\lim_{x \to 0^+} \frac{3[\ln(1 - x)]^2}{\sec^2 x - \cos x} \frac{-1}{1 - x} = \lim_{x \to 0^+} \frac{-3[\ln(1 - x)]^2}{\sec^2 x - \cos x} = \lim_{x \to 0^+} \frac{-3\cos^2 x[\ln(1 - x)]^2}{1 - \cos^3 x}$$

$$= \lim_{x \to 0^+} \frac{-3\cos^2 x}{1 + \cos x + \cos^2 x} \frac{[\ln(1 - x)]^2}{1 - \cos x} = \lim_{x \to 0^+} -\frac{[\ln(1 - x)]^2}{1 - \cos x}$$

Since

$$\lim_{x \to 0^+} \frac{-3\cos^2 x}{1 + \cos x + \cos^2 x} = -1$$

Similarly,as $x \to 0^+$, we have $1 - \cos x \to 0$, and $[\ln(1-x)]^2 \to 0$
By L'Hopital's rule , we get

$$\lim_{x\to 0^+} -\frac{[\ln(1-x)]^2}{1-\cos x} = \lim_{x\to 0^+} \frac{2[\ln(1-x)]}{\sin x}\frac{1}{1-x} = \lim_{x\to 0^+} \frac{2[\ln(1-x)]}{\sin x} \quad (3\%)$$

Since

$$\lim_{x\to 0^+} \frac{1}{1-x} = 1$$

Similarly,as $x \to 0^+$, we have $\sin x \to 0$, and $[\ln(1-x)] \to 0$
By L'Hopital's rule , we get

$$\lim_{x\to 0^+} \frac{2[\ln(1-x)]}{\sin x} = \lim_{x\to 0^+} \frac{2}{\cos x}\frac{-1}{1-x} = -2 \quad (5\%)$$

(c)
Note that

$$\lim_{n\to\infty} (1+\frac{1}{\ln n})^{n^\alpha} = \lim_{x\to\infty} (1+\frac{1}{\ln x})^{x^\alpha} \quad \textit{where} \quad \alpha > 0$$

Let $y = (1+\frac{1}{\ln x})^{x^\alpha}$. Then $\ln y = x^\alpha \ln(1+\frac{1}{\ln x}) = \frac{\ln(1+\frac{1}{\ln x})}{x^{-\alpha}}$ For $\alpha > 0$, we have as $x \to \infty$,$\ln(1+\frac{1}{\ln x}) \to 0$,and $x^{-\alpha} \to 0$
By L'Hopital's rule , we get

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{\ln x})}{x^{-\alpha}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{\ln x}}\frac{-1}{(\ln x)^2}\frac{1}{x}}{-\alpha x^{-\alpha-1}} = \lim_{x\to\infty} \frac{x^\alpha}{\alpha \ln x(1+\ln x)} \quad (2\%)$$

Also, we have as $x \to \infty$,$\alpha \ln x(1+\ln x) \to \infty$,and $x^\alpha \to \infty$
By L'Hopital's rule , we get

$$\lim_{x\to\infty} \frac{x^\alpha}{\alpha \ln x(1+\ln x)} = \lim_{x\to\infty} \frac{\alpha x^{\alpha-1}}{\alpha(\frac{1}{x}+\frac{2}{x}\ln x)} = \lim_{x\to\infty} \frac{x^\alpha}{2\ln x+1} \quad (3\%)$$

Also, we have as $x \to \infty$,$(1+2\ln x) \to \infty$,and $x^\alpha \to \infty$
By L'Hopital's rule , we get

$$= \lim_{x\to\infty} \frac{x^\alpha}{2\ln x+1} = \lim_{x\to\infty} \frac{\alpha x^{\alpha-1}}{\frac{2}{x}} = \lim_{x\to\infty} \frac{\alpha x^\alpha}{2} = \infty \quad (\textit{since } \alpha > 0)(4\%)$$

So,

$$\lim_{x\to\infty} \ln y = \infty$$

Thus,

$$\lim_{x\to\infty} (1+\frac{1}{\ln x})^{x^\alpha} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = \infty (5\%)$$

7. (15%) The gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

(a) Show that the integral converges for $x \geq 1$.
(b) Show that the integral also converges for $0 < x < 1$.
(c) Show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

Solution:
The gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

(a) Show that the integral converges for $x \geq 1$.

Since $x \geq 1$, the integrand $f(t) = t^{x-1}e^{-t}$ is defined and continuous on $\mathbb{R}$. Also, since $\lim_{t\to\infty} \frac{e^{t/2}}{t^{x-1}} = \infty$, there exists $C > 0$ such that $e^{t/2} > t^{x-1}$ for all $t \geq C$. Thus

$$\begin{aligned}
\int_0^\infty t^{x-1}e^{-t}dt &= \int_0^C t^{x-1}e^{-t}dt + \int_C^\infty t^{x-1}e^{-t}dt \\
&\leq \int_0^C t^{x-1}e^{-t}dt + \int_C^\infty e^{-t/2}dt \\
&= \int_0^C t^{x-1}e^{-t}dt + (-2)e^{-t/2}\Big|_c^\infty \\
&< \infty.
\end{aligned}$$

Therefore the integral converges.

(b) Show that the integral also converges for $0 < x < 1$.

Since $0 \leq x < 1$, the integrand $f(t) = t^{x-1}e^{-t} \to \infty$ as $t \to \infty$. Write

$$\begin{aligned}
\int_0^\infty t^{x-1}e^{-t}dt &= \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt \\
&=: A + B
\end{aligned}$$

By a similar argument as in (a), we can show that B converges. Now, since e^{-t} is decreasing in $[0, 1]$, $t^{x-1}e^{-t} \leq t^{x-1}$. Thus,

$$A = \lim_{a\to 0^+} \int_a^1 t^{x-1}e^{-t}dt \leq \lim_{a\to 0^+} \int_a^1 t^{x-1}dt \quad = \frac{1}{x}t^x\Big|_0^1$$

Therefore the integral converges.

(c) Show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

This is done by an integration by parts:

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \\ &= -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^{x-1} e^{-t} dt \\ &= x\Gamma(x).\end{aligned}$$