SECTION 9.6: Taylor and Maclaurin Series

Exercise 1

We make use of the Maclaurin series of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \cdots .$$

The series above holds for all real number x.
Substituting $3x$ for x, and then multiply by e, we get:

$$e^{3x+1} = e + 3ex + \frac{3^2 e}{2} x^2 + \frac{3^3 e}{3!} x^3 + \cdots = \sum_{n=0}^{\infty} \frac{3^n e}{n!} x^n.$$

The resulting Maclaurin series is valid for any real x since the original Maclaurin series holds for any real x.

Exercise 3

Solution 1:

Using the relations between trigonometric functions and the exponential function, we obtain:

$$\begin{aligned}
\sin\left(x - \frac{\pi}{4}\right) &= \frac{1}{2i}\left(e^{i(x-\frac{\pi}{4})} - e^{-i(x-\frac{\pi}{4})}\right) \\
&= \frac{1}{2i}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) e^{ix} - \frac{1}{2i}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) e^{-ix} \\
&= \left(-\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} i\right) e^{ix} - \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} i\right) e^{-ix} \\
&= \sum_{n=0}^{\infty} \left(-\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} i\right) \frac{i^n}{n!} x^n - \sum_{n=0}^{\infty} \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} i\right) \frac{(-i)^n}{n!} x^n \\
&= \sum_{k=0}^{\infty} \frac{-\sqrt{2}}{2} \frac{i^{2k}}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{-\sqrt{2} i}{2} \frac{i^{2k+1}}{(2k+1)!} x^{2k+1} \\
&= \frac{\sqrt{2}}{2} \sum_{k=0}^{\infty} \left(-\frac{(-1)^k}{(2k)!} x^{2k} + \frac{(-1)^k}{(2k+1)!} x^{2k+1}\right)
\end{aligned}$$

(The fifth equality holds since $i^k = (-i)^k$ when k is even; and $i^k = -(-i)^k$ when k is odd).

The Maclaurin series holds for all real value x since only exponential functions are involved, and they are valid for all real value.

Solution 2:

Observe that $\sin(x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2}(\sin x - \cos x)$, we immediately get:

$$\sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\sum_{k=0}^{\infty}\left(-\frac{(-1)^k}{(2k)!}x^{2k} + \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right).$$

Exercise 5

Substitute $\frac{x}{3}$ for x in the Maclaurin series of $\sin x$, then multiply by x^2, we obtain:

$$x^2 \sin\frac{x}{3} = x^2\left(\frac{x}{3} - \frac{1}{3!}\frac{x^3}{3^3} + \frac{1}{5!}\frac{x^5}{3^5} - \cdots\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+1}(2n+1)!}x^{2n+3}.$$

The resulting Maclaurin series is valid for every real x since the Maclaurin series of the $\sin x$ holds for every real x.

Exercise 8

Substitute $5x^2$ for x in the Maclaurin series of $\tan^{-1} x$, we obtain:

$$\tan^{-1}(5x^2) = 5x^2 - \frac{(5x^2)^3}{3} + \frac{(5x^2)^5}{5} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{2n+1}x^{4n+2}.$$

As we know the Maclaurin series of $\tan^{-1} x$ converges for all $-1 \leq x \leq 1$, the Maclaurin series of $\tan^{-1}(5x^2)$ converges if and only if $-1 \leq 5x^2 \leq 1$, equivalently, if $0 \leq x^2 \leq \frac{1}{5} \iff -\frac{\sqrt{5}}{5} \leq x \leq \frac{\sqrt{5}}{5}$.

Exercise 12

Substitute $2x^2$ for x in the Maclaurin series of e^x, we have:

$$e^{2x^2} = 1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^{2n}$$

Therefore,

$$\frac{e^{2x^2} - 1}{x^2} = \frac{1}{x^2}\sum_{n=1}^{\infty}\frac{2^n}{n!}x^{2n} = \sum_{n=1}^{\infty}\frac{2^n}{n!}x^{2n-2}.$$

Note that the function $f(x) = \frac{e^{2x^2}-1}{x^2}$ is not defined at $x = 0$, but it has a limit at $x = 0$ (this can be confirmed by examining the constant term of the Maclaurin series of $f(x)$ or by using the l'Hôpital's rule):

$$\lim_{x\to 0}\frac{e^{2x^2}-1}{x^2}=\lim_{u\to 0}\frac{e^{2u}-1}{u}=\lim_{u\to 0}\frac{2e^{2u}}{1}=2.$$

If we define $f(0)=2$, the Maclaurin series converges for any real value x, as the Maclaurin series of e^x does.

Exercise 17

Solution 1:

Compute the nth derivative of $f(x)$, we have $f(\pi)=-1$ and:

$$\begin{aligned}
f'(x)&=-\sin x, & f'(\pi)&=0\\
f''(x)&=-\cos x, & f''(\pi)&=1\\
f^{(3)}(x)&=\sin x, & f^{(3)}(\pi)&=0\\
f^{(4)}(x)&=\cos x, & f^{(4)}(\pi)&=-1\\
f^{(5)}(x)&=-\sin x, & f^{(5)}(\pi)&=0\\
&\vdots & &\vdots
\end{aligned}$$

Thus, the Taylor series for $f(x)=\cos x$ in powers of $x-\pi$ is

$$\cos x=-1+\frac{1}{2!}(x-\pi)^2-\frac{1}{4!}(x-\pi)^4+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}.$$

The series converges for all real value x by the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{(2(n+1))!}x^{2(n+1)}}{\frac{(-1)^{n+1}}{(2n)!}x^{2n}}\right|=\lim_{n\to\infty}\frac{(2n)!}{(2n+2)!}|x|^2=\lim_{n\to\infty}\frac{|x|^2}{(2n+1)(2n+2)}=0.$$

Solution 2:

Let $u=x-\pi$. The by the Maclaurin series of $\cos u$ we have:

$$\begin{aligned}
\cos x=-\cos u=-1+\frac{u^2}{2!}-\frac{u^4}{4!}+\cdots&=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}u^{2n}}{(2n)!}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}
\end{aligned}$$

Exercise 19

Let $t = \frac{x-2}{4}$. Then we have:

$$\ln(2+x) = \ln(4+(x-2)) = \ln 4 + \ln\left(1 + \frac{x-2}{4}\right) = \ln 4 + \ln(1+t).$$

So we can use the Maclaurin series of $\ln(1+t)$ to compute the required Taylor series as follows:

$$\begin{aligned}
\ln(2+x) &= \ln 4 + \ln(1+t) \\
&= 2\ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots \\
&= 2\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n \\
&= 2\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n4^n} (x-2)^n
\end{aligned}$$

Since the Maclaurin series of $\ln(1+t)$ converges when $-1 \leq t \leq 1$, and $x = 4t+2$, the resulting Taylor series of $\ln(2+x)$ converges when $-2 \leq x \leq 6$.

Exercise 22

We use a trigonometric identity to express $\cos^2 x$ in terms of $\cos 2x$, and then use addition formula for cosine to compute the Taylor series of $\cos 2x$ at $\frac{\pi}{4} = 2 \times \frac{\pi}{8}$.

$$\begin{aligned}
\cos^2 x &= \frac{1+\cos 2x}{2} \\
&= \frac{1}{2} + \frac{1}{2}\cos\left(2x - \frac{\pi}{4} + \frac{\pi}{4}\right) \\
&= \frac{1}{2} + \cos\left(2x - \frac{\pi}{4}\right)\cos\frac{\pi}{4} - \sin\left(2x - \frac{\pi}{4}\right)\sin\frac{\pi}{4} \\
&= \frac{1}{2} + \frac{\sqrt{2}}{2}\left[1 - \frac{2^2}{2!}\left(x - \frac{\pi}{8}\right)^2 + \frac{2^4}{4!}\left(x - \frac{\pi}{8}\right)^4 - \cdots\right] \\
&\quad - \frac{\sqrt{2}}{2}\left[2\left(x - \frac{\pi}{8}\right) - \frac{2^3}{3!}\left(x - \frac{\pi}{8}\right)^3 + \frac{2^5}{5!}\left(x - \frac{\pi}{8}\right)^5 - \cdots\right] \\
&= \frac{1+\sqrt{2}}{2} \\
&\quad + \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{2^{2n-1}}{(2n-1)!}\left(x - \frac{\pi}{8}\right)^{2n-1} + \frac{2^{2n}}{(2n)!}\left(x - \frac{\pi}{8}\right)^{2n}\right).
\end{aligned}$$

The resulting Taylor series is valid for every real x since the Maclaurin series of the sine and cosine function both hold for every real x.

Exercise 24

We have

$$
\begin{aligned}
\frac{x}{1+x} &= 1 - \frac{1}{1+x} = 1 - \frac{1}{2+(x-1)} = 1 - \frac{1}{2}\frac{1}{1+\frac{x-1}{2}} \\
&= 1 - \frac{1}{2}\left[1 - \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 - \cdots\right] \\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n+1}(x-1)^n.
\end{aligned}
$$

From the resulting Taylor series, we immediately see that the Taylor series converges when $|x-1| < 2 \iff -1 < x < 3$.

Exercise 28

We first need to compute the first three nonzero terms (excluding the constant term) in the Maclaurin series for $\sec x$, then we can compute the first three nonzero terms in the Maclaurin series for $\sec x \tan x$, since $(\sec x)' = \sec x \tan x$.

$$
\begin{aligned}
\sec x = (\cos x)^{-1} &= \frac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots} \\
&= 1 + \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \cdots\right) \\
&\quad + \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \cdots\right)^2 + \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots\right)^3 + \cdots \\
&= 1 + \frac{1}{2}x^2 + \left(-\frac{1}{24} + \frac{1}{4}\right)x^4 + \left(\frac{1}{720} - \frac{2}{2\times 24} + \frac{1}{8}\right)x^6 + \cdots \\
&= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots.
\end{aligned}
$$

Hence,

$$
\sec x \tan x = (\sec x)' = x + \frac{5}{6}x^3 + \frac{61}{120}x^5 + \cdots.
$$

Exercise 29

Using the Maclaurin series of $\tan^{-1} u$ and e^x, we have:

$$
\begin{aligned}
\tan^{-1}(e^x - 1) &= \tan^{-1}\left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) \\
&= \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) - \frac{1}{3}\left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right)^3 + \cdots \\
&= x + \frac{1}{2}x^2 + \left(\frac{1}{6} - \frac{1}{3} \times 1\right)x^3 + \cdots = x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots .
\end{aligned}
$$