Calculus A Solution 16-5

1. Let \( \mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k} \). And we see that the triangle lies on the plane \( x + y + z = 1 \), and hence has normal \( \hat{\mathbf{N}} = \frac{-i + j + k}{\sqrt{3}} \).

Thus by Stokes’s theorem, we have:

\[
\oint_{\mathcal{C}} xy\,dx + yz\,dy + zx\,dz = \iint_{\mathcal{D}} \mathbf{curl}\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \hat{\mathbf{N}} \cdot (\cdot) = \frac{\pi}{\sqrt{3}} \times \frac{1}{2} \times \sqrt{2} \times \frac{\sqrt{3}}{\sqrt{2}} = \frac{1}{2}
\]

2. Let \( \mathbf{F} = yi - xj + z^2k \), and \( \mathcal{S} = \{x^2 + y^2 \leq 4, z = 0\} \), thus by Stokes’s theorem:

\[
\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl}\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \iint_{\mathcal{S}} -2k \cdot k\,dxdy = -8\pi
\]

3. Define \( \mathcal{S} = \{x^2 + y^2 = a^2, z = 0\} \), then we have:

\[
\iint_{\mathcal{S}} \mathbf{curl}\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \int_0^a \int_0^{\sqrt{a^2-x^2}} -(2z+3)\,dxdy = -3\pi a^2
\]

4. We define \( \mathcal{D} = \{x^2 + y^2 = 4, y = 0\} \), and apply Stokes’s theorem twice on the equation, we get:

\[
\iint_{\mathcal{S}} \mathbf{curl}\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \mathbf{curl}\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \int_0^{2\pi} \int_0^2 3r^2\,rdrd\theta = 24\pi
\]

6. By simple trigonometry, we know that \( \sin 2t = 2 \sin t \cos t \), and hence \( \mathcal{C} \) lies on the surface \( z = 2xy \). Also, we can see that \( \mathcal{S} \) is on the cylinder \( \{x^2 + y^2 = 1\} \). We define \( \mathcal{D} = \{x^2 + y^2 = 1, z = 0\} \) and apply Stokes’s theorem twice, we get:

\[
\iint_{\mathcal{S}} \mathbf{curl}\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \mathbf{curl}\mathbf{F} \cdot \hat{\mathbf{N}}\,dS = \int_0^{2\pi} \int_0^1 3r^2\,rdrd\theta = \frac{3\pi}{2}
\]
7. We can see that the part of the paraboloid boundary on the xy plane is: \( \{ z = 0, x^2 + y^2 = 9 \} \). And hence by Stokes’s theorem, we have:

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl} \mathbf{F} \cdot \hat{N} \, dS = \iint_D (2x - 1) \, dx \, dy = 9\pi
\]

8. We can easily find that \( \mathbf{r}(t) \) lies on the plane \( x + y + z = 3 \), and it’s projection onto the xy-plane is the circle \( \mathcal{C}_1 = \{(x - 1)^2 + (y - 1)^2 = 1, z = 0\} \), and the region encircled by it is \( D = \{(x - 1)^2 + (y - 1)^2 \leq 1, z = 0\} \). Thus, by applying Stokes’s theorem we get:

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot \hat{N} \, dS = \iint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_D 2x \, dx \, dy = 2 \pi A = 2\pi
\]