6.

\[ f(x, y) = xe^{y+x^2} \quad f_1(x, y) = (2x^2 + 1)e^{y+x^2} \quad f_2(x, y) = xe^{y+x^2} \]

\[ f(x, y) \approx f(a, b) + f_1(a, b)(x-a) + f_2(a, b)(y-b) \]

Now, let \( x = 2.05 \), \( y = -3.92 \), \( a = 2 \), and \( b = -4 \)

\[ f(2.05, -3.92) \approx f(2, -4) + f_1(2, -4)(2.05 - 2) + f_2(2, -4)(-3.92 + 4) \]

\[ = 2 + 0.05 \times 9 + 0.08 \times 2 \]

\[ = 2.61 \]

10.

\[ u = x \sin(x+y) \]

\[ \Rightarrow du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = [\sin(x+y) + x \cos(x+y)]dx + x \cos(x+y)dy \]

Now, put \( x = \frac{\pi}{2} + \frac{1}{20}, \quad y = \frac{\pi}{2} - \frac{1}{30}, \quad dx = \frac{1}{20}, \quad dy = \frac{1}{30} \)

\[ u(\frac{\pi}{2} + \frac{1}{20}, \frac{\pi}{2} - \frac{1}{30}) = u(\frac{\pi}{2}, \frac{\pi}{2}) + du \]

\[ = 0 + (-\frac{\pi}{2}) \times \frac{1}{20} + (-\frac{\pi}{2}) \times (-\frac{1}{30}) \]

\[ = \frac{\pi}{120} \]

11.

We assume that the edges of this rectangular box are \( x \), \( y \), and \( z \).

Accuracy of 1% \( \Rightarrow dx = \frac{1}{100}x \), \( dy = \frac{1}{100}y \), \( dz = \frac{1}{100}z \)

(a)

Volume \( V = xyz \Rightarrow dV = yzdx + xzdy + xydz = \frac{3}{100}xyz = \frac{3}{100}V \)

The approximate maximum percentage error is 3%.

(b)

Area \( A = xy \Rightarrow dA = ydx + xdy = \frac{2}{100}xy = \frac{2}{100}A \)

The approximate maximum percentage error is 2%. Similar for others face.
Diagonal length \( L = \sqrt{x^2 + y^2 + z^2} \)

\[ dL = \frac{x}{\sqrt{x^2 + y^2 + z^2}} dx + \frac{y}{\sqrt{x^2 + y^2 + z^2}} dy + \frac{z}{\sqrt{x^2 + y^2 + z^2}} dz \]

\[ = \frac{1}{L} (xdx + ydy + zdz) = \frac{x^2 + y^2 + z^2}{100L} = \frac{L}{100} \]

The approximate maximum percentage error is 1%.

17.

\[ f(r, \theta) = (r \cos \theta, r \sin \theta) \Rightarrow Df = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

18.

\[ f(\rho, \phi, \theta) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \Rightarrow Df = \begin{pmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{pmatrix} \]

20.

\[ g(r, s, t) = (r^2 s, r^2 t, s^2 - t^2) \Rightarrow Dg = \begin{pmatrix} 2rs & r^2 & 0 \\ 2rt & 0 & r^2 \\ 0 & 2s & -2t \end{pmatrix} \]

\[ \Rightarrow Dg(1,3,3) = \begin{pmatrix} 6 & 1 & 0 \\ 6 & 0 & 1 \\ 0 & 6 & -6 \end{pmatrix} \]

Since \( g(1,3,3) = (3,3,0) \) and \( d\mathbf{x} = \begin{pmatrix} -0.01 \\ 0.02 \\ -0.03 \end{pmatrix} \), we have
\[
d g = D g(1, 3, 3) d x = \begin{pmatrix} 6 & 1 & 0 \\ 6 & 0 & 1 \\ 0 & 6 & -6 \end{pmatrix} \begin{pmatrix} -0.01 \\ 0.02 \\ -0.03 \end{pmatrix} = \begin{pmatrix} -0.04 \\ -0.09 \\ 0.3 \end{pmatrix}
\]

Therefore, \( g(0.99, 3.02, 2.97) \approx (2.96, 2.91, 0.3) \).

21.
Let \( E(x, y) = f(x, y) - f(a, b) - f_1(a, b)(x - a) - f_2(a, b)(y - b) \). \( f(x, y) \) is differentiable at \((a, b) \Rightarrow E(x, y) \to 0\) as \( x \to a\) \( y \to b \).
\[
f(x, y) - f(a, b) = E(x, y) + f_1(a, b)(x - a) + f_2(y - b) \to 0\) as \( x \to a\) \( y \to b \)
\Rightarrow \text{f(x, y) is continuous at (a, b).} \#

22.
For given \( h, k \), let \( g(t) = f(a + th, b + tk) \). We have \( g(1) = f(a + h, b + k) \), \( g(0) = f(a, b) \), and \( g(t) \) is differentiable on \((0, 1)\). By the single-variable Mean-Value Theorem,

\[
g(1) - g(0) = g'(\theta) \text{ for some } \theta \text{ satisfying } 0 < \theta < 1
\]

\[
\Rightarrow f(a + h, b + k) - f(a, b) = hf_1(a + \theta h, b + \theta k) + kf_2(a + \theta h, b + \theta k) \#
\]

Now we want to use this result to prove that \( f \) is differentiable at \((a, b)\).

Following the proof at page 705 in the textbook,
\[
\frac{|f(a + h, b + k) - f(a, b) - hf_1(a, b) - kf_2(a, b)|}{\sqrt{h^2 + k^2}}
\]
\[
\leq |f_1(a + \theta h, b + \theta k) - f_1(a, b)| + |f_2(a + \theta h, b + \theta k) - f_2(a, b)|
\]

We cannot be sure that \(f_1(a + \theta h, b + \theta k) \rightarrow f_1(a, b)\) and \(f_2(a + \theta h, b + \theta k) \rightarrow f_2(a, b)\)
because \(f(x, y)\) only has first partial derivatives continuous of the straight line
segment. So we could not use this result in place of Theorem 3 to prove Theorem 4
and hence the version of the Chain Rule given in this section.

29.
\[
p = f'(x) = \frac{3}{2 + 3x} \Rightarrow x = \frac{3 - 2p}{3p} = \frac{1}{p} - \frac{2}{3}
\]
\[
\Rightarrow f^*(p) = px - f(x) = p\left(\frac{1}{p}\right) - \ln(2 + \frac{3}{p} - 2) = 1 - \frac{2}{3}\ln 3 + \ln p
\]

32.
(a) \(p_i\) and \(\dot{q}_i\) are conjugate, and \(\frac{\partial L}{\partial \dot{q}_i}\) is implicitly determined by \(L\).

(b) Because \(p_i\) and \(\dot{q}_i\) are conjugate \(\Rightarrow \frac{\partial H}{\partial p_i} = \dot{q}_i\), \(\frac{\partial L}{\partial \dot{q}_i} = p_i\),
\[
H(q_1 \cdots q_n, p_1 \cdots p_n) = \sum_i p_i \dot{q}_i - L(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n)
\]
Taking the differential,
\[
\sum_i \frac{\partial H}{\partial \dot{q}_i} \dot{q}_i + \sum_i \frac{\partial H}{\partial p_i} \dot{p}_i = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) - \sum_i \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i
\]
Put in these conditions \(\frac{\partial H}{\partial p_i} = \dot{q}_i\), \(\frac{\partial L}{\partial \dot{q}_i} = p_i\), and \(\frac{\partial L}{\partial q_i} = \dot{p}_i\),
\[
\Rightarrow \sum_i \frac{\partial H}{\partial \dot{q}_i} \dot{q}_i + \sum_i \dot{q}_i \dot{p}_i = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) - \sum_i \dot{p}_i \ddot{q}_i - \sum_i p_i \ddot{q}_i
\]
\[
\Rightarrow \frac{\partial H}{\partial \dot{q}_i} = -\dot{p}_i
\]
So we get the Hamilton’s equations: $\frac{\partial H}{\partial p_i} = \dot{q}_i$ and $\frac{\partial H}{\partial q_i} = -\dot{p}_i$. 

(c) 

$H(p, q) = \frac{1}{2}(p^2 + q^2)$ \Rightarrow \begin{cases} \frac{\partial H}{\partial p} = p = \dot{q} \Rightarrow \dot{p} = \ddot{q} \\ \frac{\partial H}{\partial q} = q = -\dot{p} \end{cases}$

by using the Hamilton’s equations. It satisfies the differential equation $\ddot{q} + q = 0$, similarly $\ddot{p} + p = 0$, so the Hamiltonian, $\frac{1}{2}(p^2 + q^2)$, represents a harmonic oscillator. 

#