1002微甲08-12班期中考解答和評分標準

1. (10%) Determine the values of x for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}\left(1+\frac{1}{x}\right)^n$ converges absolutely, converges conditionally or diverges.

> **Solution:**
> Use Ch.9. theorem11. p.516
> $$\rho = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)\ln(n+1)}\left(1+\frac{1}{x}\right)^{n+1}}{\frac{1}{n\ln(n)}\left(1+\frac{1}{x}\right)^{n}}\right| = \lim_{n\to\infty}\left|\frac{n\ln(n)}{(n+1)\ln(n+1)}(1+\frac{1}{x})\right| = \left|1+\frac{1}{x}\right| = \left|\frac{x+1}{x}\right|$$
> $\rho < 1$
> $\left|\frac{x+1}{x}\right| < 1$, $|x+1| < |x|$
> when $x < -\frac{1}{2}$, converges absolutely
> $x = -\frac{1}{2}$, the series $= \displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}(-1)^n$ is the alternating series (Ch9. theorem 14. p.521)
> when $x = -\frac{1}{2}$, converges conditionally
> when $x > -\frac{1}{2}$, diverges

2. (10%) Evaluate $\displaystyle\lim_{x\to 0}\frac{\left(x-\tan^{-1}x\right)\left(e^{3x}-1\right)}{2x^2-1+\cos 2x}$.

> **Solution:**
> $\tan^{-1}x = x - \frac{x^3}{3} + \ldots$ (2 points)
> $e^{3x} = 1 + 3x + \ldots$ (2 points)
> $\cos 2x = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!}$ (2 points)
> $\displaystyle\lim_{x\to 0}\frac{(1-\tan^{-1}x)(e^{3x}-1)}{2x-1+\cos 2x} = \lim_{x\to 0}\frac{x^4+\ldots}{\frac{2}{3}x^4+\ldots} = \frac{3}{2}$(4 points)
> (There is no partial credit for using L'hospital law)

3. (10%) Find out the value for the series $\displaystyle\sum_{n=0}^{\infty}\frac{(\ln 2)^{2n}}{(2n)!}$.

> **Solution:**
> Since $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} + \ldots$ (2 points), $\frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$
> $1 + \frac{(\ln 2)^2}{2!} + \frac{(\ln 2)^4}{4!} + \ldots = \frac{e^{\ln 2}+e^{-\ln 2}}{2} = \frac{5}{4}$ (8 points)

4. (10%) Find the Taylor series representation for $f(x) = \ln(2+3x)$ about $x = 1$. For what values of x is this representation valid?

> **Solution:**
> $$\ln(2+3x) = \ln(5+3(x-1)) = \ln 5\left[1+\frac{3}{5}(x-1)\right] = \ln 5 + \ln\left[1+\frac{3}{5}(x-1)\right] = \ln 5 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left[\frac{3}{5}(x-1)\right]^n}{n}$$
> (6 points)
> $-1 < \frac{3}{5}(x-1) \le 1 \Rightarrow -\frac{2}{3} < x \le \frac{8}{3}$ (4 points)

5. (15%) Consider a curve $\mathbf{r}(t) = (3t-t^3)\ \mathbf{i} + 3t^2\ \mathbf{j} + (3t+t^3)\ \mathbf{k}$, $t \in \mathbb{R}$.
   (a) Find the unit tangent $\mathbf{T}(t)$.

(b) Find the arc length function $s(t)$.

(c) Find the unit normal $\mathbf{N}(t)$.

(d) Find the curvature $\kappa(t)$.

Solution:

$$\begin{aligned}
\gamma'(t) &= (3-3t^2, 6t, 3+3t^2) && \text{(2 pts)}\\
|\gamma'(t)| &= \sqrt{18+18t^4+36t^2} = 3\sqrt{2}(t^2+1) && \text{(2 pts)}\\
T(t) &= \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1}{\sqrt{2}}(\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1}, 1) && \text{(2 pts)}\\
s(t) &= \int_0^t |\gamma'(u)|du = \int_0^t 3\sqrt{2}(u^2+1)du = 3\sqrt{2}(\frac{t^3}{3}+t) && \text{(2 pts)}\\
T'(t) &= \frac{\sqrt{2}}{(t^2+1)^2}(-2t, 1-t^2, 0) && \text{(2 pts)}\\
|T'(t)| &= \frac{\sqrt{2}}{(t^2+1)^2}\sqrt{4t^2+(1-t^2)^2} = \frac{\sqrt{2}}{(t^2+1)^2}(t^2+1) = \frac{\sqrt{2}}{t^2+1} && \text{(2 pts)}\\
N(t) &= \frac{T'(t)}{|T'(t)|} = \frac{1}{t^2+1}(-2t, 1-t^2, 0) && \text{(1 pt)}\\
\kappa(t) &= \frac{|T'(t)|}{|\gamma'(t)|} = \frac{\frac{\sqrt{2}}{t^2+1}}{3\sqrt{2}(t^2+1)} = \frac{1}{3(t^2+1)^2} && \text{(2 pts)}
\end{aligned}$$

6. (20%) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Find $\lim_{(x,y)\to(0,0)} f(x,y)$. Is f continuous at $(0,0)$?

(b) Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(c) Is $\frac{\partial f}{\partial y}$ continuous at $(0,0)$?

(d) Find $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0)$.

(e) Find the directional derivative of f at $(1,1)$ in the direction of $5\,\mathbf{i} + 12\,\mathbf{j}$.

(f) Find the maximal rate of change of f at $(1,1)$ and the direction in which it occurs.

Solution:

(3 pts)(a)

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} |\frac{x^3}{x^2+y^2}| \le \lim_{(x,y)\to(0,0)} |x||\frac{x^2}{x^2+y^2}| \le \lim_{(x,y)\to(0,0)} |x| = 0 \\
&\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0 = f(0,0)
\end{aligned}$$

So f is continuous at $(0,0)$.

(4 pts)(b)

For $(x,y) \neq (0,0)$,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{x^4+3x^2y^2}{(x^2+y^2)^2}, \\
\frac{\partial f}{\partial y}(x,y) &= \frac{-2x^3y}{(x^2+y^2)^2}.
\end{aligned}$$

And

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 1,$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{h} = 0.$$

(3 pts)(c) Let $x=\cos\theta, y=\sin\theta$. Then

$$\lim_{(x,y)\to(0,0)}\frac{\partial f}{\partial y}(x,y) = \lim_{(x,y)\to(0,0)}\frac{-2x^3y}{(x^2+y^2)^2}$$
$$= \lim_{r\to 0}\frac{-2r^4\cos^3\theta\sin\theta}{r^4} = -2\cos^3\theta\sin\theta$$

Hence, $\frac{\partial f}{\partial y}$ is not continuous at $(0,0)$.

(4 pts)(d)

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0)-\frac{\partial f}{\partial y}(0,0)}{h} = 0.$$
$$\frac{\partial^2 f}{\partial y\partial x}(0,0) = \lim_{k\to 0}\frac{\frac{\partial f}{\partial x}(0,k)-\frac{\partial f}{\partial x}(0,0)}{k}$$
$$= \lim_{k\to 0}\frac{-1}{k} \quad \text{does not exist.}$$

(3 pts) (e)

$$u = \frac{5}{13}i+\frac{12}{13}j, \quad \nabla f(x,y) = (\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$$
$$\Rightarrow D_u f(1,1) = \nabla f(1,1)\cdot u = (1,\frac{-1}{2})\cdot(\frac{5}{13},\frac{12}{13}) = \frac{-1}{13}.$$

(3 pts) (f) maximal rate of change $= |\nabla f(1,1)| = \frac{\sqrt{5}}{2}$. The direction vector is parallel to $\nabla f(1,1)$.

7. (15%) Let C be the curve of intersection of the paraboloid $z = x^2+\frac{1}{2}xy+\frac{y^2}{4}$ and the circular cylinder $x^2+y^2=13$.
   (a) Find a parametric equation for the tangent line to C at the point $(3,2,13)$.
   (b) An object is moving along C. If the x-coordinate is increasing at the rate of 4 cm/sec, how fast is the z-coordinate changing at the instant when $x=3$ cm and $y=2$ cm.

Solution:
(a) [8%]
Let $f(x,y,z) = x^2+\frac{1}{2}xy+\frac{y^2}{4}-z$, $g(x,y,z)=x^2+y^2-13$ and the curve $C=C(s)$.
$\nabla f(x,y,z) = (2x+\frac{1}{2}y, \frac{1}{2}x+\frac{1}{2}y, -1) \Rightarrow \nabla f(3,2,13) = (7,\frac{5}{2},-1)$. ....(2 points)
$\nabla g(x,y,z) = (2x,2x,0) \Rightarrow \nabla g(3,2,13) = (6,4,0)$. ....(2 points)
Since $\nabla f(3,2,13) \perp$ *the tangent vector* $C'(s)|_{(3,2,13)}$ and $\nabla g(3,2,13)\perp$ *the tangent vector* $C'(s)|_{(3,2,13)}$
$\Rightarrow$ *the tangent vector* $C'(s)$ // $\nabla f(3,2,13)\times\nabla g(3,2,13)$.
And $\nabla f(3,2,13)\times\nabla g(3,2,13) = (7,\frac{5}{2},-1)\times(6,4,0) = (4,-6,13)$. ....(2 points)
Therefore, the tangent line equation is $\frac{x-3}{4} = \frac{y-2}{-6} = \frac{z-13}{13}$. ....(2 points)

(b) [7%]
At the point $(3,2,13)$, we have the rate $\frac{dx}{dt} = 4$ cm/sec.
Take $\frac{d}{dt}$ on the equation $x^2+y^2=13 \Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$ ....(2 points)

$\Rightarrow \frac{dy}{dt}|_{(3,2,13)} = (\frac{-x}{y}\frac{dx}{dt})|_{(3,2,13)} = \frac{-3}{2}\cdot 4 = -6$ cm/sec. ....(1 points)

Take $\frac{d}{dt}$ on the equation $z = x^2 + \frac{1}{2}xy + \frac{y^2}{4}$

$\Rightarrow \frac{dz}{dt} = 2x\frac{dx}{dt} + \frac{1}{2}\frac{dx}{dt}y + \frac{1}{2}x\frac{dy}{dt} + \frac{y}{2}\frac{dy}{dt}$ ....(2 points)

$\Rightarrow \frac{dz}{dt}|_{(3,2,13)} = (2x\frac{dx}{dt} + \frac{1}{2}\frac{dx}{dt}y + \frac{1}{2}x\frac{dy}{dt} + \frac{y}{2}\frac{dy}{dt})|_{(3,2,13)}$

$= 6\cdot 4 + \frac{1}{2}\cdot 4\cdot 2 + \frac{1}{2}\cdot 3\cdot(-6) + \frac{2}{2}\cdot(-6) = 13$ cm/sec. ....(2 points)

8. (10%) Let x, y, u and v be related by the equations $\begin{cases} xyuv = 1 \\ x+y+u+v = 0. \end{cases}$ Find $\left(\frac{\partial y}{\partial x}\right)_u$.

Solution:

令

$$F(x,y,u,v) = xyuv - 1 = 0,$$
$$G(x,y,u,v) = x+y+u+v = 0.$$

法一： 由公式

$$\begin{aligned}
\left(\frac{\partial y}{\partial x}\right)_u &= -\frac{\frac{\partial(F,G)}{\partial(v,x)}}{\frac{\partial(F,G)}{\partial(v,y)}} \text{ (分子分母各 2 分; } v \text{ 寫成 } u \text{ 者以下不給分)} \\
&= -\frac{\begin{vmatrix} xyu & yuv \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} xyu & xuv \\ 1 & 1 \end{vmatrix}} \text{ (分子分母各 2 分)} \\
&= -\frac{xyu - yuv}{xyu - xuv} \text{ (2 分)} \\
&= \frac{y(x-v)}{x(v-y)}. \text{ (由題意知 } u \neq 0) \quad \square
\end{aligned}$$

法二： Holding u,

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yuv + x\frac{\partial y}{\partial x}uv + xyu\frac{\partial v}{\partial x} = 0, \text{ (3 分) and}$$
$$\frac{\partial G}{\partial x} = 0 \Rightarrow 1 + \frac{\partial y}{\partial x} + \frac{\partial v}{\partial x} = 0. \text{ (3 分)}$$

By Crammer's Rule,

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{\begin{vmatrix} -yuv & xyu \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} xuv & xyu \\ 1 & 1 \end{vmatrix}} \text{ (2 分)} \\
&= \frac{y(x-v)}{x(v-y)}. \text{ (2 分)} \quad \square
\end{aligned}$$

小錯誤扣 2 分; 對非 x 的變數做偏微分至多得 4 分。

法三： 由

$$x + y + u + \frac{1}{xyu} = 0, \text{ (2 分)}$$

視 u 為常數, 對 x 偏微分得

$$1 + \frac{\partial y}{\partial x} + \frac{-1}{(xyu)^2}\cdot(yu + x\frac{\partial y}{\partial x}u) = 0, \text{ (完全正確得 6 分)}$$

同乘 x^2y^2u, 得

$$x^2y^2u + x^2y^2u\frac{\partial y}{\partial x} - y - x\frac{\partial y}{\partial x} = 0,$$

故

$$\frac{\partial y}{\partial x} = \frac{y - x^2y^2u}{x^2y^2u - x} = \frac{y - xy \cdot \frac{1}{v}}{xy \cdot \frac{1}{v} - x} = \frac{y(v-x)}{x(y-v)}. \text{ (2 分)} \quad \square$$

9. (10%) Find and classify the critical points of the function $f(x, y) = x^4 + y^4 - 4xy$.

Solution:
令

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 - 4y = 0 \Rightarrow y = x^3 \\ \dfrac{\partial f}{\partial y} = 4y^3 - 4x = 0 \Rightarrow x = y^3 \end{cases} \text{ (2 分)} \Rightarrow x = y^3 = x^9$$

$$\Rightarrow x(x^4+1)(x^2+1)(x+1)(x-1) = 0$$
$$\Rightarrow x = 0, 1, -1.$$

得 critical points:

$$(0,0), (1,1), (-1,-1). \text{ (3 分)}$$

設二階測試

$$D(x,y) = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} \text{ (2 分)},$$

計算

$$D(0,0) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0 \Rightarrow (0,0) \text{ is a saddle point. (1 分)}$$

$$D(1,1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 128 > 0, \text{ and } f_{xx}(1,1) = 12 > 0$$
$$\Rightarrow f(1,1) = -2 \text{ is a minimum. (1 分)}$$

$$D(-1,-1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 128 > 0, \text{ and } f_{xx}(-1,-1) = 12 > 0$$
$$\Rightarrow f(-1,-1) = -2 \text{ is a minimum. (1 分)} \quad \square$$

10. (10%) If the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a, b > 0)$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse?

Solution:
Since $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ enclosed $x^2 + y^2 = 2y$, and the central of the circle is located at $(0,1)$. The minimal ellipse that can enclosed the cirlce is tangent to the circle, and the tangent point (x, y) has one possible solution for y variable. Hence, replace y variable into one of the two equation we got

$$\frac{2y - y^2}{a^2} + \frac{y^2}{b^2} = 1$$

Rearrange into quadratic form we have $(a^2 - b^2)y^2 + 2b^2y - a^2b^2 = 0$.We require the determiant $D = \sqrt{b^2 - 4ac} = 0$, i.e, $b^2 - a^2(b^2 - a^2) = 0$. This is our constrain equation.(3 points).Under this constrain, we need to derive the minimal area πab. This come to consider Lagrangian multiplier method.

Then our Lagrangian equation is

$$L(x, y, \lambda) = \pi ab + \lambda[b^2 - a^2(b^2 - a^2)]$$

(1 point)We take derivative along a and b variable, and the minimal point (a, b) will satisfy following system

$$\begin{gathered}\pi b + \lambda(-2ab^2 + 4a^3) = 0 \\ \pi a + \lambda(2b - 2ba^2) = 0 \\ b^2 = a^2(b^2 - a^2)\end{gathered}$$

(3 points)Solve this system we got $a = \sqrt{3/2}, b = 3/\sqrt{2}, \lambda = -\pi/\sqrt{3}$.The minimal area is $\frac{3\sqrt{3}}{2}\pi$.

Remark *Constrain equation cost 3 points, and Lagrangian equation for 1 points. Answer and system equations cost 3 points resp.*