1002微甲01-05班期末考解答和評分標準

1. Let $a > 0$ be a positive constant.

   (a) (10 points) Given a function f continuous on $[0, a]$, reduce the iterated integral

$$\int_0^a du \int_0^u dv \int_0^v f(w)dw$$

   to a one-dimensional ordinary Riemann integral $\int_0^a H(w)dw$. Find H in terms of f.

   (b) (4 points) Evaluate $\int_0^a du \int_0^u dv \int_0^v e^{(w-a)^3} dw$.

Solution:

(a)
Answer is

$$\int_0^a f(w)dw \int_w^a du \int_w^u dv = \int_0^a \frac{1}{2}(w-a)^2 f(w)dw$$

or

$$\int_0^a f(w)dw \int_w^a dv \int_v^a du = \int_0^a \frac{1}{2}(w-a)^2 f(w)dw$$

So

$$H(w) = \frac{1}{2}(w-a)^2 f(w)$$

(10 points)

(If you draw the right graph, you obtain 2 point. And integral range is right only for u or v , you can get 3 points)

(b)

$$\int_0^a du \int_0^u dv \int_0^v f(w)dw = \int_0^a \frac{1}{2}(w-a)^2 e^{(w-a)^3} dw$$

(1 points)

Let $t = (w-a)^3 \Rightarrow dt = 3(w-a)^2 dw$

It implies

$$\frac{1}{6}\int_{-a^3}^0 e^t dt = \frac{1}{6}(1 - e^{-a^3})$$

(3 points)

2. (14 points) Show that the vector field $\mathbf{F} = x\sin 2z\ \mathbf{i} + y\sin 2z\ \mathbf{j} + \cos 2z\ \mathbf{k}$ is solenoidal in $\mathbb{R}^3$ and find a vector potential $\mathbf{H} = P(y,z)\ \mathbf{i} + Q(x,z)\ \mathbf{j}$ satisfying $\mathbf{H}(1,1,0) = \mathbf{j}$ and $\mathbf{H}(0,0,0) = \mathbf{0}$.

Solution:

$divF = sin2z + sin2z - 2sin2z = 0 \Rightarrow solenoidal$ (2 points)

vector potential means that

$curlH = F$ and $curlH = (-Q_z(x,z), P_z(y,z), Q_x(x,z) - P_y(y,z))$

So we have that

$$\begin{cases} Q_z(x.z) = -xsin2z \\ P_z(y,z) = ysin2z \\ Q_x(x,z) - P_y(y,z) = cos2z \end{cases}$$

(3 points)

By integration we obtain

$$Q(x,z) = \frac{x}{2}cos2z + A(x)$$

$$P(y,z) = \frac{-y}{2}cos2z + B(y)$$

(2 points)

$$Q_x(x,z) - P_y(y,z) = \frac{1}{2}cos2z + A'(x) - \frac{-1}{2}cos2z + B'(y) = cos2z$$

(2 points)

So we know that

$$A'(x) = B'(y) = constant = c \qquad \forall x, y$$

so we can assume $A(x) = cx + a$ and $B(y) = cy + b$

(2 points)

use the condition of H to find a,b,c

$$H(1,1,0) = P(1,0)i + Q(1,0)j = j \Rightarrow Q(1,0) = 1, P(1.0) = 0$$
$$H(0,0,0) = P(0,0)i + Q(0,0)j = 0 \Rightarrow Q(0,0) = 0, P(0.0) = 0$$

So from above arguments, we can get

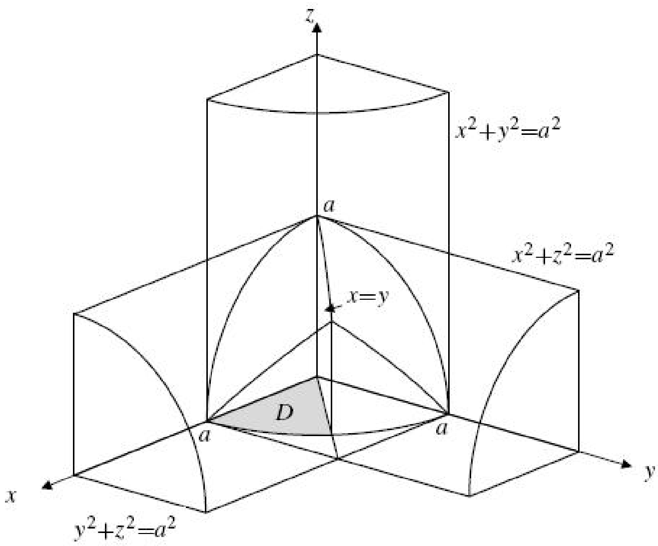
$$a = 0, b = 0, c = \frac{1}{2}$$

It implies

$$H(x,y,z) = (\frac{-ycos2z + y}{2}, \frac{xcos2z + x}{2}, 0)$$

(3 points)

3. (15 points) Find the volume of the region lying inside all three of the circular cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$, and $y^2 + z^2 = a^2$. [*Hint:* See the figure for the first octant part of the region, and use symmetry whenever possible.]



Solution:

One eighth of the required volume lies in the first octant. The eighth is divided into two equal parts by the plane $x = y$. One of these parts lies above the circular sector D in the $xy-$plane specified in polar coordinate by $0 \le r < a, 0 \le \theta < \frac{\pi}{4}$, and beneath the cylinder $z = \sqrt{a^2 - x^2}$. Thus, the total volume lying inside all three cylinders is

$$V = 16 \iint_D \sqrt{a^2 - x^2}\, dxdy$$

where $\boxed{16 : 1\%}$, $\boxed{\sqrt{a^2 - x^2} : 6\%}$. Then we use the polar coordinate to get

$$V = 16 \int_0^{\pi/4} d\theta \int_0^a \sqrt{a^2 - r^2 \cos^2\theta}\, rdr$$

where $\boxed{0 \le \theta < \frac{\pi}{4} : 2\%}$, $\boxed{0 \le r < a : 2\%}$, $\boxed{dxdy = rdrd\theta : 2\%}$

Let $u = a^2 - r^2\cos^2\theta, du = -2r\cos^2\theta dr$.

$$V = 8\int_0^{\pi/4} \frac{d\theta}{\cos^2\theta}\int_{a^2\sin^2\theta}^{a^2} u^{1/2}\,du = \frac{16a^3}{3}\int_0^{\pi/4}\frac{1-\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi/4}\left(\sec^2\theta - \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\right)d\theta = \frac{16a^3}{3}\left(\tan\theta - \frac{1}{\cos\theta} - \cos\theta\right)\Bigg|_0^{\pi/4}$$

$$= \frac{16a^3}{3}\left(1 - 0 - \sqrt{2} + 1 - \frac{1}{\sqrt{2}} + 1\right) = 16\left(1 - \frac{1}{\sqrt{2}}\right)a^3 \text{ cu. units.}$$

where $\boxed{\left(1 - \frac{1}{\sqrt{2}}\right)a^3 \,:\, 2\%}$

4. Consider the vector fields

$$\mathbf{F} = (1+x)e^{x+y}\mathbf{i} + (xe^{x+y} + 2y)\mathbf{j} - 2z\mathbf{k},$$
$$\mathbf{G} = (1+x)e^{x+y}\mathbf{i} + (xe^{x+y} + 2z)\mathbf{j} - 2y\mathbf{k}.$$

(a) (7 points) Show that $\mathbf{F}$ is conservative by finding a potential for it.

(b) (8 points) Evaluate $\int_{\mathcal{C}} \mathbf{G}\cdot d\mathbf{r}$, where $\mathcal{C}$ is given by

$$\mathbf{r} = (1-t)e^t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \quad (0 \le t \le 1)$$

by taking advantage of the similarity between $\mathbf{F}$ and $\mathbf{G}$.

Solution:
(a) $F = (1+x)e^{x+y}i + (xe^{x+y} + 2y)j - 2zk = \nabla(xe^{x+y} + y^2 - z^2)$ (7points).

(b) $\int_c G\cdot dr = \int_c F\cdot dr + \int_c 2(z-y)(j+k)\cdot dr$ (2points)
$= (xe^{x+y} + y^2 - z^2)\Big|_{(1,0,0)}^{(0,1,2)} + 2\int_0^1 (2t-1)(1+2)dt$ (4 points)
$= -3 - e + 3 = -e$ (2points)

5. (14 points) Let $\mathbf{F} = \left(\sqrt{x^2+y^2} - \frac{x}{1+y^2}\right)\mathbf{i} + \left(e^x + \tan^{-1}y\right)\mathbf{j}$ and $\mathcal{C}$ be the positively oriented cardioid $r = 1 + \cos\theta$. Find $\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,ds$.

Solution:
Let

$$P = \sqrt{x^2+y^2} - \frac{x}{1+y^2}, \quad Q = e^x + tan^{-1}y$$

By Green's Theorem,

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{n}\,ds = \iint_D (P_x + Q_y)\,dA \qquad (4\%)$$

$$= \iint_D [(\frac{x}{\sqrt{x^2+y^2}} - \frac{1}{1+y^2}) + (\frac{1}{1+y^2})]\,dA$$

$$= \iint_D \frac{x}{\sqrt{x^2+y^2}}\,dA \qquad (2\%)$$

$$= \int_0^{2\pi} \int_0^{1+cos\theta} \frac{rcos\theta}{r} rdrd\theta \qquad (4\%)$$

$$= \int_0^{2\pi} cos\theta \; [\frac{r^2}{2}]_0^{1+cos\theta} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (cos\theta + 2cos^2\theta + cos^3\theta) d\theta$$

$$= \int_0^{2\pi} (\frac{1+cos2\theta}{2}) d\theta$$

$$= \pi \qquad (4\%)$$

$$ps. \quad \frac{1}{2} \int_0^{2\pi} (cos\theta + cos^3\theta) d\theta = 0$$

6. Let $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.
   (a) (4 points) Find curl $\mathbf{F}$
   (b) (10 points) Evaluate $\oint_{\mathcal{C}} ydx + zdy + xdz$, where $\mathcal{C}$ is the intersection curve of the surface $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$ oriented counterclockwise when viewed from positive z-axis.

Solution:
(a.) (4 points)

$$curl(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\vec{i} - \vec{j} - \vec{k}$$

(b.)
Method 1:
C is the boundary of surfaces: $x^2 + y^2 + z^2 = a^2$ and $S : x + y + z = 0$
the normal vector of S is $\vec{n} = \dfrac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$ (3 points),
by stokes theorem ,

$$\oint_c ydx + zdy + xdz = \int_s curl(\vec{F}) \cdot \vec{n} ds = -\sqrt{3} \int_s ds \text{ (4 points,one equal sign is 2 points)}$$

S is a circular disk whose radius is a. Area of S=πa^2.
$-\sqrt{3} \int_s ds$=$-\sqrt{3}$(area of S)=$-\sqrt{3}\pi a^2$
Thus,

$$\oint_c ydx + zdy + xdz = -\sqrt{3}\pi a^2 \text{ (3 points)}$$

Method 2:
C is projected on x-y plane to get the new cuve $\tilde{C}$
$\tilde{C}$: $x^2 + y^2 + (-x - y)^2 = a^2$, also, $x^2 + y^2 + xy = \dfrac{1}{2}a^2$

$\tilde{C}$ is oriented clockwisely in x-y plane.
(the equation and orientation of $\tilde{C}$, 3points)

$$\oint_c ydx + zdy + xdz = \oint_{\tilde{C}} [ydx - (x+y)dy + x(-dx - dy)] = \oint_{\tilde{C}} [ydx - 2xdy] + [-xdx - ydy] \text{ (1 point)}$$

However, $xdx + ydy = 0$ has potential $\frac{1}{2}(x^2 + y^2)$,thus,

$$\oint_{\tilde{C}} [-xdx - ydy] = 0 \text{ (1 point)}$$

By Green theorem,

$$\oint_c ydx + zdy + xdz = \oint_{\tilde{C}} [ydx - 2xdy]$$

by Green theorem,

$$\oint_{\tilde{C}} [ydx - 2xdy] = \int_{x^2+y^2+xy \le \frac{1}{2}a^2} [-2-1]dxdy \text{ (2 points)}$$

$=-3(\text{area of } x^2 + y^2 + xy \le \frac{1}{2}a^2) = -3\frac{1}{\sqrt{3}}\pi a^2 = -\sqrt{3}\pi a^2$ (3points)

7. (14 points) Evaluate the flux of $\mathbf{F}(x,y,z) = \left(x^2 + \sin(y^3 + 2z^2)\right)\mathbf{i} + (e^{x^2} + y^2)\mathbf{j} + (3+x)\mathbf{k}$ upward across the surface S defined by $x^2 + y^2 + z^2 = 2az + 3a^2$, $z \ge 0$, where $a > 0$ is a constant.

Solution:
We call the area inside this boundary S.

$$S = \{x^2 + y^2 + (z-a)^2 \le 4a^2, z \ge 0\} \text{ (1pt)}$$

Then the boundary of S is

$$\partial S = S_1 + S_2,$$

where S_1 is the range we want to calculate on, and $S_2 = \{x^2 + y^2 = 3a^2, z = 0\}$. Note that the normal direction of S_2 is downward. Then by Divergence theorem,

$$\iiint_S \nabla \cdot \boldsymbol{F} dV = \text{The flux over } \partial S \text{ (2pts)}$$

$$= \oiint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n} dS + \oiint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n} dS \text{ (3pts) ,}$$

$$\nabla \cdot \boldsymbol{F} = 2x + 2y.$$

Thus

$$\iiint_S \nabla \cdot \boldsymbol{F} dV = 2(\bar{x} + \bar{y}) \iiint_S dV = 0 \text{ (3pts)}$$

(Note that if you really calculate the integration, you must use the correct range of S (2pts), sphere coordinate may cause some problems about the range of ϕ and will easily omit the region $\{x^2 + y^2 \le \frac{3}{4}(z-a)^2, 0 \le z \le a\}$.)
Hence

$$\oiint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n} dS = -\oiint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n} dS.$$

Note that the normal vector of S_2 is downward, that is, $\boldsymbol{n} = (0, 0, -1)$. Thus

$$\begin{aligned}
-\oiint_{S_2} F \cdot \boldsymbol{n} dS &= -\oiint_{S_2} \boldsymbol{F} \cdot (0,0,-1) dS \\
&= -\oiint_{S_2} -(3+x) dS \\
&= (3+\bar{x}) \oiint_{S_2} dS \\
&= (3+0) * \text{Area of } S_2 \\
&= 3 * 3a^2\pi \\
&= 9a^2\pi \text{ (3pts)}
\end{aligned}$$

Thus the answer we want to calculate is $9a^2\pi$.