1001微乙01-05班期中考解答和評分標準

1. (10%) 計算 $\lim_{x\to\infty}(\frac{x-1}{x+1})^x$ 。

   Sol:

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{-2}{x+1}\right)^x &= \lim_{x\to\infty}\left[(1+\frac{-2}{x+1})^{x+1}(1+\frac{-2}{x+1})^{-1}\right]\\ &= \lim_{y\to\infty}\left[(1+\frac{-2}{y})^{y}(1+\frac{-2}{y})^{-1}\right]\\ &= e^{-2}\end{aligned}$$

   評分標準:

   有利用 $e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ 的概念得 7 分;依算式完整度得 7 至 10 分 。

2. (10%) 令 $f(x)=x^{\ln x}$ 。求 $f'(x)$ 。

   Sol:

   $f(x)=x^{\ln x}=e^{(\ln x)^2}\Rightarrow f'(x)=e^{(\ln x)^2}\cdot 2\ln x\cdot\frac{1}{x}=2\ln x\cdot x^{(\ln x)-1}.$

   (There is no partial credit)

3. (10%) 令 $f(x)=\tan^{-1}(\sqrt{x})\cdot\tan(x^2)$ 。求 $f'(x)$ 。

   Sol:

   Let $f(x)=\arctan(\sqrt{x})\cdot\tan(x^2)$. Try to find $f'(x)$

$$\begin{aligned}f'(x) &= \arctan(\sqrt{x})'\cdot\tan(x^2)+\arctan(\sqrt{x})\cdot\tan(x^2)' \quad \text{The Prodcut Rule : 2pts}\\ &= \left(\frac{1}{1+(\sqrt{x})^2}\cdot\sqrt{x}'\right)\cdot\tan(x^2)+\arctan(\sqrt{x})\cdot\left(\sec^2(x^2)\cdot x^{2'}\right) \quad \text{The Chain Rules : 2pts + 2pts}\\ &= \left(\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\right)\cdot\tan(x^2)+\arctan(\sqrt{x})\cdot\left(\sec^2(x^2)\cdot 2x\right) \quad \text{The Differentations : 2pts + 2pts}\end{aligned}$$

4. (10%) 說明 $f(x)=4x^3+2x^2+4x+1$ 和 $g(x)=2x^2+\cos(x)$ 僅有一個交點 。

   Sol:

   Let $F(x)=f(x)-g(x)=4x^3+4x+1-\cos x$ be the difference of $f(x)$ and $g(x)$.

   Since $F(0)=0$, $f(x)$ and $g(x)$ has an intersection at $x=0$. (3 %)

M1: Suppose there is another intersection at $x = a \neq 0$, i.e. $F(a) = 0$.

By Mean Value Theorem(1%), $\exists c$ lies between a and 0, such that $F'(c) = \dfrac{F(a) - F(0)}{a - 0} = 0 \rightarrow\leftarrow$ ($\because F'(x) = 12x^2 + 4 + \sin x > 0$). Therefore, $f(x)$ and $g(x)$ only intersect at $x = 0$ (6%).

M2: Since $F'(x) = 12x^2 + 4 + \sin x > 0$, $F(x)$ is strictly increasing. Moreover $F(0) = 0$, so $f(x)$ and $g(x)$ only intersect at $x = 0$ (7%).

5. (20%) 假設 $y^3 + xy - x = 1$ 。

(a) 求過點 $(1, 1)$ 之切線方程式 。

(b) 求 $\dfrac{d^2y}{dx^2}$ 在點 $(1, 1)$ 之值 。

Sol:

(a) If the reader can find the clues of chain rules (3 pts).

$$3y^2y' + y + xy' - 1 = 0 \quad \text{(5 pts)}$$

After that you plug in $(0, 0)$ and find $y' = 0$, thus y $= 1$. (2 pts)

(b) If you have already get (8 pts) in part A, you may get (3 pts) from starting your second order operation.

$$6y(y')^2 + 3y^2y'' + y' + y' + xy'' = 0 \quad \text{(5 pts)}$$

After that you plug in $(0, 0)$ and find $y'' = 0$. (2 pts)

6. (10%) 利用線性逼近去估計 $\ln 0.97$ 之值 。

Sol:

設 $f(x) = \ln x$.

因爲已知 $f(1) = \ln 1 = 0$ 且 0.97 與 1 很接近,

所以用 $f(x)$ 在 $x = 1$ 的切線來估計 $f(0.97) = \ln 0.97$ 的值.

由公式 $f(x) \approx f(a) + f'(a)(x - a)$

代入 $x = 0.97, \quad a = 1$. 則 $f'(1) = (\ln x)'\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$

$$\begin{aligned} \ln 0.97 = f(0.97) &\approx f(1) + f'(1)(0.97-1) \\ &= \ln 1 + 1 * (-0.03) \\ &= 0 - 0.03 = -0.03 \end{aligned}$$

評分標準:

$f(x) \approx f(a) + f'(a)(x-a) \qquad (5\%)$

$(\ln x)'|_{x=1} = \frac{1}{x}|_{x=1} = 1 \qquad (2\%)$

$\ln 0.97 \approx -0.03 \qquad (3\%)$

7. (20%) 若 $y = f(x) = x^2 + \frac{1}{x}$ 。

(a) $y = f(x)$ 在 ______________________ (區間) 遞增 。

$y = f(x)$ 在 ______________________ (區間) 遞減 。

(b) $y = f(x)$ 之極大值 (若存在的話) : ______________________ (座標) 。

$y = f(x)$ 之極小值 (若存在的話) : ______________________ (座標) 。

(c) $y = f(x)$ 在 ______________________ (區間) 凹向上 。

$y = f(x)$ 在 ______________________ (區間) 凹向下 。

(d) $y = f(x)$ 所有的漸近線爲 ______________________

______________________ 。

(e) 畫出 $y = f(x)$ 之圖形 。

Sol: $f(x) := x^2 + x^{-1}$ defined on $x \neq 0$

$$f'(x) = 2x - x^{-2} = x^{-2}(2x^3 - 1) \begin{cases} < 0, & x < 2^{-1/3}, x \neq 0 \\ > 0, & x > 2^{-1/3}. \end{cases}$$

Hence $f(x)$ is increasing on $[2^{-1/3}, \infty)$. (2%)

$f(x)$ is decreasing on $(-\infty, 0)$ and $(0, 2^{-1/3}]$. (2%)

Critical points: $x_1, 0 = f'(x_1) \Rightarrow x_1 = 2^{-1/3}$

$$f''(x) = 2 + 2x^{-3} = 2x^{-3}(1+x)(1-x+x^2) \begin{cases} > 0, & x \in (-\infty, -1), \\ = 0, & x = -1, \\ < 0, & x \in (-1, 0) \\ > 0, & x \in (0, \infty). \end{cases}$$

$f''(x_1) = 2 + 4 > 0$, $f(x_1) = 2^{-2/3} + 2^{1/3} = \frac{3}{2}2^{1/3}$.

Hence $(x, y) = (2^{-1/3}, \frac{3}{2}2^{1/3})$ is local minimum. (2%)

There are no other critical points, hence no local maximum. (2%)

f is concave up when $f'' > 0$, i.e. $x \in (-\infty, -1)$ and $x \in (0, \infty)$. (2%)

f is concave down when $f'' < 0$, i.e. $x \in (-1, 0)$. (2%)

$\lim_{x \to 0^{\pm}} f(x) = \pm\infty, \Rightarrow$ '$x = 0$' is a vertical asymptote. Since f is defined and continuous on $\{x \neq 0\}$, there is no other vertical asymptote.

Let $y = mx + b$ be an asymptote, then $m = \lim_{x \to \pm\infty} f(x)/x = \lim_{x \to \pm\infty} (x - x^{-3}) = \pm\infty$. Hence no such asymptote.

The only asymptote is $x = 0$. (4%)

The graph is

8. (10%) 若高鐵每月載客量爲 40,000 人，票價爲 1,500 元/人。高鐵公司希望調整票價，增加收益。若票價每調高 10 元，則會損失乘客 200 人。請問該如何調整票價，才能達到最大的收益？

Sol:

Let increasing x dollar then it will lost $20x$ people

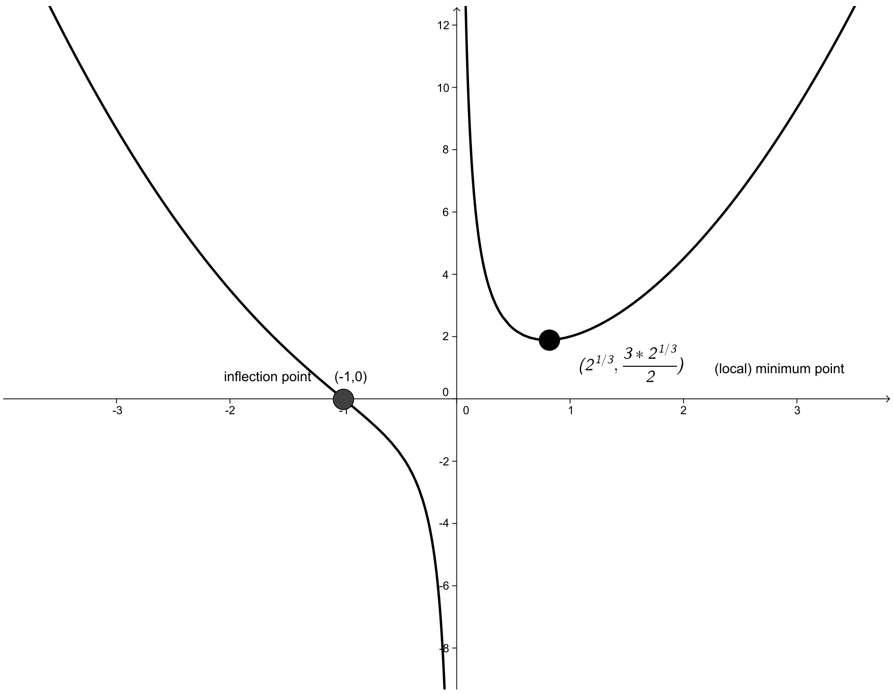
$$\begin{aligned} f(x) &= (40000 - 20x)(1500 + x) \\ &= -20x^2 + 10000x + 40000 * 1500 \end{aligned}$$

then

$$f'(x) = -40x + 10000$$

if $f'(x) = 0$ then $x = 250$ and $f'(x) = -40 < 0$ so it has max value

Hence increasing 250 dollar.



(4%)

The standard of anwser:

if you get the $f(x) = -20x^2 + 10000x + 40000 * 1500$ then 6 points.

if you get the $x = 250$ then 4 points.