1001微乙01-05班期末考解答和評分標準

1. (10%) 令 $f(x)=\int_{x^2}^{\tan^{-1}x}\frac{dt}{1+t^7}$ 。求 $f'(x)$ 。

   Sol:

   Then by Fundamental Theorem of Calculus

$$\begin{aligned} f'(x) &= \frac{1}{1+(\tan^{-1}x)^7}(\tan^{-1}x)' - \frac{1}{1+(x^2)^7}(x^2)' \\ &= \frac{1}{1+(\tan^{-1}x)^7}\frac{1}{1+x^2} - \frac{2x}{1+x^{14}} \quad (10 \quad \text{points}) \end{aligned}$$

2. (15%) 求 $\int(\sqrt{1-x^2})^3\,dx$ 。

   Sol:

   令 $x=\sin\theta \qquad \because -1\le x\le 1 \qquad \therefore -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$,

   則 $dx=\cos\theta d\theta$.

$$\begin{aligned} \int(\sqrt{1-x^2})^3dx &= \int|\cos\theta|^3\cos\theta d\theta = \int\cos^4\theta d\theta \\ &= \int(\frac{1+\cos 2\theta}{2})^2 d\theta \\ &= \int(\frac{1+2\cos 2\theta+\cos^2 2\theta}{4})d\theta \\ &= \int(\frac{1+2\cos 2\theta}{4}+\frac{1+\cos 4\theta}{8})d\theta \\ &= \int(\frac{3}{8}+\frac{2\cos 2\theta}{4}+\frac{\cos 4\theta}{8})d\theta \\ &= \frac{3}{8}\theta+\frac{\sin 2\theta}{4}+\frac{\sin 4\theta}{32}+C \\ &= \frac{3}{8}\theta+\frac{2\sin\theta\cos\theta}{4}+\frac{2\sin 2\theta\cos 2\theta}{32}+C \\ &= \frac{3}{8}\theta+\frac{\sin\theta\cos\theta}{2}+\frac{4\sin\theta\cos\theta(1-2\sin^2\theta)}{32}+C \\ &= \frac{3}{8}\sin^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{8}x\sqrt{1-x^2}(1-2x^2)+C \\ &= \frac{3}{8}\sin^{-1}x+\frac{1}{8}x\sqrt{1-x^2}(5-2x^2)+C, \quad \text{where } C \text{ is a constant.} \end{aligned}$$

   評分標準:

   變數變換成 $\int\cos^4\theta d\theta$ 或 $-\int\sin^4\theta d\theta\cdots\cdots(4\%)$

變換後到積分出來爲 θ 的函數 $\cdots\cdots\cdots$ (7%)

把 θ 的函數換回成 x 的函數 $\cdots\cdots\cdots$ (4%)

3. (20%) (a) 求 $\displaystyle\int_0^1 \frac{dx}{x^2-x+1}$ 。(10%) (b) 求 $\displaystyle\int \frac{x}{1+x^3}\,dx$ 。(10%)

Sol:

(a)

$$\begin{aligned}
\int_0^1 \frac{dx}{x^2-x+1} &= \int_0^1 \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} && (2\%)\\
&= \frac{4}{3}\int_0^1 \frac{\frac{\sqrt{3}}{2}d\frac{2}{\sqrt{3}}x}{\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)^2+1} && (3\%)\\
&= \frac{2}{\sqrt{3}}\Big[\tan^{-1}\big(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\big)\Big]_0^1 && (2\%)\\
&= \frac{2}{\sqrt{3}}\Big(\tan^{-1}\frac{1}{\sqrt{3}}-\tan^{-1}-\frac{1}{\sqrt{3}}\Big) && (1\%)\\
&= \frac{2}{\sqrt{3}}\Big(\frac{\pi}{6}+\frac{\pi}{6}\Big)\\
&= \frac{2\pi}{3\sqrt{3}} && (2\%)
\end{aligned}$$

(b) From (a), we have

$$\int \frac{1}{1-x+x^2}dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx = \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$$

Letting

$$\begin{aligned}
\frac{x}{1+x^3} = \frac{x}{(1+x)(1-x+x^2)} &= \frac{A}{1+x}+\frac{Bx+C}{1-x+x^2} \text{ (3 points)}\\
&= \frac{A(1-x+x^2)+(Bx+C)(1+x)}{(1+x)(1-x+x^2)}
\end{aligned}$$

and setting $x=0,1,-1$ into

$$x = A(1-x+x^2)+(Bx+C)(1+x),$$

we obtain

$$\begin{cases} 0 = A+C \\ 1 = A+2B+2C \\ -1 = 3A \end{cases} \Rightarrow (A,B,C) = \left(\frac{-1}{3},\frac{1}{3},\frac{1}{3}\right). \text{ (3 points)}$$

Therefore

$$
\begin{aligned}
\int \frac{x}{1+x^3}dx &= \int \frac{\frac{-1}{3}}{1+x}dx + \int \frac{\frac{1}{3}x+\frac{1}{3}}{1-x+x^2}dx \\
&= \frac{-1}{3}\int \frac{1}{1+x}dx + \frac{1}{3}\int \frac{x-\frac{1}{2}}{1-x+x^2}dx + \frac{1}{3}\int \frac{\frac{3}{2}}{1-x+x^2}dx \\
&= \frac{-1}{3}\ln|x+1| + \frac{1}{3}\cdot\frac{1}{2}\ln|x^2-x+1| + \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C' \\
&= \frac{-1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C'. \text{ (4 points)} \quad \square
\end{aligned}
$$

4. (10%) 設 A 為 $y=\sin 2x$，$y=0$ 及 $x=\frac{\pi}{2}$ 所圍成的區域。求 A 對 y 軸旋轉所得到的體積。

Sol:

$$\int_0^{\frac{\pi}{2}} 2\pi x \sin(2x)dx$$

You may get (3) points, if you write down the formula.

$$2\pi\Big(-\frac{x}{2}\cos(2x)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2x)dx\Big)$$

You may get additional (4) points, if you write down the right form of integral by part method.

$$2\pi\left(\frac{\pi}{4}+0+\frac{1}{2}(0-0)\right) = \frac{\pi}{2}$$

You may get the final (3) points, if you plug in the right boundary and get the right answer.

5. (10%) 求 $y=\frac{x^3}{6}+\frac{1}{2x}$，$1\le x\le 2$ 的曲線長度。

Sol:

$$
\begin{aligned}
&\int_1^2 \sqrt{1+(y')^2}\mathrm{d}x \qquad \text{(3 points)} \\
&= \int_1^2 \sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\mathrm{d}x \\
&= \int_1^2 \sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}\mathrm{d}x \qquad \text{(4 points)} \\
&= \int_1^2 \left(\frac{x^2}{2}+\frac{1}{2x^2}\right)\mathrm{d}x \\
&= \frac{17}{12} \qquad \text{(3 points)}
\end{aligned}
$$

6. (15%) (a) 求 $\sin x$ 在 $x=0$ 的 7 次泰勒多項式及餘項。(10%)

(b) 求 $\lim\limits_{x\to 0}\dfrac{\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!}}{x^7}$。(5%)

Sol:

(a) $P_7(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$ (7%)

$R_7(x) = \dfrac{\sin\xi\cdot x^8}{8!}$, for ξ lies between 0 and x (3%).

(b) $\lim\limits_{x\to 0}\dfrac{\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!}}{x^7} = \lim\limits_{x\to 0}\dfrac{-\frac{x^7}{7!} + \frac{\sin\xi\cdot x^8}{8!}}{x^7} = \lim\limits_{x\to 0}(-\dfrac{1}{7!} + \dfrac{\sin\xi\cdot x}{8!}) = -\dfrac{1}{7!}$ (5%)

7. (10%) 利用 $\sqrt{\dfrac{81}{121}\times\dfrac{80}{81}} = \dfrac{4}{11}\sqrt{5}$ 來求 $\sqrt{5}$ 的近似值，使誤差 $< 10^{-4}$。

Sol:

consider binomial expansion.

$$\sqrt{5} \;=\; \frac{9}{4}\Big[1 + (\frac{1}{2})(\frac{-1}{81}) + (\frac{1}{2})(\frac{1}{2})(\frac{-1}{2})(\frac{-1}{81})(\frac{-1}{81})((1+\xi)^{\frac{-3}{2}})\Big] \quad (5\%)$$

where (ξ between 0 and $\dfrac{-1}{81}$)

then $\sqrt{5} \;=\; \dfrac{9}{4}[1 + (\dfrac{1}{2})(\dfrac{-1}{81})] = \dfrac{161}{72}$ (2%)

Note that the error$\leq |\,(\dfrac{1}{2})(\dfrac{1}{2})(\dfrac{-1}{2})(\dfrac{-1}{81})(\dfrac{-1}{81}) * \mathbf{2}\,| < 10^{-4}$ (3%)

since (ξ between 0 and $\dfrac{-1}{81}$)

8. (10%) 求 $\lim\limits_{x\to\frac{\pi}{2}^-}\left(\dfrac{1}{\frac{\pi}{2}-x}\right)^{\cos x}$。(“左極限” $x\to\dfrac{\pi}{2}^-$ 意指 $x<\dfrac{\pi}{2}$，x 趨近於 $\dfrac{\pi}{2}$)

Sol:

$$
\begin{aligned}
&\lim_{x\to(\pi/2)^-}\left(\frac{1}{\pi/2-x}\right)^{\cos x}\\
&=\lim_{x\to(\pi/2)^-}\exp\left[-\cos x\cdot\ln(\pi/2-x)\right]\quad(+1\text{pts}):\ \text{Taking logarithm}\\
&=\exp\left[\lim_{x\to(\pi/2)^-}\frac{-\ln(\pi/2-x)}{\sec x}\right]\\
&=\exp\left[\lim_{x\to(\pi/2)^-}\frac{1}{(\pi/2-x)\sec x\tan x}\right]\quad(+3\text{pts}):\ \text{LHospital's Rule}\\
&=\exp\left[\lim_{x\to(\pi/2)^-}\frac{\cos^2 x}{\pi/2-x}\cdot\lim_{x\to(\pi/2)^-}\frac{1}{\sin x}\right]\\
&=\exp\left[\lim_{x\to(\pi/2)^-}\frac{2\cos x\sin x}{1}\cdot 1\right]\quad(+5\text{pts}):\ \text{LHosptial's Rule}\\
&=\exp\left[\frac{2\cdot 0\cdot 1}{1}\cdot 1\right]=\exp(0)=1\quad(+1\text{pts}):\ \text{The Answer}
\end{aligned}
$$