1001微甲08-12班期末考

1. (12%) Let $U(f, P_n)$ be the upper Riemann sum for $f(x) = \dfrac{1}{x}$ on $[1, 2]$ for partition P_n with division points $x_i = 2^{\frac{i}{n}}$ for $0 \le i \le n$.

   (a) Write out $U(f, P_n)$ and then evaluate $\lim\limits_{n\to\infty} U(f, P_n)$.

   (b) Let $L(f, P_n)$ be the lower Riemann sum. Find $\lim\limits_{n\to\infty} L(f, P_n)$.

Sol:

(a) For $x_i = 2^{\frac{i}{n}}$, since f is decreasing and continoous on $[1, 2]$, we have

$$\begin{aligned}
U(f, P_n) &= \sum_{i=0}^{n-1} (\max_{x_i \le x \le x_{i+1}} f(x))(x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} \frac{1}{x_i}(x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} 2^{\frac{-i}{n}} (2^{\frac{i+1}{n}} - 2^{\frac{i}{n}}) \\
&= \sum_{i=0}^{n-1} (2^{\frac{1}{n}} - 1) \\
&= n(2^{\frac{1}{n}} - 1)
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} n(2^{\frac{1}{n}} - 1) &= \lim_{n\to\infty} \frac{(2^{\frac{1}{n}} - 1)}{\frac{1}{n}} \\
&= \lim_{x\to 0} \frac{(2^x - 1)}{x} = \ln 2 \quad \text{with} \quad x = \frac{1}{n}
\end{aligned}$$

(b) For $x_i = 2^{\frac{i}{n}}$, since f is decreasing and continoous on $[1, 2]$, similarly ,we have

$$\begin{aligned}
L(f, P_n) &= \sum_{i=0}^{n-1} (\min_{x_i \le x \le x_{i+1}} f(x))(x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} f(x_{i+1})(x_{i+1} - x_i) = \sum_{i=0}^{n-1} \frac{1}{x_{i+1}}(x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} 2^{\frac{-i-1}{n}} (2^{\frac{i+1}{n}} - 2^{\frac{i}{n}}) = \sum_{i=0}^{n-1} (1 - 2^{\frac{-1}{n}}) \\
&= n(1 - 2^{\frac{-1}{n}})
\end{aligned}$$

So,

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} n(1 - 2^{\frac{-1}{n}}) = \lim_{n\to\infty} \frac{(1 - 2^{\frac{-1}{n}})}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{(2^{\frac{-1}{n}}) - 1}{\frac{-1}{n}} = \lim_{y\to 0} \frac{(2^y - 1)}{y} = \ln 2 \quad \text{with} \quad y = \frac{-1}{n}$$

2. (11%) Find $\lim_{n\to\infty} \ln\left(\frac{1}{n} \times \frac{2}{n} \times \cdots \times \frac{n}{n}\right)^{\frac{1}{n}}$.

Sol:

We have

$$\lim_{n\to\infty} \ln(\frac{1}{n} \times \frac{2}{n} \times \cdots \times \frac{n}{n})^{\frac{1}{n}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \ln(\frac{k}{n}) = \int_0^1 \ln x \quad dx$$

note this is an improper integral

$$= \lim_{a\to 0} \int_a^1 \ln x \quad dx = \lim_{a\to 0} (x \ln x \Big|_a^1 - \int_a^1 dx)$$

$$= \lim_{a\to 0} (-a \ln a - (1 - a)) = -1 - \lim_{a\to 0} a \ln a$$

Note that

$$\lim_{a\to 0} a \ln a = \lim_{a\to 0} \frac{\ln a}{\frac{1}{a}} = \lim_{a\to 0} \frac{\frac{1}{a}}{\frac{-1}{a^2}} = \lim_{a\to 0} -a = 0$$

(By L'Hospital's rule)

Hence

$$\lim_{a\to 0} \int_a^1 \ln x \quad dx = -1$$

3. (11%) Find the derivative $\frac{d}{dx} \int_1^x \frac{e^{\frac{1}{t}}}{t^2}(x - t)\, dt$.

Sol:

$$\frac{d}{dx} \int_1^x \frac{e^{\frac{1}{t}}}{t^2}(x - t)\, dt = \frac{d}{dx}\Big[x \int_1^x \frac{e^{\frac{1}{t}}}{t^2}\, dt - \int_1^x \frac{e^{\frac{1}{t}}}{t}\, dt\Big]$$

$$= \frac{d}{dx}\Big[x \int_1^x \frac{e^{\frac{1}{t}}}{t^2}\, dt\Big] - \frac{d}{dx}\Big[\int_1^x \frac{e^{\frac{1}{t}}}{t}\, dt\Big]$$

$$= \int_1^x \frac{e^{\frac{1}{t}}}{t^2}\, dt + x \cdot \frac{e^{\frac{1}{x}}}{x^2} - \frac{e^{\frac{1}{x}}}{x}$$

$$= \int_1^x \frac{e^{\frac{1}{t}}}{t^2}\, dt$$

$$= -e^{\frac{1}{t}}\Big|_1^x = e - e^{\frac{1}{x}}$$

4. (11%) Find the integral $\int \frac{dx}{x^3+x^2+x}$.

Sol:

$$\frac{1}{x(x^2+x+1)} = \frac{}{x} + \frac{}{x^2+x+1}$$

$x = 0$

$$\begin{aligned}
\frac{1}{x(x^2+x+1)} &= \frac{1}{x} + \frac{}{x^2+x+1} \\
\frac{1}{x(x^2+x+1)} &= \frac{1}{x} + \frac{-x-1}{x^2+x+1} \\
\frac{1}{x(x^2+x+1)} &= \frac{1}{x} + \frac{-x-1/2}{x^2+x+1} + \frac{-1/2}{(x+1/2)^2+1/4}
\end{aligned}$$

take integral

$$\int \frac{dx}{x^3+x^2+x} = \ln|x| - \frac{1}{2}\ln(x^2+x+1) - (1/2)(2/\sqrt{3})\tan^{-1}(\frac{2}{\sqrt{3}}(x+1/2)) + C$$

5. (11%) Determine the real number a so that the integral $\int_0^2 \left[\frac{1}{x\sqrt{x^2+4}} - \frac{1}{ax}\right] dx$ converges.

Sol:

解法一:

考慮積分

$$\int \frac{1}{x\sqrt{x^2+4}} dx.$$

令

$$x = 2\tan\theta,\ dx = 2\sec^2\theta d\theta,$$

則

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2+4}} dx &= \int \frac{1}{2\tan\theta \cdot 2|\sec\theta|} \cdot 2\sec^2\theta d\theta \\
&= \int \frac{1}{2}\csc\theta d\theta,\ \text{if}\ \sec\theta > 0 \\
&= -\frac{1}{2}\ln|\csc\theta + \cot\theta| + C \\
&= -\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{x} + \frac{2}{x}\right| + C.
\end{aligned}$$

所以

$$
\begin{aligned}
&\int_0^2 \left(\frac{1}{x\sqrt{x^2+4}} - \frac{1}{ax}\right) dx \text{ (has a singularity at } x=0) \\
&= \lim_{b\to 0^+} \left(-\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}+2}{x}\right| - \frac{1}{a}\ln|x|\right)\Bigg|_b^2 \\
&= \lim_{b\to 0^+} \left(-\frac{1}{2}\ln\left|\sqrt{x^2+4}+2\right| + \left(\frac{1}{2}-\frac{1}{a}\right)\ln|x|\right)\Bigg|_b^2 \\
&= \begin{cases} \text{diverges, if } a \neq 2, \\ \lim_{b\to 0^+} \left(-\frac{1}{2}\ln\left|\sqrt{x^2+4}+2\right|\right)\Bigg|_b^2, \text{ if } a=2 \end{cases} \\
&= \begin{cases} \text{diverges, if } a \neq 2, \\ -\frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{2}\right| \text{ if } a=2. \end{cases} \qquad \square
\end{aligned}
$$

解法二:

$$
\begin{aligned}
&\frac{1}{x\sqrt{x^2+4}} - \frac{1}{ax} \text{ (has a singularity at } x=0) \\
&= \frac{a-\sqrt{x^2+4}}{ax\sqrt{x^2+4}} \\
&= \frac{a\sqrt{x^2+4}-(x^2+4)}{ax(x^2+4)}
\end{aligned}
$$

分母在 0 時 vanishing order 是 1, 分子在 0 時有 vanishing order 是當 $2a-4=0$ 時; 所以只有 $a=2$ 才有可能可以積分.

O $a=2$, 則

$$
\begin{aligned}
\text{原式} &= \int_0^2 \left(\frac{1}{x\sqrt{x^2+4}} - \frac{1}{2x}\right) dx \\
&= \lim_{b\to 0^+} \left(-\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}+2}{x}\right| - \frac{1}{2}\ln|x|\right)\Bigg|_b^2 \text{ (計算同解法一)} \\
&= \lim_{b\to 0^+} \left(-\frac{1}{2}\ln\left|\sqrt{x^2+4}+2\right|\right)\Bigg|_b^2 \\
&= -\frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{2}\right|. \quad \square
\end{aligned}
$$

解法三:

令

$$u=\sqrt{x^2+4},\ u^2=x^2+4,\ 2udu=2xdx,$$

則:

$$\begin{aligned}\int_0^2\left(\frac{1}{x\sqrt{x^2+4}}-\frac{1}{ax}\right)dx &= \int_0^2\frac{a-\sqrt{x^2+4}}{ax\sqrt{x^2+4}}dx\\ &= \int_0^2\frac{a-\sqrt{x^2+4}}{ax^2\sqrt{x^2+4}}\cdot xdx\\ &= \int_2^{2\sqrt{2}}\frac{a-u}{a(u^2-4)u}\cdot udu\\ &= \frac{1}{a}\int_2^{2\sqrt{2}}\frac{a-u}{u^2-4}du\\ &= \frac{1}{a}\int_2^{2\sqrt{2}}\left(\frac{-2-a}{4}\cdot\frac{1}{u+2}+\frac{a-2}{4}\cdot\frac{1}{u-2}\right)du.\end{aligned}$$

由於

$$\int_2^{2\sqrt{2}}\frac{1}{u+2}du=\ln|u+2|\big|_2^{2\sqrt{2}}=\ln\left(\frac{\sqrt{2}+1}{2}\right),$$

而

$$\int_2^{2\sqrt{2}}\frac{1}{u-2}du=\lim_{b\to 2^+}\ln|u-2|\big|_b^{2\sqrt{2}}=\infty,$$

故知欲使原式收斂, 若且唯若 $a=2$. □

6. (11%) Find the volume of solid generated by revolving the curve $y=e^{-x}\sin x,\ x\geq 0$, about the x-axis.

Sol:

$$\begin{aligned}V &= \int_0^\infty \pi(e^{-x}\sin x)^2dx\\ &= \pi\int_0^\infty e^{-2x}(\sin x)^2dx\\ &= \pi\Big[-\frac{1}{2}e^{-2x}(\sin x)^2\Big|_0^\infty+\frac{1}{2}\int_0^\infty e^{-2x}\sin 2xdx\Big]\\ &= \frac{\pi}{2}\int_0^\infty e^{-2x}\sin 2xdx\end{aligned}$$

$$\begin{aligned}\frac{\pi}{2}\int_0^\infty e^{-2x}\sin 2xdx &= \frac{\pi}{2}\Big[-\frac{1}{2}e^{-2x}\cos 2x\Big|_0^\infty-\int_0^\infty e^{-2x}\cos 2xdx\Big]\\ &= \frac{\pi}{2}\Big[\frac{1}{2}-\frac{1}{2}e^{-2x}\sin 2x\Big|_0^\infty-\int_0^\infty e^{-2x}\sin 2xdx\Big]\\ &= \frac{\pi}{4}-\frac{\pi}{2}\int_0^\infty e^{-2x}\sin 2xdx\end{aligned}$$

Thus

$$\int_0^{\infty} e^{-2x} \sin 2x dx = \frac{1}{4}$$

so

$$V = \frac{\pi}{2} \int_0^{\infty} e^{-2x} \sin 2x dx = \frac{\pi}{8}$$

7. (11%) Find the areas of the surface obtained by rotating the curve $y = \cos x$, $0 \leq x \leq \frac{\pi}{2}$ about the $x-$axis.

   Sol:

   The required volume is

$$\begin{aligned}
&2\pi \int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin^2 x}\, dx \\
&= 2\pi \int_0^{\frac{\pi}{4}} \sec^3 \theta \, d\theta \quad (\sin x = \tan \theta) \\
&= 2\pi [\frac{1}{2} \sec x \tan x - \frac{1}{2} \ln | \sec x + \tan x |] \Big|_0^{\frac{\pi}{4}} \\
&= \pi[\sqrt{2} + \ln(1 + \sqrt{2})]
\end{aligned}$$

8. (20%) Consider the cardioid $r = 1 - 2\cos\theta$.

   (a) Find the slope of the tangents to the cardioid at the origin.

   (b) Find the area of the region between the inner and outer loops of the cardioid.

   Sol:

   (a) At the origin, $r = 1 - 2\cos\theta = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

   method 1:

   $r = 0 \Rightarrow \frac{dy}{dx} = \tan\theta = \sqrt{3}$ or $-\sqrt{3}$

   methos 2:

   $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\sin\theta\cos\theta - \sin\theta}{\cos\theta + 2\sin^2\theta - 2\cos^2\theta}$

   Plugging $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ into the eqution, we obtain $\frac{dy}{dx} = \sqrt{3}\, or\, -\sqrt{3}$

(b)

$$\begin{aligned} A(a,b) &= \int_a^b \frac{r^2}{2} d\theta \\ &= \frac{1}{2}\int_a^b (1-4\cos\theta+4\cos^2\theta)d\theta \\ &= \int_a^b \frac{1}{2}(1-4\cos\theta)+(1+\cos 2\theta)d\theta \\ &= \left[\frac{1}{2}(3\theta-4\sin\theta+\sin 2\theta)\right]_a^b \end{aligned}$$

$$\begin{aligned} \text{Area} &= A(\frac{\pi}{3},\frac{5\pi}{3}) - A(-\frac{\pi}{3},\frac{\pi}{3}) \\ &= (2\pi+\frac{3\sqrt{3}}{2}) - (\pi-\frac{3\sqrt{3}}{2}) \\ &= \pi+3\sqrt{3} \end{aligned}$$

9. (11%) Find the length of one arc of the cycloid $x = rt - r\sin t,\ y = r - r\cos t$.

Sol:

$dx = r(1-\cos t)dt$

$dy = r\sin t dt$

$$\begin{aligned} s &= \int \sqrt{(dx)^2+(dy)^2} = \int_0^{2\pi} r\sqrt{(1-cost)^2+(\sin^2 t)}\, dt \\ &= \int_0^{2\pi} r\sqrt{2-2\cos t}\, dt = \int_0^{2\pi} r\sqrt{4\sin^2\frac{t}{2}}\, dt \\ &= \int_0^{2\pi} 2r|\sin\frac{t}{2}|\, dt = \int_0^{2\pi} 2r\sin\frac{t}{2}\, dt \\ &= (-4r\cos\frac{t}{2})\Big|_0^{2\pi} = 8r \end{aligned}$$

10. (11%) Solve the initial-value problem $\begin{cases} y' + y\sec^2 x = xe^{-\tan x} \\ y(0) = \pi. \end{cases}$

Sol:

$\mu(x) = \int \sec^2 x\, dx = \tan x$

$y(x) = e^{-\mu(x)} \int e^{\mu(x)}(xe^{-\tan x})\, dx = e^{-\tan x}\int x\, dx = e^{-\tan x}(\frac{1}{2}x^2 + \mathcal{C})$

$x = 0 \Rightarrow \mathcal{C} = \pi$

$y(x) = e^{-\tan x}(\frac{1}{2}x^2 + \pi)$