INVARIANCE OF GROMOV-WITTEN THEORY UNDER A SIMPLE FLOP

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ABSTRACT. In this work, we continue our study initiated in [11]. We show that the generating functions of Gromov-Witten invariants *with ancestors* are invariant under a simple flop, for all genera, after an analytic continuation in the extended Kähler moduli space.

The results presented here give the first evidence, and the only one not in the toric category, of the invariance of full Gromov-Witten theory under the K-equivalence (crepant transformation).

0. INTRODUCTION

0.1. **Statement of the main results.** Let X be a smooth complex projective manifold and $\psi : X \to \bar{X}$ a flopping contraction in the sense of minimal model theory, with $\bar{\psi} : Z \cong \mathbb{P}^r \to pt$ the restriction map to the extremal contraction. Assume that $N_{Z/X} \cong \mathscr{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}$. It was shown in [11] that a simple $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$ exists and the graph closure $[\bar{\Gamma}_f] \in A^*(X \times X')$ induces a correspondence $\mathscr{F}$ which identifies the Chow motives $\hat{X}$ of X and $\hat{X}'$ of X'. Furthermore, the big quantum cohomology rings, or equivalently genus zero Gromov-Witten invariants with 3 or more insertions, are invariant under a simple flop, after an analytic continuation in the extended Kähler moduli space.

The goal of the current paper is to extend the results of [11] to all genera. In the process we discovered the natural framework in the *ancestor* potential

$$\mathscr{A}_X(\bar{t}, s) := \exp \sum_{g=0}^{\infty} \hbar^{g-1} \overline{F}_g^X(\bar{t}, s),$$

which is a formal series in the Novikov variables $\{q^\beta\}_{\beta \in NE(X)}$ defined in the stable range $2g + n \geq 3$. See Section 1 for the definitions.

The main results of this paper are the following theorems.

Theorem 0.1. *The total ancestor potential $\mathscr{A}_X$ (resp. $\mathscr{A}_{X'}$) is analytic in the extremal ray variable q^ℓ (resp. $q^{\ell'}$). They are identified via $\mathscr{F}$ under a simple flop, after an analytic continuation in the extended Kähler cone $\omega \in H^{1,1}_{\mathbb{R}}(X) + i(\mathcal{K}_X \cup \mathscr{F}^{-1}\mathcal{K}_{X'})$ via*

$$q^\ell = e^{2\pi i (\omega.\ell)},$$

where $\mathcal{K}_X$ (resp. $\mathcal{K}_{X'}$) is the Kähler cone of X (resp. X').

There are extensive discussions of analytic continuation and the Kähler moduli in Section 3. We note that the *descendent* potential is in general *not* invariant under $\mathscr{F}$ (c.f. [11], §3). The descendents and ancestors are related via a simple transformation ([7, 5], c.f. Proposition 1.1), but the transformation is in general not compatible with $\mathscr{F}$. Nevertheless we do have

Theorem 0.2. *For a simple flop f, any generating function of mixed invariants of f-special type*

$$\langle \tau_{k_1,\bar{l}_1}\alpha_1, \cdots, \tau_{k_n,\bar{l}_n}\alpha_n\rangle_g,$$

with $2g+n \geq 3$, is invariant under $\mathscr{F}$ up to analytic continuation under the identification of Novikov variables $\mathscr{F}q^\beta = q^{\mathscr{F}\beta}$.

Here a mixed insertion $\tau_{k_j,\bar{l}_j}\alpha_j$ consists of descendents ψ_j^k and ancestors $\bar{\psi}_j^l$. Given $f : X \dashrightarrow X'$ with exceptional loci $Z \subset X$ and $Z' \subset X'$, a mixed invariant is of f-special type if for every insertion $\tau_{k_j,\bar{l}_j}\alpha_j$ with $k_j \geq 1$ we have $\alpha_j.Z = 0$. Theorem 0.1 follows from an application of Theorem 0.2 when no descendent is present.

0.2. **Outline of the contents.** Section 1 contains some basic definitions as well as special terminologies in Gromov-Witten theory used in the article. One of the main ingredients of our proof of invariance of the the higher genus Gromov–Witten theory is Givental's *quantization formalism* [5] for *semisimple* Frobenius manifolds. This is reviewed in Section 2.

Another main ingredient, in comparing Gromov-Witten theory of X and X', is the degeneration analysis. We generalize the genus zero results of the degeneration analysis in [11] to *ancestor potentials* in *all genera*. The analysis allows us to reduce the proofs of Theorem 0.1 (and 0.2) from flops of X to flops of the local model $\mathbb{P}_Z(N_{Z/X} \oplus \mathscr{O})$.

To keep the main idea clear, we choose to work on local models first in Section 4 and postpone the degeneration analysis till section 5. The local models are semi-Fano toric varieties and localizations had been effectively used to solve the genus zero case. The idea is to utilize Givental's quantization formalism on the local models to derive the invariance in higher genus, up to analytic continuation, from our results [11] in genus zero.

In doing so, the key point is that local models have *semisimple quantum cohomology*, and we trace the effect of analytic continuation carefully during the process of quantization. The issues of the analyticity of the Frobenius manifolds and the precise meaning of the analytic continuation involved in this study is discussed in Section 3 before we discuss local models.

The proofs of our main results Theorem 0.1 and 0.2, as well as the degeneration analysis, are presented in section 5.

0.3. **Some remarks on the crepant transformation conjecture.** A morphism $\psi : X \to \bar{X}$ is called a *crepant resolution*, if X is smooth and $\bar{X}$ is $\mathbb{Q}$-Gorenstein such that $\psi^* K_{\bar{X}} = K_X$. When $\bar{X}$ admits a resolution by a smooth Deligne-Mumford stack (orbifold) $\mathfrak{X}$, there is a well-defined orbifold Gromov-Witten theory due to Chen-Ruan. The *crepant transformation conjecture* asserts a close relationship between the Gromov-Witten theory of X and that of $\mathfrak{X}$.

Crepant resolution conjecture, as formulated in [2], still uses descendent potentials rather than the ancestor potentials, as proposed in [8]. Yet ancestors often enjoy better properties than the corresponding descendents, as exploited by E. Getzler [4].

Since different crepant resolutions are related by a crepant (K-equivalent) transformation, e.g. a flop, the conjecture must be consistent with a transformation under a flop (c.f. [16]). Although the descendent potentials can be obtained from the ancestor potentials via a simple transformation, this very transformation actually spoils the invariance under $\mathscr{F}$. The insistence in the descendents may introduce unnecessary complication in the formulation of the conjecture. This is especially relevant in the stronger form of the conjecture when the orbifolds satisfy the Hard Lefschetz conditions.

Our result suggests that a more natural framework to study crepant transformation conjecture is to use ancestors rather than descendents. We leave the interested reader to consult [2] and references therein.

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1. Descendent and ancestor potentials

1.1. **The ancestor potential.** For the stable range $2g + n \geq 3$, let

$$\sigma := \mathrm{ft} \circ \mathrm{st} : \overline{M}_{g,n+l}(X, \beta) \to \overline{M}_{g,n}$$

be the composition of the *stabilization morphism* $\mathrm{st} : \overline{M}_{g,n+l}(X, \beta) \to \overline{M}_{g,n+l}$ defined by forgetting the map and the *forgetful morphism* $\mathrm{ft} : \overline{M}_{g,n+l} \to \overline{M}_{g,n}$ defined by forgetting the last l points. The *ancestors* are defined to be

$$\bar{\psi}_j := \sigma^* \psi_j \tag{1.1}$$

for $j = 1, \dots, n$. The class $\bar{\psi}_j$ depends on l and n. For simplicity we suppress l and n from the notation when no confusion is likely to arise.

Let $\{T_\mu\}$ be a basis of $H^*(X, \mathbb{Q})$. Denote $\bar{t} = \sum_{\mu,k} \bar{t}^\mu_k \bar{\psi}^k T_\mu$, $s = \sum_\mu s^\mu T_\mu$, and let

$$\begin{aligned}\overline{F}^X_g(\bar{t}, s) &= \sum_{n,l,\beta} \frac{q^\beta}{n!l!} \langle \bar{t}^n, s^l \rangle_{g,n+l,\beta} \\ &= \sum_{n,l,\beta} \frac{q^\beta}{n!l!} \int_{[\overline{M}_{g,n+l}(X,\beta)]^{\mathrm{vir}}} \prod_{j=1}^{n} \sum_{k,\mu} \bar{t}^\mu_k \bar{\psi}^k_j \, \mathrm{ev}^*_j \, T_\mu \prod_{j=n+1}^{n+l} \sum_\mu s^\mu \, \mathrm{ev}^*_j \, T_\mu\end{aligned}$$

be the generating function of genus g ancestor invariants.

The ancestor potential is defined to be the formal expression

$$\mathscr{A}_X(\bar{t}, s) := \exp \sum_{g=0}^{\infty} \hbar^{g-1} \overline{F}_g^X(\bar{t}, s).$$

Note that $\mathscr{A}$ depends on s (variables on the Frobenius manifold), in addition to $\bar{t} = \sum \bar{t}_k^\mu T_\mu z^k$ (variables on the "Fock space"). It is analogous to the formal descendent potential

$$\mathscr{D}_X(t) = \exp \sum_{g=0}^{\infty} \hbar^{g-1} F_g^X(t),$$

where $t = \sum_{\mu,k} t_k^\mu \psi^k T_\mu$ and $F_g^X(t) = \sum_{n,\beta} \langle t^n \rangle_{g,n,\beta}\, q^\beta / n!$ is the genus g generating function of the descendent invariants.

Let j be one of the first n marked points such that $\bar{\psi}_j$ is defined. Let D_j be the (virtual) divisor on $\overline{M}_{g,n+l}(X, \beta)$ defined by the image of the gluing morphism

$$\sum_{\beta'+\beta''=\beta} \sum_{l'+l''=l} \overline{M}_{0,\{j\}+l'+\bullet}(X, \beta') \times_X \overline{M}_{g,(n-1)+l''+\bullet}(X, \beta'') \to \overline{M}_{g,n+l}(X, \beta),$$

where $\bullet$ represents the gluing point; $\overline{M}_{g,(n-1)+l''+\bullet}(X, \beta'')$ carries all first n marked points *except the j-th one*, which is carried by $\overline{M}_{0,\{j\}+l'+\bullet}(X, \beta')$. Ancestor and descendent invariants are related by the simple geometric equation

$$(\psi_j - \bar{\psi}_j) \cap [\overline{M}_{g,n+l}(X, \beta)]^{\mathrm{vir}} = [D_j]^{\mathrm{vir}}. \tag{1.2}$$

This can be easily seen from the definitions of ψ and $\bar{\psi}$. The morphism π in (1.1) contracts only rational curves during the processes of forgetful and stabilization morphisms. The (virtual) difference of ψ and $\bar{\psi}$ is exactly D_j.

1.2. **The mixed invariants.** We will consider more general *mixed invariants* with mixed ancestor and descendent insertions. Denote by

$$\langle \tau_{k_1,\bar{l}_1} \alpha_1, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n \rangle_{g,n,\beta}$$

the invariants with mixed descendent and ancestor insertion $\psi_j^{k_j} \bar{\psi}_j^{l_j} \operatorname{ev}_j^* \alpha_j$ at the j-th marked point and let

$$\langle \tau_{k_1,\bar{l}_1} \alpha_1, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n \rangle_g(s) := \sum_{l,\beta} \frac{q^\beta}{l!} \langle \tau_{k_1,\bar{l}_1} \alpha_1, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n, s^l \rangle_{g,n+l,\beta},$$

$$\langle \tau_{k_1,\bar{l}_1} \alpha_1, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n \rangle_g(\bar{t}, s) := \sum_{m,l,\beta} \frac{q^\beta}{m!l!} \langle \tau_{k_1,\bar{l}_1} \alpha_1, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n, \bar{t}^m, s^l \rangle_{g,(n+m)+l,\beta}.$$

to be the generating functions.

Equation (1.2) can be rephrased in terms of these generating functions.

Proposition 1.1. *In the stable range* $2g + n \geq 3$*, for* $(k_1, l_1) = (k+1, l)$*,*

$$\begin{aligned} &\langle \tau_{k+1,\bar{l}}\, \alpha_1, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n \rangle_g(\bar{t}, s) \\ &\quad = \langle \tau_{k,\overline{l+1}}\, \alpha_1, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n \rangle_g(\bar{t}, s) \\ &\qquad + \sum_\nu \langle \tau_k\, \alpha_1, T_\nu \rangle_0(s)\, \langle \tau_{\bar{l}}\, T^\nu, \cdots, \tau_{k_n,\bar{l}_n} \alpha_n \rangle_g(\bar{t}, s) \end{aligned} \tag{1.3}$$

where $\cdots$ *denotes the same list of mixed insertions.*

In fact, only one special type of the mixed invariants will be needed. Let (X, E) be a smooth pair with $j : E \hookrightarrow X$ a smooth divisor, which we call the divisor at infinity. At the i-th marked point, if $k_i \neq 0$, then we require that $\alpha_i = \varepsilon_i \in j_* H^*(E) \subset H^*(X)$. This type of invariants will be called *mixed invariants of special type* and the marked points with $k_i \neq 0$ will be called *marked points at infinity*.

For a birational map $f : X \dashrightarrow X'$ with exceptional locus $Z \subset X$, a mixed invariant is said to be *of f-special type* if $\alpha.Z = 0$ for every insertion $\tau_{k,\bar{l}}\, \alpha$ with $k \neq 0$. When (X_{loc}, E) comes form the local model of (X, Z), namely $X_{loc} := \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathscr{O})$ with E being the infinity divisor, these two notions of special type agree.

Proposition 1.1 will later be used (c.f. Theorem 4.5) in the following setting. Suppose that under a flop $f : X \dashrightarrow X'$ we have invariance of ancestor generating functions. To extend the invariance to allow also descendents we may reduce the problem to the $g = 0$ case and with at most one descendent insertion $\tau_k\, \alpha$. For local models, it is important that the invariants are of special type to ensure the invariance.

2. Review of Givental's quantization formalism

In this section we recall Givental's axiomatic Gromov-Witten theory. As it is impossible to include all background material, this is mainly to fix the notations. The reader may consult [8, 9, 13] for the details.

2.1. **Formal ingredients in the geometric Gromov-Witten theory.** For a projective smooth variety X, Gromov-Witten theory of X consists of the following ingredients

(i) $H := H^*(X, \mathbb{C})$ is a $\mathbb{C}$-vector space, assumed of rank N. Let $\{T_\mu\}_{\mu=1}^N$ be a basis of H and $\{s^\mu\}_{\mu=1}^N$ be the dual coordinates with $\partial/\partial s^\mu = T_\mu$. Set $T_1 = \mathbf{1} \in H^0(X)$, the (dual of) fundamental class. H carries a symmetric bilinear form, the Poincaré pairing,

$$(\cdot, \cdot) : H \otimes H \to \mathbb{C}.$$

Define $g_{\mu\nu} := (T_\mu, T_\nu)$ and $g^{\mu\nu}$ to be the inverse matrix.

(ii) Let $\mathscr{H}_t := \bigoplus_{k=0}^\infty H$ be the infinite dimensional complex vector space with basis $\{T_\mu \psi^k\}$. $\mathscr{H}_t$ has a natural $\mathbb{C}$-algebra structure:

$$T_\mu \psi^{k_1} \otimes T_\nu \psi^{k_2} \mapsto (T_\mu \cup T_\nu) \psi^{k_1+k_2}.$$

Let $\{t^\mu_k\}$, $\mu = 1, \dots, N$, $k = 0, \dots, \infty$, be the dual coordinates of the basis $\{T_\mu \psi^k\}$. We note that at each marked point, the descendent insertion is $\mathscr{H}_t$-valued. Let

$$t := \sum_{k,\mu} t^\mu_k T_\mu \psi^k$$

denote a general element in the vector space $\mathscr{H}_t$.

(iii) The generating function of descendents

$$F^X_g(t) := \sum_{n,\beta} \frac{q^\beta}{n!} \langle t, \dots, t \rangle_{g,n,\beta}$$

is a formal function on $\mathscr{H}_t$ with coefficient in the Novikov ring. (The convergence holds for local models, c.f. Section 3, which is the only case we need in the quantization process.)

(iv) H carries a (big quantum cohomology) ring structure. Let $s^\mu = t^\mu_0$ and $F_0(s) = F_0(t)|_{t_k=0,\ \forall k>0}$. The ring structure is defined by

$$T_{\mu_1} *_s T_{\mu_2} := \sum_{\nu,\nu'} \frac{\partial^3 F_0(s)}{\partial s^{\mu_1} \partial s^{\mu_2} \partial s^\nu} g^{\nu\nu'} T_{\nu'}.$$

1 is the identity element of the ring. In the subsequent discussions, the subscript s of $*_s$ will be dropped when the context is clear.

(v) The Dubrovin connection ∇_z on the tangent bundle TH is defined by

$$\nabla_z := d - z^{-1} \sum_\mu ds^\mu (T_\mu *).$$

The quantum cohomology differential equation

$$\nabla_z S = 0 \tag{2.1}$$

has a fundamental solution $S = (S_{\mu,\nu}(s, z^{-1}))$, an $N \times N$ matrix-valued function, in (formal) power series of z^{-1} satisfying the conditions

$$S(s, z^{-1}) = Id + O(z^{-1}) \quad \text{and} \quad S^*(s, -z^{-1}) S(s, z^{-1}) = Id, \tag{2.2}$$

where * denotes the adjoint with respect to $(\cdot, \cdot)$.

(vi) The non-equivariant genus zero Gromov-Witten theory is graded, i.e. with a *conformal* structure. The grading is determined by an *Euler field* $E \in \Gamma(T_X)$,

$$E = \sum_\mu (1 - \frac{1}{2} \deg T_\mu) s^\mu \frac{\partial}{\partial s^\mu} + c_1(T_X). \tag{2.3}$$

2.2. **Semisimple Frobenius manifolds.** The concept of Frobenius manifolds was originally introduced by B. Dubrovin. We assume that the readers are familiar with the definitions of the Frobenius manifolds. See [10] Part I for an introduction. The quantum product $*$, together with Poincaré pairing, and the special element $\mathbf{1}$, defines on H a Frobenius manifold structure $(QH, *)$.

A point $s \in H$ is called a *semisimple* point if the quantum product on the tangent algebra $(T_s H, *_s)$ at $s \in H$ is isomorphic to $\bigoplus_1^N \mathbb{C}$ as an algebra. $(QH, *)$ is called semisimple if the semisimple points are (Zariski) dense in H. If $(QH, *)$ is semisimple, it has idempotents $\{\epsilon_i\}_1^N$

$$\epsilon_i * \epsilon_j = \delta_{ij}\epsilon_i.$$

defined up to S_N permutations. The *canonical coordinates* $\{u^i\}_1^N$ is a local coordinate system on H near s defined by $\partial/\partial u^i = \epsilon_i$. When the Euler field is present, the canonical coordinates are also uniquely defined up to signs and permutations. We will often use the *normalized* form $\tilde{\epsilon}_i = \epsilon_i / \sqrt{(\epsilon_i, \epsilon_i)}$.

Lemma 2.1. *$\{\epsilon_i\}$ and $\{\tilde{\epsilon}_i\}$ form orthogonal bases.*

Proof. $(\epsilon_i, \epsilon_j) = (\epsilon_i * \epsilon_i, \epsilon_j) = (\epsilon_i, \epsilon_i * \epsilon_j) = (\epsilon_i, \delta_{ij}\epsilon_i) = \delta_{ij}(\epsilon_i, \epsilon_i)$. □

When the quantum cohomology is semisimple, the quantum differential equation (2.1) has a fundamental solution of the following type

$$\mathbf{R}(s, z) := \Psi(s)^{-1} R(s, z) e^{\mathbf{u}/z},$$

where $(\Psi_{\mu i}) := (T_\mu, \tilde{\epsilon}_i)$ is the transition matrix from $\{\tilde{\epsilon}_i\}$ to $\{T_\mu\}$; $\mathbf{u}$ is the diagonal matrix $(\mathbf{u}_{ij}) = \delta_{ij} u^i$. The main information of $\mathbf{R}$ is carried by $R(s, z)$, which is a (formal) power series in z. One notable difference between $S(s, z^{-1})$ and $R(s, z)$ is that the former is a (formal) power series in z^{-1} while the latter is a (formal) power series in z. See [5] and Theorem 1 in Chapter 1 of [10].

2.3. **Preliminaries on quantization.** Let $\mathscr{H}_\mathbf{q} := H[z]$. Let $\{T_\mu z^k\}_{k=0}^\infty$ be a basis of $\mathscr{H}_\mathbf{q}$, and $\{\mathbf{q}_k^\mu\}$ the dual coordinates. We define an isomorphism of $\mathscr{H}_\mathbf{q}$ to $\mathscr{H}_t$ as an affine vector space via a *dilaton shift* "$t = \mathbf{q} + z\mathbf{1}$":

$$t_k^\mu = \mathbf{q}_k^\mu + \delta^{\mu 1}\delta_{k1}. \tag{2.4}$$

The cotangent bundle $\mathscr{H} := T^*\mathscr{H}_\mathbf{q}$ is naturally isomorphic to the H-valued Laurent series in z^{-1}, $H[\![z^{-1}]\!]$. It has a natural symplectic structure

$$\Omega = \sum_{k,\mu,\nu} g_{\mu\nu}\, d\mathbf{p}_k^\mu \wedge d\mathbf{q}_k^\nu$$

where $\{\mathbf{p}_k^\mu\}$ are the dual coordinates in the fiber direction of $\mathscr{H}$ in the natural basis $\{T_\mu(-z)^{-k-1}\}_{k=0}^\infty$. In this way,

$$\Omega(f, g) = \mathrm{Res}_{z=0}(f(-z), g(z)).$$

To quantize an infinitesimal symplectic transformation on $(\mathscr{H}, \Omega)$, or its corresponding quadratic hamiltonians, we recall the standard Weyl quantization. An identification $\mathscr{H} = T^* \mathscr{H}_{\mathbf{q}}$ of the symplectic vector space $\mathscr{H}$ (the *phase space*) as a cotangent bundle of $\mathscr{H}_{\mathbf{q}}$ (the *configuration space*) is called a polarization. The "Fock space" will be a certain class of functions $f(\hbar, \mathbf{q})$ on $\mathscr{H}_{\mathbf{q}}$ (containing at least polynomial functions), with additional formal variable $\hbar$ ("Planck's constant"). The classical observables are certain functions of $\mathbf{p}, \mathbf{q}$. The quantization process is to find for the phase space of the "classical mechanical system" on $(\mathscr{H}, \Omega)$ a "quantum system" on the Fock space such that the classical observables, like the hamiltonians $h(\mathbf{q}, \mathbf{p})$ on $\mathscr{H}$, are quantized to become operators $\widehat{h}(\mathbf{q}, \partial/\partial\mathbf{q})$ on the Fock space.

Let $A(z)$ be an $\mathrm{End}(H)$-valued Laurent formal series in z satisfying

$$\Omega(Af, g) + \Omega(f, Ag) = 0,$$

for all $f, g \in \mathscr{H}$. That is, $A(z)$ defines an infinitesimal symplectic transformation. $A(z)$ corresponds to a quadratic "polynomial" hamiltonian [1] $P(A)$ in $\mathbf{p}, \mathbf{q}$

$$P(A)(f) := \frac{1}{2}\Omega(Af, f).$$

Choose a *Darboux coordinate system* $\{\mathbf{q}_k^i, \mathbf{p}_k^i\}$ so that $\Omega = \sum d\mathbf{p}_k^i \wedge d\mathbf{q}_k^i$. The quantization $P \mapsto \widehat{P}$ assigns

$$\begin{aligned} &\widehat{1} = 1, \quad \widehat{\mathbf{p}_k^i} = \sqrt{\hbar}\frac{\partial}{\partial \mathbf{q}_k^i}, \quad \widehat{\mathbf{q}_k^i} = \mathbf{q}_k^i/\sqrt{\hbar}, \\ &\widehat{\mathbf{p}_k^i\mathbf{p}_l^j} = \widehat{\mathbf{p}_k^i}\widehat{\mathbf{p}_l^j} = \hbar\frac{\partial}{\partial \mathbf{q}_k^i}\frac{\partial}{\partial \mathbf{q}_l^j}, \\ &\widehat{\mathbf{p}_k^i\mathbf{q}_l^j} = \mathbf{q}_l^j\frac{\partial}{\partial \mathbf{q}_k^i}, \quad \widehat{\mathbf{q}_k^i\mathbf{q}_l^j} = \mathbf{q}_k^i\mathbf{q}_l^j/\hbar, \end{aligned} \tag{2.5}$$

In summary, the quantization is the process

$$\begin{array}{ccccc} A & \mapsto & P(A) & \mapsto & \widehat{P(A)} \\ \text{inf. sympl. transf.} & \mapsto & \text{quadr. hamilt.} & \mapsto & \text{operator on Fock sp..} \end{array}$$

It can be readily checked that the first map is a Lie algebra isomorphism: The Lie bracket on the left is defined by $[A_1, A_2] = A_1A_2 - A_2A_1$ and the Lie bracket in the middle is defined by Poisson bracket

$$\{P_1(\mathbf{p}, \mathbf{q}), P_2(\mathbf{p}, \mathbf{q})\} = \sum_{k,i} \frac{\partial P_1}{\partial \mathbf{p}_k^i}\frac{\partial P_2}{\partial \mathbf{q}_k^i} - \frac{\partial P_2}{\partial \mathbf{p}_k^i}\frac{\partial P_1}{\partial \mathbf{q}_k^i}.$$

The second map is close to be a Lie algebra homomorphism. Indeed

$$[\widehat{P}_1, \widehat{P}_2] = \widehat{\{P_1, P_2\}} + \mathscr{C}(P_1, P_2),$$

[1] Due to the nature of the infinite dimensional vector spaces involved, the "polynomials" here might have infinite many terms, but the degrees in $\mathbf{p}$ and $\mathbf{q}$ are at most 2.

where the cocycle $\mathscr{C}$, in orthonormal coordinates, vanishes except

$$\mathscr{C}(\mathbf{p}_k^i\mathbf{p}_l^j, \mathbf{q}_k^i\mathbf{q}_l^j) = -\mathscr{C}(\mathbf{q}_k^i\mathbf{q}_l^j, \mathbf{p}_k^i\mathbf{p}_l^j) = 1 + \delta^{ij}\delta_{kl}.$$

Example 2.2. Let $\dim H = 1$ and $A(z)$ be multiplication by z^{-1}. It is easy to see that $A(z)$ is infinitesimally symplectic.

$$\begin{aligned} P(z^{-1}) &= -\frac{\mathbf{q}_0^2}{2} - \sum_{m=0}^{\infty} \mathbf{q}_{m+1}\mathbf{p}_m \\ \widehat{P(z^{-1})} &= -\frac{\mathbf{q}_0^2}{2} - \sum_{m=0}^{\infty} \mathbf{q}_{m+1}\frac{\partial}{\partial \mathbf{q}_m}. \end{aligned} \tag{2.6}$$

Note that one often has to quantize symplectic transformations. Following the common practice in physics, define

$$\widehat{e^{A(z)}} := e^{\widehat{A(z)}}, \tag{2.7}$$

for $A(z)$ an infinitesimal symplectic transformation.

2.4. **Ancestor potentials via quantization.** Let N be the rank of $H = H^*(X)$ and $\mathscr{D}_N(\mathbf{t}) = \prod_{i=1}^N \mathscr{D}_{pt}(t^i)$ be the descendent potential of N points, where

$$\mathscr{D}_{pt}(t^i) \equiv \mathscr{A}_{pt}(t^i) := \exp \sum_{g=0}^{\infty} \hbar^{g-1} F_g^{pt}(t^i)$$

is the total descendent potential of a point and $t^i = \sum_k t_k^i z^k$.

Suppose that $(QH, *)$ is semisimple, then the ancestor potential can be reconstructed from the $\mathscr{D}_N(\mathbf{t})$ via the the quantization formalism.

Since $\{\tilde{\epsilon}_i\}$ defines an *orthonormal basis* for $T_sH \cong H$ (for s a semisimple point), the dual coordinates $(\mathbf{p}_k^i, \mathbf{q}_k^i)$ of the basis $\{\tilde{\epsilon}_i z^k\}_{k\in\mathbb{Z}}$ for $\mathscr{H}$ form a Darboux coordinate system. The coordinate system $\{t_k^i\}$ is related to $\{\mathbf{q}_k^i\}$ by the dilaton shift (2.4). Note that $\partial/\partial \mathbf{q}_k^i = \partial/\partial t_k^i$.

The following beautiful formula was first formulated by Givental [5]. Many special cases have since been solved in [1, 9, 6]. It was completely established by C. Teleman in a recent preprint [15]. In this paper, we will only need Givental's conjecture for smooth *semi-Fano toric varieties* (c.f. [6]).

Theorem 2.3. *For X a smooth variety with semisimple* $(QH(X), *)$,

$$\mathscr{A}_X(\bar{t}, s) = e^{\bar{c}(s)}\widehat{\Psi}^{-1}(s)\widehat{R}_X(s,z)\widehat{e^{\mathbf{u}/z}}(s)\mathscr{D}_N(\mathbf{t}), \tag{2.8}$$

where $\bar{c}(s) = \frac{1}{48}\log\det(\epsilon_i, \epsilon_j)$.

Note that it is not very difficult to check that $\log R_X(s,z)$ defines an infinitesimal symplectic transformation. See e.g. [5, 10]. $\widehat{R}_X(s,z)$ is then defined via (2.7). By Example 2.2, $\widehat{e^{\mathbf{u}/z}}$ is also well-defined. Since the quantization involves only the z variable, $\widehat{\Psi}^{-1}(s)$ really is the transformation from the coordinates with respect to the normalized canonical frame to flat coordinates. No quantization is needed.

Remark 2.4. The operator $\widehat{e^{\mathbf{u}/z}}$ can be removed from the above expression. It is shown in [5] that the string equation implies that $\widehat{e^{\mathbf{u}/z}}\mathscr{D}_N = \mathscr{D}_N$.

3. Analytic Continuations

We discuss the issues of analyticity of the Frobenius manifolds and analytic continuations involved in the study of the flop $f : X \dashrightarrow X'$.

3.1. **Review of the genus zero theory.** Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop with $\mathscr{F}$ being the graph correspondence. This subsection rephrases the analytic continuation of big quantum rings proved in [11] in more algebraic terms.

Let NE_f be the cone of curve classes $\beta \in NE(X)$ with $\mathscr{F}\beta \in NE(X')$, i.e. the classes which are effective on both sides. Let

$$\mathbf{f}(q) = \frac{q}{1-(-1)^{r+1}q}$$

be the rational function coming from the generating function of three points Gromov-Witten invariants attached to the extremal ray $\ell \subset Z \cong \mathbb{P}^r$ with positive degrees. Namely for any $i, j, k \in \mathbb{N}$ with $i + j + k = 2r + 1$,

$$\mathbf{f}(q^\ell) = \sum_{d\geq 1} \langle h^i, h^j, h^k\rangle_{0,3,d\ell}\, q^{d\ell},$$

where h denotes a class in X which restricts to the hyperplane class of Z.

Gromov-Witten invariants take value in the Novikov ring

$$N(X) = \widehat{\mathbb{C}[NE(X)]}$$

(formal series in q^β, $\beta \in NE(X)$), which is the I-adic completion with I being the maximal ideal generated by $NE(X)\backslash\{0\}$. [2]

Define the ring

$$\mathscr{R} = \widehat{\mathbb{C}[NE_f]}[\mathbf{f}(q^\ell)], \tag{3.1}$$

which can be regarded as certain algebraization of $N(X)$ in the q^ℓ variable. Notice that $\mathscr{R}$ is canonically identified with its counterpart $\mathscr{R}' = \widehat{\mathbb{C}[NE'_f]}[\mathbf{f}(q^{\ell'})]$ for X' under $\mathscr{F}$ since $\mathscr{F}NE_f = NE'_f$ and

$$\mathscr{F}\mathbf{f}(q^\ell) = (-1)^r - \mathbf{f}(q^{\ell'}) \tag{3.2}$$

(via $\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r$).

Theorem 3.1. *The genus zero n-point functions with $n \geq 3$ lie in $\mathscr{R}$:*

$$\langle\alpha\rangle^X \in \mathscr{R}$$

for all $\alpha \in H^*(X)^{\oplus n}$. Moreover $\mathscr{F}\langle\alpha\rangle^X = \langle\mathscr{F}\alpha\rangle^{X'}$ in $\mathscr{R}'$.

---

[2] The notation $\hat{}$ in this section always means completion in the I-adic topology and should not be confused with quantization used in the previous section.

Proof. This is the main result of [11] except the statement that $\langle \alpha \rangle^X \in \mathscr{R}$. This in tern will follow from a closer look at the proof of $\mathscr{F}\langle \alpha \rangle^X = \langle \mathscr{F}\alpha \rangle^{X'}$ give there. The argument below assumes familiarity with [11].

The degeneration analysis in §4 of [11] implies that

$$\langle \alpha \rangle^{\bullet X} = \sum_\mu m(\mu) \sum_I \langle \alpha_1 \mid \varepsilon_I, \mu \rangle^{\bullet (Y,E)} \langle \alpha_2 \mid \varepsilon^I, \mu \rangle^{\bullet (\tilde{E},E)},$$

which decomposes *absolute* invariants into *relative* ones on the *blow-up* $Y = \mathrm{Bl}_Z X = \bar{\Gamma}_f \subset X \times X'$ and on the *local model* $\tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathscr{O})$; here $\langle \cdot \rangle^\bullet$ denotes invariants with possibly disconnected domain curves. This formula, which involves *deformation to the normal cone*, will be reviewed in Section 5 where a generalization to all genera is presented.

Under the projections $\phi : Y \to X$ and $\phi' : Y \to X'$, the graph correspondence is given by $\mathscr{F} = \phi'_* \circ \phi^*$. The variable q^{β_1} for $\beta_1 \in NE(Y)$ is identified with $q^{\phi_* \beta_1} \in NE(X)$. If q^{β_1} appears in a summand with *contact type* μ, then $(E.\beta_1) = |\mu| \geq 0$ (the contact order). Also if ℓ'_Y is the ruling on $E \cong \mathbb{P}^r \times \mathbb{P}^r \subset Y$ which projects to $\ell' \subset X'$, then

$$\phi^* \phi_* \beta_1 = \beta_1 + (E.\beta_1)\ell'_Y$$

(since $N_{E/Y} \cong \mathscr{O}(-1,-1)$ and $(E.\phi^*\phi_*\beta_1) = 0$). This implies that

$$\mathscr{F}\phi_*\beta_1 = \phi'_*\beta_1 + |\mu|\ell' \in NE(X').$$

Hence $\beta_1 \in NE_f$ and $\langle \alpha_1 \mid \varepsilon_I, \mu \rangle^{\bullet (Y,E)} \in \widehat{\mathbb{C}[NE_f]}$.

To compare $\mathscr{F}\langle \alpha \rangle^{\bullet X}$ and $\langle \mathscr{F}\alpha \rangle^{\bullet X'}$, by [11], Proposition 4.4 we may assume that $\alpha_1 = \alpha'_1$ and $\alpha'_2 = \mathscr{F}\alpha_2$. Thus the problem is reduced to the local model $\tilde{E}$ which has $NE(\tilde{E}) = \mathbb{Z}_+\ell + \mathbb{Z}_+\gamma$ with γ being the fiber line class of $\tilde{E} \to Z$. Denote $\beta = d\ell + d_2\gamma \in NE(X)$.

The relative invariants $\langle \alpha_2 \mid \varepsilon^I, \mu \rangle^{\bullet (\tilde{E},E)}$ are converted to the absolute descendent invariants of f-special type on $\tilde{E}$ by solving *triangular linear systems* arising from the degeneration formula inductively (c.f. Proposition 5.3 where this is generalized to the case allowing also ancestors in α_2).

Now $H^*(\tilde{E}) = \mathbb{Z}[h, \xi]/\langle h^{r+1}, (\xi - h)^{r+1}\xi \rangle$ is generated by divisors where h is the hyperplane class of Z and ξ is the class of E. By a virtual dimension count, for each $\alpha \in \tau_\bullet H^*(\tilde{E})^{\oplus n}$, $\langle \alpha \rangle^{\tilde{E}}_\beta \neq 0$ for at most one d_2. Then the process in [11] (§5, Theorem 5.6) via the *reconstruction theorem* and induction on $d_2 \geq 0$ shows that there are indeed only two basic relations which together generate all the analytic continuations and lead to the $\mathscr{F}$-invariance theorem.

The first relation is (3.2), which is the origin of analytic continuation: For $d_2 = 0$, the 3-point functions for extremal rays is given by **f**. The constant $(-1)^r$ is responsible for the *topological defect*. Another relation comes from the *quasi-linearity* $\mathscr{F}\langle \tau_k \xi a \rangle^X = \langle \tau_k \xi' \mathscr{F} a \rangle^{X'}$ for one point f-special invariants

([11], Lemma 5.4). This is an identity of *small J functions* in $\mathbb{C}[NE'_f]$:

$$\mathscr{F} J^{\tilde{E}}.\xi a = J^{\tilde{E}'}.\xi' \mathscr{F} a$$

where no analytic continuation is needed.

For $D \in \mathrm{Pic}(\tilde{E})$, the *power operator* δ_D is defined by $\delta_D q^\beta = (D.\beta) q^\beta$. Then under the basis $\{h, \xi\}$ of $H^2(\tilde{E})$ we have dual basis $\{\ell, \gamma\}$ of $H_2(\tilde{E})$ and

$$\delta := \delta_h = q^\ell \frac{d}{dq^\ell}. \tag{3.3}$$

The reconstruction shows that the desired analytic continuations arise from finite $\mathbb{C}[NE_f]$-linear combinations of $\delta^m \mathbf{f}$'s with $m \geq 0$.

It remains to show that $\delta^m \mathbf{f}$ is a polynomial in $\mathbf{f}$. This follows easily from

$$\delta \mathbf{f} = \mathbf{f} + (-1)^{r+1} \mathbf{f}^2$$

and $\delta(\mathbf{f}_1 \mathbf{f}_2) = (\delta \mathbf{f}_1) \mathbf{f}_2 + \mathbf{f}_1 \delta \mathbf{f}_2$ by induction on m. □

3.2. **Integral structure on local models.** For $X = \tilde{E}$, the second half of above proof shows that

$$\langle \alpha \rangle \in \mathbb{C}[NE_f][\mathbf{f}] =: \mathscr{R}_{loc} \tag{3.4}$$

without the need of taking completion, where $NE_f = \mathbb{Z}_+ \gamma + \mathbb{Z}_+ (\gamma + \ell)$.

In fact for a given set of insertions α and genus g, the virtual dimension count shows that the *contact weight* $d_2 := (E.\beta)$ is fixed among all $\beta = d_1 \ell + d_2 \gamma$ in the series $\langle \alpha \rangle_g^X$. Hence for $g = 0$ we must have

$$\langle \alpha \rangle^X = q^{d_2 \gamma} (p_0(\mathbf{f}) + q^\ell p_1(\mathbf{f}) + \cdots + q^{d_2 \ell} p_{d_2}(\mathbf{f}))$$

for certain polynomials $p_i(\mathbf{f}) \in \mathbb{Q}[\mathbf{f}]$.

In particular $\langle \alpha \rangle$ is an analytic function over the *extended Kähler cone* $\omega \in \mathcal{K}_X^{\mathbb{C}} \cup \mathscr{F}^{-1} \mathcal{K}_{X'}^{\mathbb{C}}$ (where $\mathcal{K}_X^{\mathbb{C}} := H_{\mathbb{R}}^{1,1}(X) + i\mathcal{K}_X$ is the *complexified Kähler cone*) via the identification [3]

$$q^\beta = e^{2\pi i (\omega.\beta)}. \tag{3.5}$$

Thus analytic continuation can be taken in the traditional complex analytic sense or as isomorphisms in the ring $\mathscr{R}_{loc} \cong \mathscr{R}'_{loc}$.

3.3. **Analytic structure on the Frobenius manifolds.** The Frobenius manifold corresponding to X is a priori a formal scheme, given by the formal completing $\widehat{H}_X$ of $H^*(X, \mathbb{C})$ at the origin, with values in the Novikov ring. The divisor axiom implies that one may combine the $H^{1,1}(X)$ directions of

[3]In string theory, the identification of *weights* $q^\beta = e^{2\pi i(\omega.\beta)}$ is essential in matching the A model and B model moduli spaces in mirror symmetry (c.f. [3]). It is generally believed that the GW theory converges in the "large radius limit", i.e. when $\mathrm{Im}\, \omega$ is large.

the Frobenius manifold and the Novikov variables into a *formal completion* at the *boundary point* $q = 0$ of

$$\frac{H^{1,1}(X)}{H^{1,1}(X) \cap (2\pi i.H^2(X,\mathbb{Z}))} \subset \frac{H^2(X,\mathbb{C})}{2\pi i H^2(X,\mathbb{Z})} \cong (\mathbb{C}^\times)^{h^2}. \tag{3.6}$$

Indeed, let $s = s' + s_1$ be a point in the Frobenius manifold with $s_1 \in H^2(X,\mathbb{C})$. The divisor axiom says that

$$\langle \alpha \rangle_\beta (s' + s_1)\, q^\beta = \langle \alpha \rangle_\beta (s')\, q^\beta e^{(s_1.\beta)}. \tag{3.7}$$

Compared with (3.5), this suggests an identification of q^β with $e^{(s_1.\beta)}$ which leads to (3.6). This identification can be done at the analytic level when the convergence of big quantum ring is known. In practice when the convergence is known only in some variables q^β's, partial identification can still be made in certain $H^{1,1}$ directions.

Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop and h be a divisor class dual to the extremal ray ℓ, i.e. $(h.\ell) = 1$. Then $H^2(X,\mathbb{C}) = \mathbb{C}h \oplus H^2(X,\mathbb{C})^{\perp_\ell}$. Theorem 3.1 gives an analytic structure on $\widehat{H}_X$ in the h-direction:

Corollary 3.2. *(i) The Frobenius manifold structure on $\widehat{H}_X$ can be extended to*

$$H_X := \widehat{H}_X^{\perp_\ell} \times (\mathbb{P}^1_{q^\ell} \setminus (-1)^{r+1}).$$

(ii) $H_X \cong H_{X'}$.
(iii) If X is the local model, H_X is an analytic manifold.

Proof. Theorem 3.1 says that, as polynomial functions of $\mathbf{f}$, all invariants are defined on $\mathbf{f} \in \mathbb{C}$. Equivalently, as rational functions of q^ℓ, all invariants are defined on $\mathbb{P}^1 \setminus (-1)^{r+1}$. For $s_1 \in H^2(X,\mathbb{C})$, $s_1 = th + h'$ with $(h'.\ell) = 0$. Then the identification $q^\ell = e^{(s_1.\ell)} = e^{(th,\ell)} = e^t$ in (3.1) is used to replace $\mathbb{C}h$. This proves (i). (ii) follows from (i), and (iii) from Section 3.2 □

Corollary 3.2 and results of the previous subsections show that the Frobenius manifold structures on the quantum cohomology of X and X' are isomorphic. The former is a series expansion of analytic functions at $q^\ell = 0$, and the latter at $q^\ell = \infty$. Considered as a one-parameter family

$$H_X \to \mathbb{P}^1_{q^\ell} \setminus (-1)^{r+1},$$

it produces a family of product structure on $\widehat{H}_X^{\perp_\ell} \otimes \widehat{\mathbb{C}[NE_f]}$. At two special points 0 and ∞, the Frobenius structure specializes to the big quantum cohomology modulo extremal rays of X and of X' respectively. The term "analytic continuation" used in this paper can be understood in this way.

4. LOCAL MODELS

We move to the study of local models. The semisimplicity of the Frobenius manifolds and the quantization formalism are used to reduce the invariance of Gromov-Witten theory to the semi-classical (genus zero) case.

4.1. **Semisimplicity of big quantum ring for local models.** Toric varieties admits a nice big torus action and its equivariant cohomology ring is always semisimple, hence as a deformation the equivariant big quantum cohomology ring (the Frobenius manifold) is also semisimple. Givental's quantization formalism works in the equivariant setting, hence one way to prove the higher genus invariance for local models is to extend results in [11] to the equivariant setting. This can in principle be done, but here we take a direct approach which requires no more work.

Lemma 4.1. *For* $X = \mathbb{P}_{\mathbb{P}^r}(\mathscr{O}(-1)^{\oplus(r+1)} \oplus \mathscr{O})$, $QH^*(X)$ *is semisimple.*

Proof. By [3], the proof of Proposition 11.2.17 and [11], Lemma 5.2, the small quantum cohomology ring is given by Batyrev's ring (though X is only semi-Fano). Namely for $q_1 = q^\ell$ and $q_2 = q^\gamma$,

$$QH^*_{small}(X) \cong \mathbb{C}[h,\xi][q_1,q_2]/(h^{r+1} - q_1(\xi - h)^{r+1}, (\xi - h)^{r+1}\xi - q_2).$$

Solving the relations, we get the eigenvalues of the quantum multiplications $h*$ and $\xi*$:

$$h = \eta^j \omega^i q_1^{\frac{1}{r+1}} q_2^{\frac{1}{r+2}} (1 + \omega^i q_1^{\frac{1}{r+1}})^{-\frac{1}{r+2}}, \quad \xi = \eta^j q_2^{\frac{1}{r+2}} (1 + \omega^i q_1^{\frac{1}{r+1}})^{\frac{r+1}{r+2}} \tag{4.1}$$

for $i = 0, 1, \cdots, r$ and $j = 0, 1, \cdots, r+1$. where ω and η are the $(r+1)$-th and the $(r+2)$-th root of unity respectively. As these eigenvalues of $h*$ (resp. $\xi*$) are all different, we see that $h*$ and $\xi*$ are semisimple operators, hence $QH^*_{small}(X)$ is semisimple.

This proves that the formal Frobenius manifold $(QH^*, *)$ is semisimple at the origin $s = 0$. Since semisimplicity is an open condition, the formal Frobenius manifold $QH^*(X)$ is also semisimple. □

Remark 4.2. The Batyrev ring for any smooth projective toric variety, whether or not equal to the small quantum ring, is always semisimple.

4.2. **Invariance of mixed invariants of special type.**

Proposition 4.3. *For the local models, the correspondence* $\mathscr{F}$ *for a simple flop induces, after the analytic continuation, an isomorphism of the ancestor potentials.*

Proof. Since a flop induces K-equivalence, by (2.3) the Euler vector fields of X and X' are identified under $\mathscr{F}$. By Theorem 3.1 and Lemma 4.1, X and X' give rise to isomorphic semisimple *conformal* formal Frobenius manifolds over $\mathscr{R}$ (or rather $\mathscr{R}_{loc}$):

$$QH^*(X) \cong QH^*(X')$$

under $\mathscr{F}$. The first statement then follows from Theorem 2.3, the quantization formula, since all the quantities involved are uniquely determined by the underlying abstract Frobenius structure.

To be more explicit, to compare $\mathscr{F}\mathscr{A}_X$ with $\mathscr{A}_{X'}$ is equivalent to compare $\mathscr{F}(\widehat{\Psi}_X^{-1}\widehat{R}_X)e^{\widehat{\mathbf{u}}/z}$ with $\widehat{\Psi}_{X'}^{-1}\widehat{R}_{X'}e^{\widehat{\mathbf{u}'}/z}$, and $\mathscr{F}\bar{c}$ with $\bar{c}'$. Recall that

$$\epsilon_i := \partial_{u^i}, \quad \tilde{\epsilon}_i := \frac{\epsilon_i}{\sqrt{(\epsilon_i, \epsilon_i)}}.$$

Lemma 4.4. *$\mathscr{F}$ sends canonical coordinates on X to canonical coordinates on X': $\mathscr{F}\epsilon_i = \epsilon'_i$, $\mathscr{F}\tilde{\epsilon}_i = \tilde{\epsilon}'_i$. Moreover, $\bar{c}$, Ψ and $\mathbf{u}$ transform covariantly under $\mathscr{F}$.*

Proof. As $\mathscr{F}$ preserves the big quantum product, $\mathscr{F}$ sends idempotents $\{\epsilon_i\}$ to idempotents $\{\epsilon'_i\}$. Since the canonical coordinates are uniquely defined for *conformal* Frobenius manifolds (up to S_N permutation which is fixed by $\mathscr{F}$), $\mathscr{F}$ takes canonical coordinates on X to those on X'. Furthermore, $\mathscr{F}$ preserves the Poincaré pairing [11], hence that $\mathscr{F}\tilde{\epsilon}_i = \tilde{\epsilon}'_i$.

The $\mathscr{F}$ covariance of $\bar{c}(s) = \frac{1}{48}\log\det(\epsilon_i, \epsilon_j)$, the matrix $\mathbf{u}_{ij} = (\delta_{ij}u^i)$ and the matrix $\Psi_{\mu i} = (T_\mu, \tilde{\epsilon}_i)$ also follow immediately. For example,

$$\mathscr{F}\Psi_{\mu i} = (\mathscr{F}T_\mu, \mathscr{F}\tilde{\epsilon}_i)$$

again by the $\mathscr{F}$ covariance of Poincaré pairing. □

The lemma implies that the Darboux coordinate systems on $\mathscr{H}^X$ and $\mathscr{H}^{X'}$ defined in Section 2.4 via $\tilde{\epsilon}_i$ and $\tilde{\epsilon}'_i$ respectively are compatible under $\mathscr{F}$. By the definition of the quantization process (2.5), which assigns differential operators $\partial/\partial \mathbf{q}^i_k$'s in an universal manner under a Darboux coordinate system, it clearly commutes with $\mathscr{F}$. It is thus enough to prove the invariance of the semi-classical counterparts, or equivalently the "covariance" of the corresponding matrix functions, under $\mathscr{F}$. Note that all the invariance and covariance are up to an analytic continuation.

Therefore, one is left with the proof of the covariance of the R matrix under $\mathscr{F}$, after analytic continuation. Namely $\mathscr{F}R(s) = R'(\mathscr{F}s)$.

This follows from the uniqueness of R for semisimple formal conformal Frobenius manifolds. To be explicit, recall that in the proof of [10], Theorem 1, the formal series $R(s,z) = \sum_{n=0}^\infty R_n(s)z^n$ of the R matrix is recursively constructed by $R_0 = \mathrm{Id}$ and the following relation in canonical coordinates:

$$(R_n)_{ij}(du^i - du^j) = [(\Psi d\Psi^{-1} + d)R_{n-1}]_{ij}. \tag{4.2}$$

Applying $\mathscr{F}$ to it, we get $\mathscr{F}R_n = R'_n$ by induction on n. □

In order to generalize Proposition 4.3 to simple flops of general smooth varieties, which will be carried out in the next section by degeneration analysis, we have to allow descendent insertions at the infinity marked points, i.e. those marked points where the cohomology insertions come from $j_*H^*(E) \subset H^*(X)$.

Theorem 4.5. *For the local models, the correspondence $\mathscr{F}$ for a simple flop induces, after the analytic continuation, an isomorphism of the generating functions of mixed invariants of special type in the stable range.*

Proof. Using Proposition 4.3 and 1.1 and by induction on the power k of descendent, the theorem is reduced to the case of $g = 0$ and with exactly one descendent insertion. It is of the form $\langle \tau_k\alpha, T_\nu\rangle_0(s)$ with $k \geq 0$ and by our assumption $\alpha \in j_*H^*(E)$. This series is a formal sum of subseries

$$\langle \tau_k\alpha, T_\nu, T_{\mu_1}, \cdots T_{\mu_l}\rangle_{0,2+l}$$

with $l \geq 0$ ($n = 2 + l \geq 2$), which are sums over $\beta \in NE(\tilde{E})$. Each such series supports a unique $d_2 \geq 0$ in $\beta = d_1\ell + d_2\gamma$.

If $d_2 = 0$ then the series (resp. its counterpart in $X' = \tilde{E}'$ which supports the same d_2) is trivial for $d_1 \geq 1$ (resp. $d_1' \geq 1$ where $\beta' = d_1'\ell'$) since α is supported in E and the extremal curves are supported in Z (resp. Z').

For the remaining case $\beta = 0$, Since $\overline{M}_{0,n}(X, 0) \cong \overline{M}_{0,n} \times X$, we have

$$\langle \tau_k a_1, a_2, \cdots, a_n \rangle_{0,n,0} = \int_{\overline{M}_{0,n}} \psi_1^k \times \int_X a_1 \cdots a_n. \tag{4.3}$$

It is non-trivial only if $k = \dim \overline{M}_{0,n} = n - 3$, and then

$$\int_X \alpha T_\nu T_{\mu_1} \cdots T_{\mu_l} = \int_{X'} \mathscr{F}\alpha \mathscr{F} T_\nu \mathscr{F} T_{\mu_1} \cdots \mathscr{F} T_{\mu_l}$$

since the flop f restricts to an isomorphism on E.

If $d_2 > 0$, the invariance follows from [11], Theorem 5.6. □

We will generalize the theorem into the form of Theorem 0.2 by removing the local model condition after we discuss the degeneration formula.

Remark 4.6. The proof of [11], Theorem 5.6 is by induction on d_2 and n, which is based on (1) the reconstruction theorem, (2) the case $d_2 = 0$ and (3) the case $n = 1$ (quasi-linearity). However the discussion there on $d_2 = 0$ was not explicitly addressed. In particular, $\beta = 0$ terms was ignored. In that case, the f-special invariants are either zero or reduced to $\mathscr{F}$-invariant constants as above. The arguments there are thus valid with this noted.

Remark 4.7. By section 4.1 and the proof of Proposition 4.3, the canonical coordinates u^i's, idempotents ϵ_i's, hence the transition matrix Ψ and the R matrix all lie in some integral extension $\tilde{\mathscr{R}}_{loc}$ of $\mathscr{R}_{loc}$. It is interesting to know whether all genus g ancestor n-point generating functions take value in $\tilde{\mathscr{R}}_{loc}$ and $\mathscr{F}\langle \tau_{\bar{l}} \alpha \rangle_g^X = \langle \tau_{\bar{l}} \mathscr{F} \alpha \rangle_g^{X'}$ in $\tilde{\mathscr{R}}_{loc}$. This is plausible from Theorem 2.3 since the quantization process requires no further extensions. In fact explicit calculation suggests that $\langle \tau_{\bar{l}} \alpha \rangle_g^X$ might belong to $\mathscr{R}_{loc}$.

5. Degeneration analysis

Let $f : X \dashrightarrow X'$ be a simple $\mathbb{P}^r$ flop with $\mathscr{F}$ being the graph correspondence. To prove Theorem 0.2, we need to show that

$$\mathscr{F}\langle \tau_{k,\bar{l}} \alpha \rangle_g^X = \langle \tau_{k,\bar{l}} \mathscr{F} \alpha \rangle_g^{X'}$$

up to analytic continuation, for all $\tau_{k,\bar{l}}\alpha = (\tau_{k_1,\bar{l}_1}\alpha^1, \ldots, \tau_{k_n,\bar{l}_n}\alpha^n)$ being of f-special type (in the stable range $2g + n \geq 3$).

We follow the strategy employed in [11], §4 to apply the degeneration formula [12] to reduce the problem to local models. The two changes are

(1) to generalize primary invariants to ancestors (and descendents);
(2) to generalize genus zero invariants to arbitrary genus.

(2) is almost immediate while (1) needs more explanations. We will focus on the necessary changes and refer to [11], §4 for complementary details.

5.1. **Mixed relative invariants and the degeneration formula.** Given a pair (Y, E) with $E \hookrightarrow Y$ a smooth divisor, let $\Gamma = (g, n, \beta, \rho, \mu)$ with $\mu = (\mu_i) \in \mathbb{N}^\rho$ a partition of $(\beta.E) = |\mu| := \sum_{i=1}^{\rho} \mu_i$. For $A \in H^*(Y)^{\oplus n}$, $k, l \in \mathbb{Z}_+^n$ and $\varepsilon \in H^*(E)^{\oplus \rho}$, we require that $2g + n + \rho \geq 3$ if $l \neq 0$, and then the *mixed relative invariant* of stable maps with topological type Γ (i.e. with contact order μ_i in E at the i-th contact point) is given by

$$\langle \tau_{k,\bar{l}} A \mid \varepsilon, \mu \rangle_\Gamma^{(Y,E)} = \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} \Big(\prod_{j=1}^{n} \psi_j^{k_j} \bar{\psi}_j^{l_j} e_{Y,j}^* A^j \Big) \cup e_E^* \varepsilon,$$

where $e_{Y,j} : \overline{M}_\Gamma(Y, E) \to Y$, $e_E : \overline{M}_\Gamma(Y, E) \to E^\rho$ are evaluation maps on marked points and contact points respectively.

The descendent ψ_j is defined in the usual manner as $c_1(\mathscr{L}_j)$, with $\mathscr{L}_j \to \overline{M}_\Gamma(Y, E)$ being the cotangent line at the j-th marked point for $j = 1, \ldots, n$.

The ancestors are defined by $\bar{\psi}_j := \sigma_\Gamma^* \psi_j$ for $j = 1, \ldots, n$, where

$$\sigma_\Gamma : \overline{M}_\Gamma(Y, E) \longrightarrow \overline{M}_{g,n+\rho} \tag{5.1}$$

is the stabilization morphism which forgets the maps. Now $\psi_j = c_1(\mathscr{L}_j)$ with $\mathscr{L}_j \to \overline{M}_{g,n+\rho}$. We consider ancestors *only* at the n marked points.

If $\Gamma = \coprod_\pi \Gamma^\pi$, the relative invariants with possibly disconnected domain curves are defined by the product rule:

$$\langle \tau_{k,\bar{l}} A \mid \varepsilon, \mu \rangle_\Gamma^{\bullet(Y,E)} = \prod\nolimits_\pi \langle (\tau_{k,\bar{l}} A)^\pi \mid \varepsilon^\pi, \mu^\pi \rangle_{\Gamma^\pi}^{(Y,E)}.$$

It is set to be zero if some ancestor in the right hand side is *undefined*. This is the case when there is a π with $l_{\Gamma^\pi} \neq 0$ but $g_{\Gamma^\pi} = 0$, $n_{\Gamma^\pi} = \rho_{\Gamma^\pi} = 1$.

Consider a degeneration $W \to \mathbb{A}^1$ of a trivial family with $W_t \cong X$ for $t \neq 0$ and $W_0 = Y_1 \cup Y_2$ a simple normal crossing. All classes $\alpha \in H^*(X, \mathbb{Z})^{\oplus n}$ have global lifting and for each lifting the restriction $\alpha(0)$ on W_0 is defined. Let $j_i : Y_i \hookrightarrow W_0$ be the inclusion maps for $i = 1, 2$. The lifting can be encoded by (α_1, α_2) with $\alpha_i = j_i^* \alpha(0)$.

Let $\{\varepsilon_i\}$ be a basis of $H^*(E)$ with $\{\varepsilon^i\}$ its dual basis. $\{\varepsilon_I\}$ forms a basis of $H^*(E^\rho)$ with dual basis $\{\varepsilon^I\}$ where $|I| = \rho$, $\varepsilon_I = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_\rho}$.

The *degeneration formula* expresses the absolute invariants of X in terms of the relative invariants of the two smooth pairs (Y_1, E) and (Y_2, E):

Theorem 5.1 ([12]). *Assume that $2g + n \geq 3$ if $l \neq 0$, then*

$$\langle \tau_{k,\bar{l}} \alpha \rangle_{g,n,\beta}^X = \sum_I \sum_{\eta \in \Omega_\beta} C_\eta \langle \tau_{k,\bar{l}}^1 \alpha_1 \mid \varepsilon_I, \mu \rangle_{\Gamma_1}^{\bullet(Y_1,E)} \langle \tau_{k,\bar{l}}^2 \alpha_2 \mid \varepsilon^I, \mu \rangle_{\Gamma_2}^{\bullet(Y_2,E)}. \tag{5.2}$$

Here $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ is an *admissible triple* which consists of (possibly disconnected) topological types $\Gamma_i = \coprod_{\pi=1}^{|\Gamma_i|} \Gamma_i^\pi$ with the same partition μ

of contact order under the identification I_ρ of contact points. The marked points in Γ_1 and Γ_2 are labeled by $x_1, x_2, \ldots, x_n$ and the gluing

$$\Gamma_1 +_{I_\rho} \Gamma_2$$

has type (g, n, β) and is connected. In particular, $\rho = 0$ if and only if that one of the Γ_i is empty. The total genus g_i, total number of marked points n_i and the total degree $\beta_i \in NE(Y_i)$ satisfy the splitting relations $g - 1 = g_1 - |\Gamma_1| + g_2 - |\Gamma_2| + \rho$, $n = n_1 + n_2$, and (β_1, β_2) is a lifting of β.

The constants $C_\eta = m(\mu)/|\mathrm{Aut}\,\eta|$, where $m(\mu) = \prod \mu_i$ and $\mathrm{Aut}\,\eta = \{\, \sigma \in S_\rho \mid \eta^\sigma = \eta \,\}$. We denote by Ω the set of equivalence classes of all admissible triples; by Ω_β and Ω_μ the subset with fixed degree β and fixed contact order μ respectively.

Notation: Throughout this section, we use the convention that in (5.2) the valid insertions $\tau_{k_j,\bar{l}_j}(\alpha_i)^j$, $j \in \{1, 2, \ldots, n\}$ used in $\langle \tau^i_{k,\bar{l}} \alpha_i \mid \varepsilon, \mu \rangle^{\bullet(Y_i,E)}_{\Gamma_i}$ correspond to those marked points x_j's appeared in Γ_i.

Proof. Theorem 5.1 follows from the *degeneration formula for virtual moduli cycles* proved by Li in [12], with special attention paid on ancestors:

$$[\overline{M}_\Gamma(W_0)]^{virt} = \sum_{\eta \in \Omega} C_\eta \Phi_{\eta *} \Delta^! \Big([\overline{M}_{\Gamma_1}(Y_1, E)]^{virt} \times [\overline{M}_{\Gamma_2}(Y_2, E)]^{virt} \Big)$$

where $\Delta : E^\rho \times E^\rho \to E^\rho$ is the diagonal. The descendents obey the same formula clearly. For ancestors, we investigate the gluing diagram for η:

$$\begin{array}{ccc} \overline{M}_{\Gamma_1}(Y_1, E) \times_{E^\rho} \overline{M}_{\Gamma_2}(Y_2, E) & \xrightarrow{\Phi_\eta} & \overline{M}_\Gamma(W_0) \\ \Big\downarrow{\scriptstyle \sigma_1 \times \sigma_2} & & \Big\downarrow{\scriptstyle \sigma} \\ \overline{M}_{\Gamma_1^\circ} \times \overline{M}_{\Gamma_2^\circ} & \xrightarrow{G_\eta} & \overline{M}_{g,n} \end{array} \tag{5.3}$$

where $\overline{M}_{\Gamma_i^\circ}$ is the moduli of stable curves with topological type Γ_i forgetting β_i. The vertical maps are stabilization's which define ancestors.

When the commutative diagram (5.3) exits, by the functoriality of the construction of moduli cycles, the ancestors obey the formula on that component as well. This applies to those η even if some of the connected components in $\overline{M}_{\Gamma_i^\circ}$ do not exist — as long as no ancestors are attached to those (unstable) marked points. In fact, $\Phi_\eta^* \sigma^* \psi_j$ is then easily seen to be the ancestor of the component $\overline{M}_{\Gamma_i^\pi}(Y_i, E)$ containing the marked point x_j.

It remains to consider the case that there is a Γ_i^π with a marked point x_j such that $l_j \neq 0$, $g_{\Gamma_i^\pi} = 0$ and $n_{\Gamma_i^\pi} = \rho_{\Gamma_i^\pi} = 1$. We need to show that the corresponding contribution vanishes. But then it is clear that $\Phi_\eta^* \sigma^* \psi_j = 0$ on $\overline{M}_{\Gamma_i^\pi}(Y_i, E)$ since $\Phi_\eta^* \sigma^* \mathscr{L}_j$ is trivial there. $\square$

5.2. **Reduction to relative local models.** The first step is to apply deformation to the normal cone

$$W = \mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{A}^1 \to \mathbb{A}^1.$$

$W_0 = Y_1 \cup Y_2$, $Y_1 = Y = \mathrm{Bl}_Z X \xrightarrow{\phi} X$ and $Y_2 = \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathscr{O}) \xrightarrow{p} Z$. $E = Y \cap \tilde{E}$ is the ϕ exceptional divisor as well as the infinity divisor of $\tilde{E}$.

Similar construction applies to X':

$$W' = \mathrm{Bl}_{Z'\times\{0\}} X' \times \mathbb{A}^1 \to \mathbb{A}^1.$$

$W'_0 = Y'_1 \cup Y'_2$, $Y'_1 = Y' = \mathrm{Bl}_{Z'} X' \xrightarrow{\phi'} X'$, $Y'_2 = \tilde{E}' = \mathbb{P}_{Z'}(N_{Z'/X'} \oplus \mathscr{O}) \xrightarrow{p'} Z'$ and $E' = Y' \cap \tilde{E}'$. By the construction of $\mathbb{P}^r$ flops we have $(Y, E) = (Y', E')$. For simple $\mathbb{P}^r$ flops we even have an abstract isomorphism $\tilde{E} \cong \tilde{E}'$ as both are $\mathbb{P}_{\mathbb{P}^r}(\mathscr{O}(-1)^{\oplus(r+1)} \oplus \mathscr{O})$. However $W_0 \ncong W'_0$ since the gluing of $\tilde{E}$ to Y along $E \cong \mathbb{P}^r \times \mathbb{P}^r$ differs from the one of $\tilde{E}'$, with the $\mathbb{P}^r$ factors switched.

In fact, the flop f induces $f_{loc} : X_{loc} = \tilde{E} \dashrightarrow X'_{loc} = \tilde{E}'$, the projective local model of f, which is again a simple $\mathbb{P}^r$ flop.

Define the generating series for genus g (connected) relative invariants

$$\langle A \mid \varepsilon, \mu \rangle_g^{(\tilde{E},E)} := \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A \mid \varepsilon, \mu \rangle_{g,\beta_2}^{(\tilde{E},E)} q^{\beta_2} \tag{5.4}$$

and the one for all genera with possibly disconnected domain curves

$$\langle A \mid \varepsilon, \mu \rangle^{\bullet(\tilde{E},E)} := \sum_{\Gamma;\, \mu_\Gamma = \mu} \frac{1}{|\mathrm{Aut}\,\Gamma|} \langle A \mid \varepsilon, \mu \rangle_\Gamma^{\bullet(\tilde{E},E)} q^{\beta^\Gamma} \hbar^{g^\Gamma - |\Gamma|}. \tag{5.5}$$

Here for connected invariants of genus g we assign the $\hbar$-weight $\hbar^{g-1}$, while for disconnected ones we simply assign the product weights.

Proposition 5.2 (Reduction to relative local models)**.** *To prove*

$$\mathscr{F} \langle \tau_{k,\bar{l}}\, \alpha \rangle_g^X = \langle \tau_{k,\bar{l}}\, \mathscr{F} \alpha \rangle_g^{X'}$$

for all $\alpha \in H^*(X)^{\oplus n}$ and $k, l \in \mathbb{Z}_+^n$, it suffices to show

$$\mathscr{F} \langle \tau_{k,\bar{l}}\, A \mid \varepsilon, \mu \rangle_{g_0}^{(\tilde{E},E)} = \langle \tau_{k,\bar{l}}\, \mathscr{F} A \mid \varepsilon, \mu \rangle_{g_0}^{(\tilde{E}',E)}$$

for all $A \in H^*(\tilde{E})^{\oplus n}$, $k, l \in \mathbb{Z}_+^n$, $\varepsilon \in H^*(E)^{\oplus \rho}$, contact type μ, and all $g_0 \le g$.

Proof. For the n-point mixed generating function

$$\langle \tau_{k,\bar{l}}\, \alpha \rangle^X = \sum_g \langle \tau_{k,\bar{l}}\, \alpha \rangle_g^X \hbar^{g-1} = \sum_{g;\, \beta \in NE(X)} \langle \tau_{k,\bar{l}}\, \alpha \rangle_{g,\beta}^X q^\beta \hbar^{g-1},$$

the degeneration formula gives

$$\begin{aligned}
&\langle \tau_{k,\bar{l}}\alpha\rangle^X \\
&= \sum_{\beta\in NE(X)} \sum_{\eta\in\Omega_\beta} \sum_I C_\eta \langle \tau^1_{k,\bar{l}}\alpha_1 \mid \varepsilon_I, \mu\rangle^{\bullet(Y_1,E)}_{\Gamma_1} \langle \tau^2_{k,\bar{l}}\alpha_2 \mid \varepsilon^I, \mu\rangle^{\bullet(Y_2,E)}_{\Gamma_2} q^{\phi^*\beta}\, \hbar^{g-1} \\
&= \sum_\mu \sum_I \sum_{\eta\in\Omega_\mu} C_\eta \left(\langle \tau^1_{k,\bar{l}}\alpha_1 \mid \varepsilon_I, \mu\rangle^{\bullet(Y_1,E)}_{\Gamma_1} q^{\beta_1}\, \hbar^{g^{\Gamma_1}-|\Gamma_1|}\right) \\
&\qquad\qquad \times \left(\langle \tau^2_{k,\bar{l}}\alpha_2 \mid \varepsilon^I, \mu\rangle^{\bullet(Y_2,E)}_{\Gamma_2} q^{\beta_2}\, \hbar^{g^{\Gamma_2}-|\Gamma_2|}\right) \hbar^\rho,
\end{aligned}$$

where $(\alpha_1, \alpha_2) \in H^*(Y_1)^{\oplus n} \times H^*(Y_2)^{\oplus n}$ is any cohomology lifting of α and we have used $g - 1 = \sum_i (g^{\Gamma_i} - |\Gamma_i|) + \rho$. Notice that $\beta = \phi_*\beta_1 + p_*\beta_2$ and we identify $q^{\beta_1} = q^{\phi_*\beta_1}$, $q^{\beta_2} = q^{p_*\beta_2}$ throughout our degeneration analysis.

We consider also absolute invariants $\langle \tau_{k,\bar{l}}\alpha\rangle^{\bullet X}$ with product weights in $\hbar$. Then by comparing the order of automorphisms,

$$\langle \tau_{k,\bar{l}}\alpha\rangle^{\bullet X} = \sum_\mu m(\mu) \sum_I \langle \tau^1_{k,\bar{l}}\alpha_1 \mid \varepsilon_I, \mu\rangle^{\bullet(Y_1,E)} \langle \tau^2_{k,\bar{l}}\alpha_2 \mid \varepsilon^I, \mu\rangle^{\bullet(Y_2,E)}\, \hbar^\rho. \tag{5.6}$$

To compare $\mathscr{F}\langle \tau_{k,\bar{l}}\alpha\rangle^{\bullet X}$ and $\langle \tau_{k,\bar{l}}\mathscr{F}\alpha\rangle^{\bullet X'}$, by [11], Proposition 4.4, we may assume that $\alpha_1 = \alpha'_1$ and $\alpha'_2 = \mathscr{F}\alpha_2$. This choice of cohomology lifting identifies the relative invariants of (Y, E) and those of $(Y', E') = (Y, E)$ with the same topological types. It remains to compare

$$\langle \tau^2_{k,\bar{l}}\alpha_2 \mid \varepsilon^I, \mu\rangle^{\bullet(\tilde{E},E)} \quad \text{and} \quad \langle \tau^2_{k,\bar{l}}\alpha_2 \mid \varepsilon^I, \mu\rangle^{\bullet(\tilde{E}',E)}.$$

We further split the sum into connected invariants. Let Γ^π be a connected part with the contact order μ^π induced from μ. Denote $P : \mu = \sum_{\pi\in P} \mu^\pi$ a partition of μ and $P(\mu)$ the set of all such partitions. Then

$$\langle \tau_{k,\bar{l}} A \mid \varepsilon, \mu\rangle^{\bullet(\tilde{E},E)} = \sum_{P\in P(\mu)} \prod_{\pi\in P} \sum_{\Gamma^\pi} \frac{1}{|\mathrm{Aut}\,\mu^\pi|} \langle \tau_{k,\bar{l}} A \mid \varepsilon^\pi, \mu^\pi\rangle^{(\tilde{E},E)}_{\Gamma^\pi} q^{\beta^{\Gamma^\pi}}\, \hbar^{g^{\Gamma^\pi}-1}.$$

In the summation over Γ^π, the only index to be summed over is β^{Γ^π} on $\tilde{E}$ and the genus. This reduces the problem to $\langle (\tau_{k,\bar{l}} A)^\pi \mid \varepsilon^\pi, \mu^\pi\rangle^{(\tilde{E},E)}_g$.

Instead of working with all genera, the proposition follows from the same argument by reduction modulo $\hbar^g$. □

5.3. Further reduction to local absolute invariants.

Proposition 5.3 ([11], Proposition 4.8). *For the local simple flop $\tilde{E} \dashrightarrow \tilde{E}'$, to prove*

$$\mathscr{F}\langle \tau_{\bar{l}} A \mid \varepsilon, \mu\rangle^{(\tilde{E},E)}_g = \langle \tau_{\bar{l}} \mathscr{F} A \mid \varepsilon, \mu\rangle^{(\tilde{E}',E)}_g$$

for all $A \in H^*(\tilde{E})^{\oplus n}$, $l \in \mathbb{Z}^n_+$, and weighted partitions (ε, μ), it suffices to show for mixed invariants of special type

$$\mathscr{F}\langle \tau_{\bar{l}} A, \tau_k \varepsilon\rangle^{\tilde{E}}_{g_0} = \langle \tau_{\bar{l}} \mathscr{F} A, \tau_k \varepsilon\rangle^{\tilde{E}'}_{g_0}$$

for all A, l, ε and $k \in \mathbb{Z}^\rho_+$, and all $g_0 \le g$.

Proof. The proof in [11] works, so we only outline it. We apply deformation to the normal cone for $Z \hookrightarrow \tilde{E}$ to get $W \to \mathbb{A}^1$. Then $W_0 = Y_1 \cup Y_2$ with

$$\pi : Y_1 \cong \mathbb{P}_E(\mathscr{O}_E(-1,-1) \oplus \mathscr{O}) \to E$$

being a $\mathbb{P}^1$ bundle and $Y_2 \cong \tilde{E}$. Let $\bar{\gamma}$ be the π-fiber curve class.

We prove the theorem by induction on $(g, |\mu|, n, \rho)$ with ρ in the reverse ordering. Without loss of generality we assume that $\varepsilon = \varepsilon_I$. The idea, inspired by [14], is to degenerate a suitable absolute invariant of f-special type (with virtual dimension matches) so that the desired relative invariant appears as the main term. The same procedure in [11] leads to

$$\langle \tau_{\bar{l}} A, \tau_{\mu_1 - 1}\varepsilon_{i_1}, \ldots, \tau_{\mu_\rho - 1}\varepsilon_{i_\rho} \rangle_g^{\bullet \tilde{E}} = \sum_{\mu'} m(\mu') \times$$
$$\sum_{I'} \langle \tau_{\mu_1 - 1}\varepsilon_{i_1}, \ldots, \tau_{\mu_\rho - 1}\varepsilon_{i_\rho} \mid \varepsilon^{I'}, \mu' \rangle_0^{\bullet (Y_1, E)} \langle \tau_{\bar{l}} A \mid \varepsilon_{I'}, \mu' \rangle_g^{(\tilde{E}, E)} + R,$$

where R denotes the remaining terms which either have lower genus or have total contact order smaller than $d_2 = |\mu| = |\mu'|$ or have number of insertions fewer than n on the $(\tilde{E}, E)$ side or the invariants on $(\tilde{E}, E)$ are disconnected ones. R is $\mathscr{F}$-invariant by induction.

For the main terms, $\deg e_I - \deg e_{I'} = \rho - \rho'$ by the virtual dimension count. Also the integrals on (Y_1, E) turn out to be fiber integrals ($\beta_1 = d_2 \bar{\gamma}$) and this allows to conclude that $\deg e_I \le \deg e_{I'}$ and then $\rho \le \rho'$. The terms $\langle \tau_{\bar{l}} A \mid \varepsilon_{I'}, \mu' \rangle_g^{(\tilde{E}, E)}$ with $\rho' > \rho$ are handled by induction. The case $\rho' = \rho$ in fact leads to $e_{I'} = e_I$. Thus there is a single term remaining, which is

$$C(\mu) \langle \tau_{\bar{l}} A \mid \varepsilon_I, \mu \rangle^{(\tilde{E}, E)}$$

with $C(\mu) \neq 0$. The proposition then follows by induction. □

Proof of Main Theorems. We only need to prove Theorem 0.2.

By Proposition 5.2, the theorem is reduced to the relative local case. Moreover, the special type assumption implies that for any insertion $\tau_{k_j, \bar{l}_j} \alpha_j$ with nontrivial descendent ($k_j \ge 1$) we may represent α_j by a cycle with support disjoint from Z. Thus we may select the cohomology lifting of α_j to be $(\alpha_j, 0)$. To avoid trivial invariants this insertion only contributes to the (Y_1, E) side in the degeneration formula. Hence the theorem is reduced to the case of relative invariants on local model $\tilde{E} = \mathbb{P}_{\mathbb{P}^r}(\mathscr{O}(-1)^{\oplus(r+1)} \oplus \mathscr{O})$ with at most ancestor insertions.

Now by Propositions 5.3, the theorem is further reduced to the case of mixed invariants of f-special type with *non-trivial* appearance of ε:

$$\langle \tau_{\bar{l}} A, \tau_k \varepsilon \rangle_g^{\tilde{E}}.$$

The 2-point case with $g = 0$ and $d_2 = 0$ is zero (for $d_1 = 0$ by (4.3), and for $d_1 > 0$ by $\varepsilon \in j_* H^*(E)$). If $d_2 > 0$ the $\mathscr{F}$-invariance follows from [11], Theorem 5.6 (c.f. Remark 4.6). All other cases are in the stable range $2g + n \ge 3$ which follow form Theorem 4.5. The proof is complete. □

Remark 5.4. The proof also shows that f-special invariants with *non-trivial descendent* are $\mathscr{F}$-invariant even for $2g+n<3$, i.e. $(g,n)=(0,1)$ or $(0,2)$.

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