Some Algebraic Aspects of Hodge Theory

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Degenerations

In this talk, we will discuss how to study a degeneration. First, let us look at the definition.

Definition

A degeneration is a proper, surjective map

$$f : X \to \triangle,$$

where X is a Kähler manifold and $\triangle$ a unit disk. The map f has maximal rank at each point in $\triangle^* = \triangle \setminus \{0\}$. We call

- $X_t = f^{-1}(t)$ a smooth, or generic fiber for all $t \neq 0$, and
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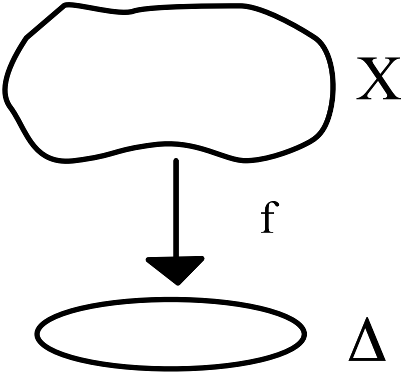
Degenerations (continues)



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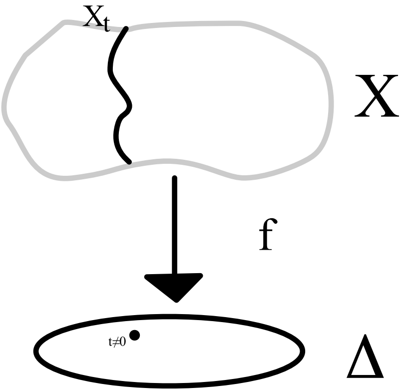

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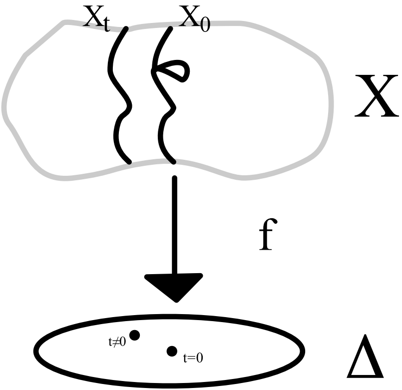

Degenerations (continues)

Fix $f' : X' \to \triangle^*$, where

- $X' = X \setminus \{X_0\}$
- $\triangle^* = \triangle \setminus \{0\}$

One might get different singular fibers from the same f'. For example, we can blow-up or blow-down X_0 and keep X' unchaged. Therefore, we want to study some invariants of the fibers. In Hodge theory, the invariants will be Hodge structures and Mixed Hodge Structures.

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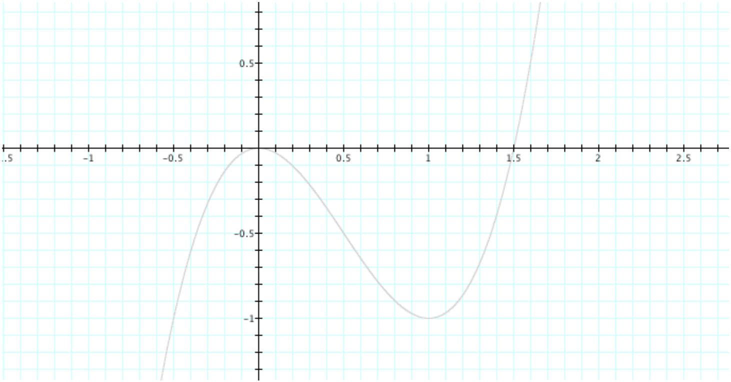
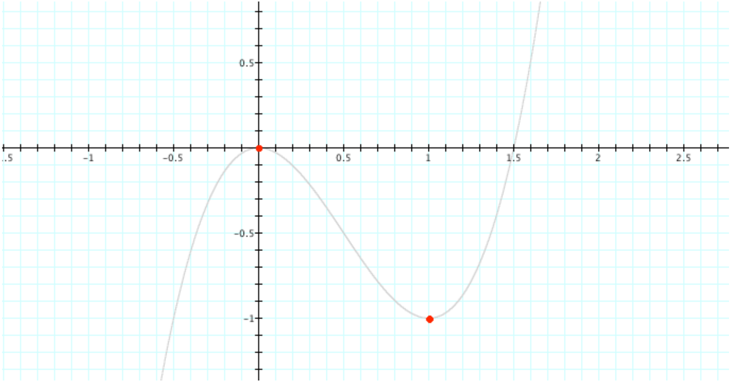
Naive Ideas

We want to know how $f(x) = 2x^3 - 3x^2$ looks like.



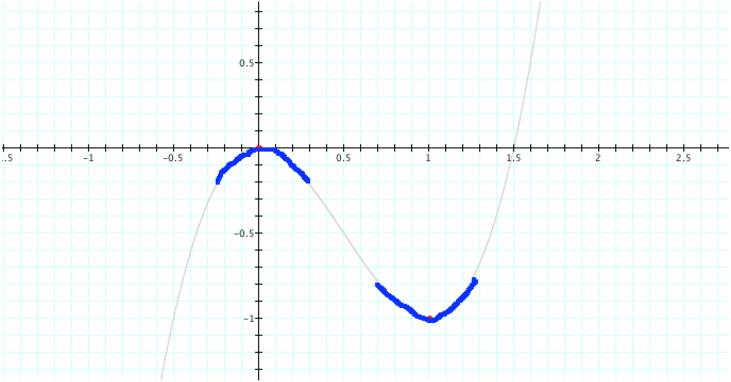
Naive Ideas

$f'(x) = 6x^2 - 6x = 0$. We got two critical points: 0,1



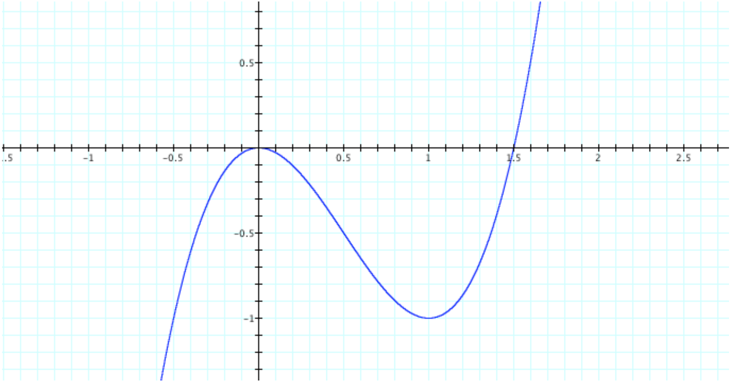
Naive Ideas

See how the function looks like around the critical points.



Naive Ideas

We get the function.



Motivation

- We can get a “good cut” for a reasonable good fibration.

Theorem (Donaldson, 1999)

Any Symplectic manifold admits a Lefschetz pencil.

- Mumford’s GIT.

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Study the Singular Fiber

Let $f : X \to \triangle$ be a degeneration.

$$\begin{array}{ccccc} & & & & \boxed{\text{study}} \\ X_t & \text{———} & \mathrm{H}^n(X_t) & \longleftarrow & \text{Hodge Structure or MHS} \\ & & & & \Downarrow \\ X_0 & \text{———} & \mathrm{H}^0(X_t) & \longleftarrow & \text{put a MHS on it} \end{array}$$

Non-abelian Cohomology

abelian

$$\mathrm{H}^1(X_t) = \mathrm{Hom}(\pi_1(\mathrm{X_t}), \mathbb{C})$$

non-abelian

$$\mathrm{H}^1_G(X_t) = \mathrm{Hom}(\pi_1(\mathrm{X_t}), \mathrm{G})//$$

We shall consider $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C}), \ldots$.

algebraic	analytic
$\mathrm{H}^1(X_t) = \mathrm{Hom}(\pi_1(\mathrm{X_t}), \mathbb{C})$	$0 \to E_0(X_t) \to E_1(X_t) \to \cdots$ $\mathrm{H}^1_{\mathrm{DR}}(X_t) = \frac{\ker d_1}{\mathrm{im}\, d_0}$

de Rham Theorem

The link between the algebraic and analytic worlds.

Theorem (de Rham)

$\mathrm{H}^\bullet_{\mathrm{DR}}(X_t) \simeq \mathrm{H}^\bullet(X_t)$

"proof."

$$\mathrm{H}^1_{\mathrm{DR}}(X_t) = \frac{\ker d_1}{\mathrm{im}\, d_0} \qquad \mathrm{H}^1(X_t) = \mathrm{Hom}(\pi_1(\mathrm{X_t}), \mathbb{C})$$

$$w \in E^1(X_t) \longmapsto \int_\bullet w$$

Hodge Theorem

Theorem (Hodge)

- *X : Kähler manifold*
- $\mathrm{H}^{p,q}(X) = \dfrac{\textit{close } (p,q) \textit{ forms}}{\textit{exact } (p,q) \textit{ forms}}$

Then

$$\mathrm{H}^k(X_t) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X).$$

Hodge Structures

Definition

A Hodge structure V of weight k consists the following data:

- $V_{\mathbb{Z}}$: finitely generated abelian group
- $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$.

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

and

$$\overline{V^{p,q}} = V^{q,p}$$

Hodge Filtration

There is an equivalent way to define a Hodge structure.

Definition

A Hodge filtration on V of weight k is a decreasing filtration

$$V = F^0 V \supseteq \ldots \supseteq F^p V \supseteq F^{p+1} V \supseteq \ldots$$

satisfying

$$V \simeq F^p V \oplus \overline{F^{k-p+1}} V$$

Think of differential (p, q) forms, then

- $F^p V$ = forms have at least p dz's.
- $\overline{F^q V}$ = forms have at least q $d\bar{z}$'s.

Equivalence of the Definitions

Theorem

$$\{V^{p,q}\}_{p+q=k} \Longleftrightarrow \{F^pV\}_{p=0,1,\ldots,k}$$

"proof."

$(\Rightarrow)$

$$F^pV = \bigoplus_{t+s=k, t\geq p} = V^{k,0} \oplus V^{k-1,1} \oplus \cdots \oplus V^{p,k-p}$$

$(\Leftarrow)$

$$V^{p,q} = F^pV \cap \overline{F^qV}$$

Observation

Given a degeneration. Griffiths observers that Hodge filtrations vary holomorphically in families, whereas the (p, q) pieces generally do not. Roughly speaking, we have

$$\left| \underbrace{\boxed{\text{algebraic}}}_{V^{p,q}} \middle| \underbrace{\boxed{\text{analytic}}}_{F^p} \right|$$

Variation of Hodge Structures

Let $f : X \to \triangle$ be a degeneration. Consider $f' : X' \to \triangle^*$. We get the following local system (locally constant sheaf):

$$\mathbb{H}^k := R^k f_* \mathbb{Z} = \{\mathrm{H}^k(X_t, \mathbb{Z})\}_{t \in \triangle^*}$$

Define a holomorphic vector bundle:

$$\mathcal{H}^k := \mathbb{H}^k \otimes_{\mathbb{Z}} \mathcal{O}_{\triangle^*}.$$

Let

$$\mathcal{F}^p := \{F^p \mathrm{H}^k(X_t)\}_{t \in \triangle^*} \subseteq \mathcal{H}^k.$$

Griffiths proved that $\mathcal{F}^p$ is a holomorphic subbundle of $\mathcal{H}^k$ and the natural flat connection of $\mathcal{H}^k$ induces a map

$$\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_{\triangle^*}.$$

Variation of Hodge Structures (continues)

The previous observations give us a prototype of a variation of Hodge structure.

Definition

A variation of Hodge structure of weight k over $\triangle^*$ is a $\mathbb{Z}$-local system $\mathbb{V}$ together with a flag

$$\cdots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} \supseteq \cdots$$

of holomorphic subbundles of the flat bundle $\mathcal{V} := \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_{\triangle^*}$ which satisfies

- $\bigtriangledown : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_{\triangle^*}$
- $\{\mathcal{F}^p\}$ induces Hodge fibration at each fiber.

Mixed Hodge Structures

Definition (Degline)

A mixed Hodge structure (MHS) consists

- a finitely generated abelian group $V_{\mathbb{Z}}$
- an increasing filtration (the weight filtration)

$$\cdots \subseteq W_{m-1}V_{\mathbb{Q}} \subseteq W_m V_{\mathbb{Q}} \subseteq W_{m+1}V_Q \subseteq \cdots$$

  where $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$

- a decreasing filtration (the Hodge filtration)

$$\cdots \supseteq F^{p-1}V_{\mathbb{C}} \supseteq F^p V_{\mathbb{C}} \supseteq F^{p+1}V_C \supseteq \cdots$$

  where $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$

Mixed Hodge Structures (continues)

Definition (MHS, continues)

$(V_{\mathbb{Z}}, \{W_m V_{\mathbb{Q}}\}, \{F^p V_{\mathbb{C}}\})$ satisfies the following properties. For each m, define

$$\mathrm{Gr}_m^W V := \frac{W_m V}{W_{m-1} V}$$

$$F^p \mathrm{Gr}_m^W V := \mathrm{im}\,\{F^p V \cap W_m V \to \mathrm{Gr}_m^W V\}$$

$\{F^p \mathrm{Gr}_m^W V\}$ is a Hodge structure.

Example

Let $\mathrm{H}^{m_1}, \mathrm{H}^{m_2}, \ldots, \mathrm{H}^{m_\ell}$ are Hodge structures of weight $m_1, m_2, \ldots, m_\ell$, respectively. Suppose $m_1 < m_2 < \cdots < m_\ell$. Let

$$\mathrm{H} = \bigoplus_{i=1}^{\ell} \mathrm{H}^{m_i},$$

and

$$F^p\mathrm{H} := \bigoplus_{i=1}^{\ell} F^p\mathrm{H}^{m_i}$$

$$W_m\mathrm{H} := \bigoplus_{k \leq m} \mathrm{H}^k$$

Then $(\mathrm{H}, \{W_m\mathrm{H}\}, \{F^p\mathrm{H}\})$ is a MHS, and such a MHS is said to be split.

Example (proof)

$$\mathrm{Gr}_{\mathrm{m}}^{\mathrm{W}}\mathrm{H} = \frac{W_m\mathrm{H}}{W_{m-1}\mathrm{H}} = \frac{\bigoplus_{k\leq m}\mathrm{H}^k}{\bigoplus_{k\leq m-1}\mathrm{H}^k} = \mathrm{H}^m$$

We need to check if $F^p\mathrm{H}$ induces a Hodge filtration on $\mathrm{Gr}_{\mathrm{m}}^{\mathrm{W}}\mathrm{H}$.

$$F^p\mathrm{H} \cap \mathrm{Gr}_{\mathrm{m}}^{\mathrm{W}} = \bigoplus_{i=1}^{\ell} F^p\mathrm{H}^{m_i} \cap \mathrm{H}^m = F^p\mathrm{H}^m$$

Picard-Lefschetz Transformation

Let us go back to a degeneration $f : X \to \triangle$. $f' : X' \to \triangle^*$ is a locally trivial fibration.

Each element of the fundamental group $\pi_1(\triangle^*) = \mathbb{Z}$ of the base $\triangle^*$, induces an automorphism on both cohomology and homotopy groups. In particular, take the positive generator of $\pi_1(\triangle^*)$, we have the following associate maps:

$$T : \mathrm{H}^1(X_t) \to \mathrm{H}^1(X_t)$$
$$\hat{T} : \mathsf{H}^1_G(X_t) \to \mathsf{H}^1_G(X_t)$$

which called the Picard-Lefschetz Transformation.

Picard-Lefschetz Transformation (continues)

Theorem (Landman)

T is quasi unipotent. i..e. There exist $s, t \in \mathbb{Z}$ such that

$$(T^s - I)^t = 0$$

Define $N := \log T$. It is easy to see that N is nilpotent. ($N^d = I$ for some d.)

The Clemens-Schmid Exact Sequence

To simplify, assume each generic fiber X_t is a curve.

$$1 \to \mathrm{H}^1(X_0) \to \mathrm{H}^1(X_t) \to \mathrm{H}^1(X_t) \to \mathrm{H}_1(X_0) \to 1$$

Theorem (Clemens)

The above sequence is exact.

However, both N and T are NOT Hodge/mixed Hodge morphisms in general!

The Limit Mixed Hodge Structure

Each $H^k(X_t)$ has a Hodge structure. Schmid observed that when t approaches to zero, the Hodge structure tends to be a mixed Hodge structure.

algebraic	analytic
$\mathrm{H}^k(X_t) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}$	$\{F^p\mathrm{H}^k(X_t)\} \xrightarrow{t\to 0}$ Mixed Hodge Structure

The mixed Hodge structure is called the limit mixed Hodge structure.

The Weight Filtration

Theorem (Schmid)

The weight filtration of the limit mixed Hodge structure can be determined by the Picard-Lefschetz transformation.

$$\mathrm{H}^m(X_t) = W_{2m} \supseteq W_{2m-1} \cdots \supseteq W_0 \supseteq 0$$

- $N(W_n) \subseteq W_{n-2}$
- $N^k : \mathrm{Gr}^W_{m+k}\mathrm{H}^m(X_t) \xrightarrow{\sim} \mathrm{Gr}^W_{m-k}\mathrm{H}^m(X_t)$
- $N(W_k) = \mathrm{im}\, N \cap W_{k-2}$

Compute the Weight Filtration

It is easy to calculate the limit weight filtration. First, we get $W_2 m$ for free. Then, from the previous theorem, we have

$$N^m : \frac{W^{2m}}{W_{2m-1}} \simeq \frac{W_0}{W_{-1}} = W_0.$$

We can find W_0 and W_{2m-1} then.

$$\operatorname{im} N^m = W_0$$
$$\ker N^m = W_{2m-1}$$

The Clemens-Schmid Exact Sequence (again)

Theorem (Clemens-Schmid)

All maps on the Clemens-Schmid exact sequence are morphisms of the limit mixed Hodge structure.

example. Let X_t be a complex curve (for example, a Riemann surface of genus g).

$$\mathrm{H}^1(X_t) = W_2 \supseteq W_1 \supseteq W_0 \supseteq 0$$
$$\mathrm{im}\, N = W_0$$
$$\ker N = W_1$$

Example

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{H}^1(X_0) & \longrightarrow & \mathrm{H}^1(X_t) & \longrightarrow & \mathrm{H}^1(X_t) & \longrightarrow & \mathrm{H}_1(X_t) & \longrightarrow & 1 \\
 & & \| & & \| & & \| & & \| & & \\
 & & W_1^0 & \longrightarrow & W_2 & & W_2 & \longrightarrow & W'_0 & & \\
 & & \cup & & \cup & \searrow & \cup & & \cup & & \\
 & & W_0^0 & \longrightarrow & W_1 & & W_1 & \longrightarrow & W'_{-1} & & \\
 & & \cup & & \cup & \searrow & \cup & & \cup & & \\
 & & 0 & & W_0 & & W_0 & \longrightarrow & 0 & & \\
 & & & & \cup & & \cup & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

The Clemens-Schmid Exact Sequence

It is reasonable to consider the analog of the Clemens-Schmid exact sequence.

$$1 \to \mathrm{H}^1_G(X_0) \to \mathrm{H}^1_G(X_t) \to \mathrm{H}^1_G(X_t) \to \mathcal{M} \to 1$$

Theorem (Katzarkov, Xia, Tsai, 2003-2004)

There are counterexamples of non-abelian Clemens-Schmid exact sequence for nilpotent or irreducible representations.

Chen-Hains' Theory

Goal: We want to detect elements of $\pi_1(X, x)$ that are not visible on $\mathrm{H}_1(X)$.

Analytic: Iterated Integrals

Definition

Let $\gamma \in PM$, and $w_1, w_2, \ldots, w_r \in E^1(X)$.

$$\int_\gamma w_1 w_2 \cdots w_r = \int_{0 \leq t_1 \leq \cdots \leq t_r \leq 1} f_1(t_1) \cdots f_r(t_r)\, dt_1 \cdots dt_r$$

where $f_j(t)dt = r^* w_j$.

Homotopy Groups

algebraic: homotopy groups

step 1: Consider

$$\mathbb{C}\pi_1(X,x) = \{ \sum_{g \in \pi_1(X_t)} c_g g \mid c_g \in \mathbb{C} \}.$$

step 2: Consider the augmentation

$$\begin{aligned} \varepsilon : \mathbb{C}\pi_1(X,x) &\to \mathbb{C} \\ \sum c_g g &\mapsto \sum c_g \end{aligned}$$

Let $J = \ker \varepsilon$ and consider $\mathbb{C}\pi_1(X,x)/J^m$.

step 3: Take the completion:

$$\mathbb{C}\widetilde{\pi_1(X_t,x)} := \varprojlim \mathbb{C}\pi_1(X,x)/J^m.$$

The End.

Thank You!