The random cluster model
and
a new integration identity

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The function $I(a,b,c)$

$$\int_0^{2\pi} dx \int_0^{2\pi} dy \ln[1 - a\cos x - b\cos y - (1-a-b)\cos(x+y)]$$

(i) For $a = b = 2$ and $c = 0$,

$$I(2,2,0) = 4/\pi(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} - \cdots),$$

which gives the spanning-tree per-site entropy on rectangular lattices.
(ii) For $a = b = c = 2$,

$$I(2,2,2) = 6/\pi(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} - \cdots) + \frac{1}{2}\ln 3,$$

which gives the spanning-tree per-site entropy on triagular lattices.

(see Kasteleyn (1991), Temperley - Fisher (1961), F.Y.Wu (1977) and Glassr-Wu 2004, etc.). In general, $I(a, b, c) =?$

Consider a q-state Potts model on square lattice of N sites with interaction $K = \epsilon/kT$. Its partition function is

$$Z_N(K) = \sum \prod_{<i,j>} \exp[K\delta(\sigma_i, \sigma_j)],$$

where

$\sum$ denote the sum is over all q^N spin states

$\sigma_i = \{1, 2..., q\}, i = 1, 2, ..., N$,

$< i, j >$ is nearest-neighbor connected i and j

and δ is the Kronecker delta function.

If the constant $\epsilon > 0 (\epsilon < 0)$, we call it ferromagnetic (antiferromagnetic)case.

We can rewrite $Z_N(K)$ as following

$$
\begin{aligned}
Z_N(K) &= \sum \prod_{<i,j>} \exp[K\delta(\sigma_i, \sigma_j)] \\
&= \sum \prod_{<i,j>} \left\{ 1 + (e^K - 1)\delta(\sigma_i, \sigma_j) \right\}
\end{aligned}
$$

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&= \sum_G q^{c(G)} (e^K - 1)^l (G),
\end{aligned}
$$

where the summation is over all edge sets G of the lattice, $c(G)$ is the number of connected clusters in G including isolated points, and $l(G)$ is the number of lines in G.

Similarly, consider a q-state Potts model on triangle lattice of N sites with interactions K_1, K_2, K_3 along the three principle lattice. We have

$$
\begin{aligned}
& Z_N(K_1, K_2, K_3) \\
= & \sum \prod_{<i,j>} \exp[K_\alpha \delta(\sigma_i, \sigma_j)] \\
= & \sum_G q^{c(G)} (e^{K_1} - 1)^{l_1(G)} (e^{K_2} - 1)^{l_2(G)} (e^{K_3} - 1)^{l_3(G)}.
\end{aligned}
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We are interested in the per-site free energy of the random cluster model

$$f^{RC} = \lim_{N \to \infty} \frac{\log Z_N^{RC}(k_1, k_2, k_3)}{N}$$

Since

$$Z_N(K_1, K_2, K_3) = \sum_G q^{c(G)}(e^{K_1} - 1)^{l_1(G)}(e^{K_2} - 1)^{l_2(G)} \times (e^{K_3} - 1)^{l_3(G)}.$$

For each graph G, by Euler's relation

$$s = c(G) - N + l_1(G) + l_2(G) + l_3(G).$$

where s is the number of circuits in G. There is polygon a p such that

$$p = c(G) + s$$

Then

$$
\begin{aligned}
Z_N(K_1, K_2, K_3) &= \sum_G q^{\frac{p+N-l_1(G)-l_2(G)-l_3(G)}{2}} (e^{k_1}-1)^{l_1(G)} \\
&\quad \times (e^{k_2}-1)^{l_2(G)} (e^{k_3}-1)^{l_3(G)} \\
&= q^{\frac{N}{2}} \sum_G q^{\frac{p}{2}} (\frac{e^{k_1}-1}{q^{\frac{1}{2}}})^{l_1(G)} (\frac{e^{k_2}-1}{q^{\frac{1}{2}}})^{l_2(G)} \\
&\quad \times (\frac{e^{k_3}-1}{q^{\frac{1}{2}}})^{l_3(G)}
\end{aligned}
$$

Let

$$x_j = \frac{e^{k_j}-1}{q^{\frac{1}{2}}}, \text{ for } j = 1, 2, 3.$$

The triangular random cluster model is known (Baxter, Temperley and

Ashley, Proc. Roy. Soc. London A, 1978), to be critical at

$$\sqrt{q}x_1x_2x_3 + x_1x_2 + x_2x_3 + x_1x_3 = 1.$$

Define λ snd α_j, $j = 1, 2, 3$ by

$$\begin{aligned} \cosh\lambda &= \sqrt{q}/2. \\ x_j &= \frac{\sinh(\lambda - \alpha_j)}{\sinh\alpha_j}. \end{aligned}$$

Note that the parameters $\lambda, \alpha_j, j = 1, 2, 3$ are real if $q > 4$ and pure imaginary in the interval $(0, \pi i)$ if $q < 4$.
For convenience, if $q < 4$, define ϕ and v_j by

$$\phi = -i\lambda, \qquad v_j = -i\alpha_j, \quad j = 1, 2, 3.$$

Then

$$e^{k_j} = 1 + \frac{1}{2}[\sqrt{q(4-q)}\cot v_j - q], \qquad \text{for } q < 4.$$

and the critical point can be written as

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 2\lambda \qquad \text{for } q > 0 \\ v_1 + v_2 + v_3 &= 2\phi \qquad \text{for } q < 4. \end{aligned}$$

The per-site free energy at critical point,(Baxter, Temperley and Ashley, Proc. Roy. Soc. London A, 1978),

$$f^{RC}_{critical} = \frac{1}{2}\log q + \psi(\lambda, \alpha_1) + \psi(\lambda, \alpha_2) + \psi(\lambda, \alpha_3)$$

where

$$\psi(\lambda, \alpha) = \lambda - \alpha + \sum_{n=1}^{\infty} \frac{e^{-n\lambda}\sinh 2n(\lambda - \alpha)}{n\cosh n\lambda} \qquad \text{for } q > 4,$$

and

$$\psi(\lambda, \alpha) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sinh(\pi - \phi)x \sinh(\phi - v)x}{x \sinh \pi x \cosh \phi x} dx$$

for $q < 4$.

We first rewrite ψ as following

$$\psi(\phi, v) = \lim_{r \to \infty} \frac{1}{2} \oint_r \frac{\sinh(\pi - \phi)z \sinh(\phi - v)z}{z \sinh \pi z \cosh \phi z} dz.$$

The contour integral can be carried out by using residue formula. Since

$$\begin{aligned}
\sinh(\pi z_1) = 0 \qquad & \text{if } z_1 = ni, n = 1, 2, ...,\\
\cosh(\phi z_2) = 0 \qquad & \text{if } z_2 = \frac{\pi}{2\phi}(2m + 1), m = 0, 1, 2, ...
\end{aligned}$$

Case 1 if $\frac{\pi}{2\phi}$ is either irrational or $\frac{\pi}{2\phi} = M/N, N = 2, 4, 6, ...$(No double poles)
We have

$$\psi(\phi, v) = \sum_{n=1}^{\infty} \frac{1}{n} \tan(n\phi) \sin 2n(\phi - v)$$
$$+ \sum_{m=0}^{\infty} \frac{2}{2m+1} \cot\left[\Big(m + \frac{1}{2}\Big)\frac{\pi^2}{\phi}\right] \sin\left[(2m+1)\frac{v\pi}{\phi}\right].$$

Since

$$\tan x \sin 2(x-y) = \cos 2y - \cos 2(x-y) + \tan x \sin 2y,$$

we have

$$\psi(\phi, v) \quad = \quad \log[\frac{\sin(\phi - v)}{\sin v}] + \sum_{n=1}^{\infty} \frac{1}{n} \tan(n\phi) \sin(2nv)$$
$$+ \sum_{n=0}^{\infty} \frac{2}{2n+1} \cot[(n + \frac{1}{2})\frac{\pi^2}{\phi}] \sin[(2n+1)\frac{v\pi}{\phi}].$$

after making use of the summation identity

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln\left[2\sin\left(\frac{x}{2}\right)\right], \quad 0 < x < 2\pi.$$

Case 2 **if** $\pi/2\phi = M/N, N = 1, 3, 5, ...$

$$\psi((N\pi)/2M, v) = R_1 + R_2 + R_3,$$

where
R_1 is the sum of residues from simple pole in z_1,
R_2 is the sum of residues from double poles,
R_3 is the sum of residues from simple pole in z_2.

To compute R_1, we note that the forming of double poles excludes points $Mi,\ 3Mi,\ 5Mi, \cdots$ which divide the remaining $z_1 = ni$ into sections $n = \{1, M-1\}, \{M+1, 3M-1\}, \{3M+1, 5M-1\}, \cdots$. Then we can write

$$R_1 = R_{11} + R_{12}$$

where R_{11} is the sum of the first $M-1$ residues and R_{12} is the sum of the rest. Then

$$R_{11} = \sum_{k=1}^{M-1} \tan(k\phi)\left[\frac{\sin[2k(\phi - v)]}{k}\right], \quad \phi = \frac{N\pi}{2M}$$

$$R_{12} = \sum_{k=-(M-1)}^{M-1} \sum_{n=1}^{\infty} \tan[(2nM+k)\phi]\left[\frac{\sin[2(2nM+k)(\phi - v)]}{2nM+k}\right].$$

Expanding the sine function, we can rewrite R_{11} as

$$R_{11} = \sum_{k=1}^{M-1} \tan(k\phi)\left[\sin(2k\phi)\int_{2v}^{\pi/2k} \sin kx dx - \cos(2k\phi)\int_{0}^{2v} \cos kx dx\right].$$

To evaluate R_{12} we use $4M\phi = 2N\pi$, obtain

$$\begin{aligned} R_{12} &= \sum_{k=-(M-1)}^{M-1} \tan(k\phi) \sum_{n=1}^{\infty} \Bigg[\sin(2k\phi) \int_{2v}^{\pi/2|k|} \sin[(2nM+k)x]dx \\ &\qquad - \cos(2k\phi) \int_{0}^{2v} \cos[(2nM+k)x]dx \Bigg], \quad \phi = \frac{N\pi}{2M}. \end{aligned}$$

Thus we have

$$\begin{aligned} R_{12} &= 2\sum_{k=1}^{M-1} \tan(k\phi) \sum_{n=1}^{\infty} \Bigg[\sin(2k\phi) \int_{2v}^{\pi/2k} \sin(2nMx)\cos(kx)dx \\ &\qquad + \cos(2k\phi) \int_{0}^{2v} \sin(2nMx)\sin(kx)dx \Bigg] \end{aligned}$$

Write

$$\begin{aligned}\sum_{n=1}^{\infty}\sin(2nMx) &= \lim_{\mathcal{N}\to\infty}\sum_{n=1}^{\mathcal{N}}\sin(2nMx)\\ &= \lim_{\mathcal{N}\to\infty}\frac{\cos(Mx)-\cos[(2\mathcal{N}+1)Mx]}{2\sin Mx},\end{aligned}$$

thus, one obtains

$$\begin{aligned}R_{12} &= \sum_{k=1}^{M-1}\tan(k\phi)\Bigg[\sin(2k\phi)\int_{2v}^{\pi/2k}\cos(kx)\cot(Mx)dx\\ &\quad+\cos(2k\phi)\int_{0}^{2v}\sin(kx)\cot(Mx)dx\Bigg]\end{aligned}$$

Finally, we obtain

$$R_1 = \sum_{k=1}^{M-1} \tan(k\phi)\Big[\sin(2k\phi)\int_{2v}^{\pi/2k} \frac{\cos(M-k)x}{\sin(Mx)}dx - \cos(2k\phi)\int_0^{2v} \frac{\sin(M-k)x}{\sin(Mx)}dx\Big],$$

Now we compute R_2. Residues from double poles at $z_2 = (2m+1)Mi$ are

$$R_2 = \pi i \sum_{m=0}^{\infty} \lim_{z\to z_2} \frac{d}{dz}\left[(z-z_2)^2\left(\frac{\sinh(\pi-\phi)z\ \sinh 2(\phi-v)z}{z\sinh \pi z\ \cosh\phi z}\right)\right].$$

where $z_2 = (2m+1)Mi, \phi = N\pi/2M, N =$ odd. Define number u and integer p by

$$Mv = p\pi/2 + u, \quad \text{where} \quad 0 < u < \pi/2,\ p = 0, 1, 2, \cdots,\ p < N,$$

then

$$R_2 = \frac{2(-1)^p}{MN\pi} \sum_{m=0}^{\infty} \frac{\sin[2(2m+1)u]}{(2m+1)^2} + \frac{4(-1)^p(\phi - v)}{N\pi} \sum_{m=0}^{\infty} \frac{\cos[2(2m+1)u]}{2m+1}$$

where $Mv = p\pi/2 + u$ By the following *Identity*

Identity 1

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\sin 2(2k+1)x}{(2k+1)^2} &= \mathrm{Ti}_2(\tan x)+x\ln\cot x,\\ \sum_{k=0}^{\infty}\frac{\cos 2(2k+1)x}{2k+1} &= \frac{1}{2}\ln\cot x\ , \quad 0<x<\frac{\pi}{2}\end{aligned}$$

with $0<x<\frac{\pi}{2}$, where

$$\begin{aligned}Ti_2(a) &= \int_0^a \frac{\tan^{-1} t}{t}dt\\ &= a-\frac{a^3}{3}+\frac{a^5}{5}-\frac{a^7}{7}+\cdots,\end{aligned}$$

we get

$$R_2 = \frac{2(-1)^p}{MN\pi}\mathrm{Ti}_2(\tan u) + \frac{(-1)^p(N-p)}{MN}\ln\cot u.$$

To compute R_3, the sum of residues of simple poles in z_2, we need to exclude from $z_2 = [(2m+1)M/N]i$ the double poles $Mi,\ 3Mi,\ 5Mi, \cdots$. As a result, the remaining simple poles in z_2 are divided into sections $m = \{0, (N-3)/2\}, \{(N+1)/2, (3N-3)/2\}, \{(3N+1)/2, (5N-3)/2\}, \cdots$. The situation is similar to that of R_1, and R_3 can be similarly computed.

$$R_3 \;=\; \sum_{k=1}^{(N-1)/2} \frac{-2M}{N}\cot\left(\frac{2kM\pi}{N}\right)\int_0^{2v}\frac{\sin(\frac{2xMk}{N})}{\sin(Mx)}dx,$$

where $N = 1, 3, \cdots$. Note that $R_3 = 0$ when $N = 1$.

For $q = 2$, clearly, $\phi = \pi/4, M = 2, N = 1$ since $\cos\phi = \sqrt{q}/2$ and $\pi/2\phi = M/N$ (Case 2). First, we obtain

$$\psi\left(\frac{\pi}{4}, v\right) = \frac{1}{2}\ln[(\cot v)(\cot 2v)] + \frac{1}{\pi}\mathrm{Ti}_2(\tan 2v).$$

Therefore, the critical free energy is

$$f^{Potts}_{critical} = \frac{1}{2}\ln 2 + \sum_{\alpha=1}^{3}\left[\frac{1}{2}\ln[(\cot v_\alpha)(\cot 2v_\alpha)] + \frac{1}{\pi}\mathrm{Ti}_2(\tan 2v_\alpha)\right].$$

with critical condition $v_1 + v_2 + v_3 = 2\phi$.

On the other hand, the $q = 2$ Potts model is completely equivalent to an Ising model.

Consider an Ising model on the same triangular lattice of N site with anisotropic interactions $K_\alpha/2, \alpha = 1, 2, 3$. We have the equivalence

$$
\begin{aligned}
Z_N^{Ising} &= \sum_{\sigma=\pm 1} \prod_{<i,j>} e^{(K_\alpha/2)\sigma_i\sigma_j} \\
&= e^{-N(K_1+K_2+K_3)/2} Z_N^{Potts}\Big|_{q=2}
\end{aligned}
$$

after making use of the identity $\sigma_i\sigma_j = 2\delta(\sigma_i, \sigma_j) - 1$. It follows that their critical free energies are related by

$$
\begin{aligned}
f^{Ising} &= f^{Potts}\Big|_{q=2} - \frac{1}{2}(K_1 + K_2 + K_3) \\
&= f^{Potts}\Big|_{q=2} - \frac{1}{2}\ln\left[(\cot v_1)(\cot v_2)(\cot v_3)\right] \\
&= \frac{1}{2}\ln 2 + \sum_{\alpha=1}^{3}\left[\frac{1}{2}\ln(\cot 2v_\alpha) + \frac{1}{\pi}\mathrm{Ti}_2(\tan 2v_\alpha)\right].
\end{aligned}
$$

Since $e^{k_j} = 1 + 1/2[\sqrt{q(4-q)}\cot v_j - q]$, for $q = 2$,

$$
e^{K_\alpha} = \cot v_\alpha \ , \quad \text{or} \quad \sinh K_\alpha = \cot 2v_\alpha.
$$

It can be verified that the critical point $(v_1 + v_2 + v_3 = \pi/2)$ satisfies

$$\begin{aligned} &\cosh K_1 \cosh K_2 \cosh K_3 + \sinh K_1 \sinh K_2 \sinh K_3 \\ &\quad = \sinh K_1 + \sinh K_2 + \sinh K_3, \end{aligned}$$

In fact, the Ising free energy is known at all temperatures [Houtappel, Physica $16, 425 - 455(150)$] to be

$$\begin{aligned} f^{Ising} \;\; = \;\; & \ln 2 + \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln \Big[\cosh K_1 \cosh K_2 \cosh K_3 \\ & + \sinh K_1 \sinh K_2 \sinh K_3 - \sinh K_1 \cos\theta \\ & - \sinh K_2 \cos\phi - \sinh K_3 \cos(\theta + \phi) \Big] d\theta d\phi. \end{aligned}$$

so we can rewrite the integral as

$$f_{critical}^{Ising} = \ln 2 + \frac{1}{2} I(a, b, c)$$

with $a = \cot 2v_1, b = \cot 2v_2, c = \cot 2v_3$ and

$$\begin{aligned} I(a, b, c) &= \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln \Big[a + b + c \\ &\qquad -a \cos\theta - b\cos\phi - c\cos(\theta + \phi) \Big] \end{aligned}$$

with

$$ab + bc + ca = 1.$$

(The critical condition $v_1 + v_2 + v_3 = \pi/2$)

So

$$I(a,b,c) = -\ln 2 + \ln(abc) + \frac{2}{\pi}\Big[\mathrm{Ti}_2(a^{-1}) + \mathrm{Ti}_2(b^{-1}) + \mathrm{Ti}_2(c^{-1})\Big],$$

with

$$ab + bc + ca = 1.$$

$\left(\ a = \cot 2v_1, b = \cot 2v_2, c = \cot 2v_3\ \right)$

For the general integral $I(A,B,C)$ we can always define variables $a = AS, b = BS, c = CS$ with $S = 1/\sqrt{AB + BC + CA}$, thus, one has

$$\begin{aligned} I(A,B,C) &= -\ln S + I(a,b,c) \\ &= -\ln(2S) + \ln(abc) + \frac{2}{\pi}\Big[\mathrm{Ti}_2(a^{-1}) + \mathrm{Ti}_2(b^{-1}) + \mathrm{Ti}_2(c^{-1})\Big] \\ &= -\ln(2S) + \frac{2}{\pi}\Big[\mathrm{Ti}_2(AS) + \mathrm{Ti}_2(BS) + \mathrm{Ti}_2(CS)\Big], \end{aligned}$$

where use has been made of the identity

$$\mathrm{Ti}_2(y^{-1}) = \mathrm{Ti}_2(y) - \frac{\pi}{2}\ln y, \quad y > 0.$$

Special cases

For $q = 1$, clearly, $\phi = \pi/3, M = 3, N = 2$ since $\cos\phi = \sqrt{q}/2$ and $\pi/2\phi = M/N$ (Case 1). Then $\cot\left[(n+1/2)\pi^2/\phi\right] = 0$. So we use the following $identity$

Identity 2

$$\sum_{n=1}^{\infty} \frac{1}{n} \tan(n\pi/3) \sin 2nv = \ln \frac{\sqrt{3}\cot v + 1}{\sqrt{3}\cot v - 1}, \quad 0 < v < \pi/6.$$

We obtain

$$\psi\left(\frac{\pi}{3}, v_j\right) = \ln \frac{\sqrt{3}\cot v_j + 1}{2} = K_j$$

since $e^{K_j} = 1 + 1/2[\sqrt{q(4-q)} cot v_j - q]$. It follows that the free energy is

$$f^{RC}_{critical} = K_1 + K_2 + K_3 \ , \quad q = 1.$$

For $q = 3$ we have $\phi = \pi/6, M = 3, N = 1$ (Case 2). It is straightforward to obtain

$$
\begin{aligned}
f^{RC}_{critical}(3) \;=\;& \frac{1}{4}\ln\left(\frac{9}{8}\right) + \frac{3}{2}\ln\left(\frac{2+\sqrt{3}}{2}\right) \\
&+ \sum_{\alpha=1}^{3}\left[\frac{1}{6}\ln\left(\frac{\sqrt{3}\cot v_\alpha - 1}{\sqrt{3}\tan v_\alpha + 1}\right)\right. \\
&\left. + \frac{1}{2}\ln\left(1 + \sqrt{3}\cot 2v_\alpha\right) + \frac{2}{3\pi}\mathrm{Ti}_2(\cot 3v_\alpha)\right].
\end{aligned}
$$

for $v_1 + v_2 + v_3 = \pi/3$.