Solutions of Homework #4

1. **Claim**1 : $\lambda(E+s)=\lambda(E)$.

$$\begin{aligned}\lambda^*(E+s) &= inf\{\Sigma\lambda^*(I_n)|\bigcup I_n \supseteq E+s, I_n : open\}\\ &= inf\{\Sigma\lambda^*(I_n-s)|\bigcup(I_n-s) \supseteq E, I_n : open\}\\ &= inf\{\Sigma\lambda^*(I_n)|\bigcup(I_n-s) \supseteq E, I_n : open\}\\ &= inf\{\Sigma\lambda^*(I_n)|\bigcup I_n \supseteq E, I_n : open\} = \lambda^*(E)\end{aligned}$$

Therefore, $\lambda(E+s)=\lambda(E)$.

Claim2 : $E\in\mathcal{L}\Rightarrow E+s\in\mathcal{L}$.
Let $A\subseteq\mathbb{R}$ and $B=A-s$.

$$\begin{aligned}&\lambda^*(A\cap(E+s))+\lambda^*(A\setminus(E+s))\\ &\quad=\lambda^*((B+s)\cap(E+s))+\lambda^*((B+s)\setminus(E+s))\\ &\quad=\lambda^*((B\cap E)+s)+\lambda^*((B\setminus E)+s)\\ &\quad=\lambda^*(B\cap E)+\lambda^*(B\setminus E)=\lambda^*(B)=\lambda^*(A).\end{aligned}$$

Therefore, $E+s\in\mathcal{L}$.

Claim3 : $\lambda(rE)=|r|\lambda(E)$, $r\in\mathbb{R}$.

$$\begin{aligned}\lambda^*(rE) &= inf\{\Sigma\lambda^*(I_n)|\bigcup I_n \supseteq rE, I_n : open\}\\ &= inf\{\Sigma\lambda^*(rI_n)|\bigcup I_n \supseteq rE, I_n : open\}\\ &= |r|inf\{\Sigma\lambda^*(I_n)|\bigcup I_n \supseteq E, I_n : open\} = |r|\lambda(E).\end{aligned}$$

Therefore, $\lambda(rE)=|r|\lambda(E)$.

Claim4 : $E\in\mathcal{L}\Rightarrow rE\in\mathcal{L}$.
If $r=0$, then $rE=\{0\}\in\mathcal{L}$.
For $r\neq 0$, let $A\subseteq\mathbb{R}$ and $B=\frac{1}{r}A$.

$$\begin{aligned}&\lambda^*(A\cap(rE))+\lambda^*(A\setminus(rE))\\ &\quad=\lambda^*((rB)\cap(rE))+\lambda^*((rB)\setminus(rE))\\ &\quad=\lambda^*(r(B\cap E))+\lambda(r(B\setminus E))\\ &\quad=|r|\lambda^*(B\cap E)+|r|\lambda(B\setminus E)\\ &\quad=|r|\lambda^*(B)=\lambda^*(A)\end{aligned}$$

Therefore $rE\in\mathcal{L}$.

2. Let E be Lebesgue measurable with $\lambda(E)<\infty$. Then $\lambda(E)=\lambda^*(E)$. By the definition of λ^*, $\forall\varepsilon>0$, $\exists$ open intervals $\{A_n\}$ such that $\bigcup_{n=1}^{\infty}A_n\supset E$ and

$$\lambda(\bigcup_{n=1}^{\infty}A_n)\leq\sum_{n=1}^{\infty}\lambda(A_n)<\lambda^*(E)+\frac{\varepsilon}{2}=\lambda(E)+\frac{\varepsilon}{2} \tag{1}$$

Since $\lambda(E) < \infty$, $\sum_{n=1}^{\infty} \lambda(A_n)$ converges.
Therefore with respect to this ε, $\exists n_0 \in \mathbb{N}$ s.t.

$$\sum_{n=n_0+1}^{\infty} \lambda(A_n) < \frac{\varepsilon}{2}. \tag{2}$$

Let $G = \bigcup_{n=1}^{n_0} A_n$, then because $A_n's$ are open intervals , we can write $G = \bigcup_{k=1}^{K_0} I_k$, where I_k is a finite sequence of mutually disjoit open intervals. Thus, we have

$$G \triangle E \subset (G \setminus E) \cup (\bigcup_{n=n_0+1}^{\infty} A_n) \subset ((\bigcup_{n=1}^{\infty} A_n) \setminus E) \cup (\bigcup_{n=n_0+1}^{\infty} A_n)$$

Since E is Lebesgue measurable and by (1) (2), we have

$$\lambda(G \triangle E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

3. Let $B_1 = [0.5, 0.6)$ and

   $$B_n = \{0.a_1 a_2 \cdots a_{n-1} 5 | a_i \in I, \forall i = 1, \ldots, n-1\}, \text{ where } I = 0, 1, 2, \ldots, 9.$$

   Let $A_1 = ([0,1] \setminus B_1)$, $A_2 = (A_1 \setminus B_2)$, ..., $A_n = (A_{n-1} \setminus B_n)$.
   Therefore,
   $$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots.$$
   $\Rightarrow A = \bigcap_{n=1}^{\infty} A_n$.
   Since $\lambda(A_n) = (\frac{9}{10})^n$, $\Rightarrow \lambda(A_1) = \frac{9}{10} < \infty$. Thus we have

   $$\lambda(A) = \lambda(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \lambda(A_n) = \lim_{n\to\infty} (\frac{9}{10})^n = 0$$

4. There are two examples! Given $0 < \varepsilon < 1$.

   1^o The first stage of the construction is to subdivide $[0,1]$ into three parts. The length of the middle part is $\frac{\varepsilon}{3}$, and the remaining two parts have equal length. We denote the middle part by A_1.
   At the second stage, we subdivide each of the remaining two parts at the first stage into three parts. The length of each middle part is $\frac{\varepsilon}{9}$. Then let $A_2 = A_1 \cup$(the two middle parts at the second stage). Use similar algorithm to construct $\{A_k\}$ for all $k \in \mathbb{N}$ and observe that $A_k \nearrow$ and each A_k is measurable.
   Set $E = \bigcup_{n=1}^{\infty} A_n$.

   $$\begin{aligned} \lambda E = \lim_{n\to\infty} \lambda A_n &= \frac{\varepsilon}{3} + 2 * \frac{\varepsilon}{9} + 4 * \frac{\varepsilon}{3^3} + \cdots \\ &= \frac{\varepsilon}{3}(1 + \frac{2}{3} + (\frac{2}{3})^2 + \cdots) \\ &= \frac{\varepsilon}{3} * \frac{1}{1 - \frac{2}{3}} = \varepsilon \end{aligned}$$

   Therefore $\lambda(E) = \varepsilon$.

Claim : E is dense in $[0,1]$.

Pf :$\forall x \in [0,1]$

case$< 1 >$: $x \in E \Rightarrow \forall r > 0 B(x,r) \cap E \neq \emptyset$.

case$< 2 >$: $x \in ([0,1] \setminus E) \Rightarrow x \in (\bigcap_{n=1}^{\infty}(A_n)^c) \cap [0,1]$.

$\Rightarrow x \in A_n^c \cap [0,1]$, $\forall n$. We can estimate $\text{dist}(x, A_n)$. For example,

$$\text{dist}(x, A_n) \leq \frac{1 - \lambda(A_n)}{2^n} < \frac{1}{2^n}.$$

Thus x is a limit point of $\bigcup A_n = E$.

2^o Let $E' = (0, \varepsilon)$, and $E = E' \cup (\mathbb{Q} \cap [0,1])$.

Since $\mathbb{Q}$ is dense in $[0,1]$, E is also dense in $[0,1]$. Moreover,

$$\varepsilon = \lambda(0, \varepsilon) \leq \lambda(E) \leq \lambda(E') + \lambda(\mathbb{Q}) = \varepsilon + 0 = \varepsilon$$

Therefore $\lambda(E) = \varepsilon$.