Real Analysis Homework #3

Due 10/13

1. Let μ be a finite measure on $(X, \mathcal{S})$ and μ^* the outer measure induced by μ. Assume that $E \subseteq X$ satisfies $\mu^*(E) = \mu^*(X)$ (but not that $E \in \mathcal{S}$). Let $\mathcal{S}_E = \{A \cap E : A \in \mathcal{S}\}$, and define the set function ν on $\mathcal{S}_E$ defined by $\nu(A \cap E) = \mu(A)$. Show that $\mathcal{S}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{S}_E$. You should first prove that ν is well-defined, i.e., if $A, B \in \mathcal{S}$ with $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

2. Let X be the set of rational numbers. Denote $\mathcal{R} = \{(a, b] \cap X : -\infty \leq a \leq b \leq \infty\}$. Let $\mathcal{A}$ be the collection of finite unions of sets of $\mathcal{R}$.
(i) Prove that $\mathcal{A}$ is an algebra.
(ii) Show that $\sigma(\mathcal{A}) = 2^X$.
(iii) Let μ be a set function on $\mathcal{A}$ such that $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ for $A \neq \emptyset$. Then the extension of μ to $\sigma(\mathcal{A})$ is not unique.

3. (i) Show that $\mathcal{N}(\mu) = \{E \subset X : E \subset N \text{ for some } N \in \mathcal{S} \text{ with } \mu(N) = 0\}$ and $\mathcal{S} \vee \mathcal{N}(\mu) = \{E \cup F : E \in \mathcal{S} \text{ and } F \in \mathcal{N}(\mu)\}$.
(ii) If $\bar{\mu}$ is the completion of μ, show that a set E is in the σ-algebra on which $\bar{\mu}$ is defined iff there exist some A and C in the domain of μ with $A \subset E \subset C$ and $\mu(C \setminus A) = 0$.

4. If $(X, \mathcal{S}, \mu)$ is a measure space and $A_1, A_2, \cdots,$ are any subsets of X, let C_j be a measurable cover of A_j for $j = 1, 2, \cdots$.
(i) Show that $\bigcup_j C_j$ is a measurable cover of $\bigcup_j A_j$.
(ii) Give an example to show that $C_1 \cap C_2$ need not be a measurable cover of $A_1 \cap A_2$.