Real Analysis Homework #11

Due 12/22

1. Let $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Show that the Hardy-Littlewood maximal function $Mf(x) \in L^p(\mathbb{R}^n)$ and

$$\|Mf\|_{L^p} \leq C\|f\|_{L^p},$$

where the constant C only depends on n and p.

2. $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *lower semicontinuous* if $\{x : f(x) > a\}$ is open for every $a \in \mathbb{R}$. Let μ be a complex measure on $\mathbb{R}^n$. Define the maximal function

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(r,x))}{\lambda(B(r,x))}.$$

Show that $M\mu$ is lower semicontinuous.

3. Let f be of bounded variation on $[a, b]$. If $f = g + h$, where g is absolutely continuous and h is singular, show that

$$\int_a^b \phi df = \int_a^b \phi f' dx + \int_a^b \phi dh,$$

for any continuous function ϕ.

4. Show that the formula

$$\int_{\mathbb{R}} fg' = -\int_{\mathbb{R}} f'g$$

for integration by parts may not hold if f is of bounded variation on $\mathbb{R}$ and $g \in C^\infty(\mathbb{R})$ with compact support. (Let f be the Cantor-Lebesgue function on $[0, 1]$ and be zero elsewhere.)