Quantitative uniqueness estimates for the generalized non-stationary Stokes system

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Abstract

We study the local behavior of a solution to a generalized non-stationary Stokes system with singular coefficients in R^n with $n \geq 2$. One of our main results is a bound on the vanishing order of a nontrivial solution u satisfying the generalized non-stationary Stokes system, which is a quantitative version of the (strong) unique continuation property for u. Different from the previous known results, our unique continuation result only involves the velocity field u. Our proof relies on some delicate Carleman-type estimates. We first use these estimates to derive crucial *optimal* three-cylinder inequalities for u.

Keywords: Stokes system, Quantitative uniqueness estimates, Carleman estimates

1 Introduction

In this work we study the local behavior of the solution to a generalized non-stationary Stokes system. This generalized Stokes system includes the usual Navier-Stokes equations with velocity that can become singular at one point. Let Ω be a connected bounded domain in $\mathbf{R}^n$ with $n \geq 2$. Without loss of generality, we assume 0 is in the interior of Ω and define $B_r(x) = \{y : |y - x| < r\}$. We consider the following time-dependent Stokes systems

$$\begin{cases} \partial_t u - \Delta u + A(t,x)\cdot\nabla u + B(t,x)u + \nabla p = 0 \quad \text{in} \quad (-1,1)\times\Omega, \\ \nabla\cdot u = 0 \quad \text{in} \quad (-1,1)\times\Omega. \end{cases} \tag{1.1}$$

We assume that coefficients $A(t,x)$ and $B(t,x)$ satisfy

$$\begin{cases} |A(t,x)| \leq \lambda|x|^{-1+\epsilon}, \quad t\in(-1,1), \\ |B(t,x)| \leq \lambda|x|^{-2+\epsilon}, \quad t\in(-1,1) \end{cases} \tag{1.2}$$

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for some $\lambda > 0$ and $0 < \epsilon < 1$. Our main concern is the local vanishing behavior of $u(t,x)$ in $(-1,1)\times\Omega$.

We now describe our main results. For $s \in (-1,1)$, let $Q^{s,\tau}_{x,R} = \{(t,y) : y \in B_R(x) \subset \mathbf{R}^n,\ t \in (-\tau+s, \tau+s)\}$ Let $u \in H^1((-1,1); H^2_{loc}(\Omega))$ be a nontrivial solution of (1.1) with an appropriate pressure function $p \in L^2((-1,1); H^1_{loc}(\Omega))$. For the local case, we derive the following optimal three cylinder inequality and doubling cylinder inequality.

Theorem 1.1 *Given T and t_0 such that $0 < t_0 < T \le 1$. For any $\hat{R} < 1$ such that if $0 < R_1 < R_2 < R_3/3 < \hat{R}$ then*

$$\iint_{Q^{0,T-t_0}_{0,R_2}} |u|^2 dxdt \le \tilde{C}\left(\iint_{Q^{0,T}_{0,R_1}} |u|^2 dxdt\right)^{\kappa}\left(\iint_{Q^{0,T}_{0,R_3}} |u|^2 dxdt\right)^{1-\kappa}, \qquad (1.3)$$

where $\tilde{C}$ depends on $\epsilon, \lambda, n, T, t_0, R_2/R_3$ and

$$\kappa = \frac{\log(\frac{2R_3}{3R_2})}{4\log(\frac{4R_2}{R_1}) + \log(\frac{2R_3}{3R_2})}.$$

We remark that the optimality of (1.3) is due to the fact that $\kappa \sim -1/\log R_1$. From Theorem 1.1, we can get the following double cylinder inequality.

Theorem 1.2 *Let $\epsilon, n, \lambda, R_2, T, t_0$ and R_3 be as in Theorem 1.2. Then we have*

$$\iint_{Q^{0,T-t_0}_{0,R_2}} |u|^2 dxdt \le Ce^{4(\log\frac{4R_2}{R_1})m_1}\iint_{Q^{0,T}_{0,R_1}} |u|^2 dxdt, \qquad (1.4)$$

where

$$m_1 := 64 + \frac{1}{2} + 64\left[\log\Big(\frac{2\iint_{Q^{0,T}_{0,R_3}} |u|^2 dxdt}{\iint_{Q^{0,T-t_0}_{0,\frac{R_3}{2}}} |u|^2 dxdt}\Big)\right]$$

and $[x]$ is the largest integer small than x.

We also obtain a vanishing order of the velocity over one cylinder.

Corollary 1.3 *Let $\epsilon, n, \lambda, R_2, T, t_0$ and R_3 in Theorem 1.2 be fixed. Then we have*

$$\Big(\iint_{Q^{0,T}_{0,R_3}} |u|^2 dxdt\Big) R_1^m \le \iint_{Q^{0,T}_{0,R_1}} |u|^2 dxdt$$

for all R_1 sufficiently small, where

$$m = C_1 + C_1\log\Big(\frac{\iint_{Q^{0,T}_{0,R_3}} |u|^2 dxdt}{\iint_{Q^{0,T-t_0}_{0,R_2}} |u|^2 dxdt}\Big)$$

and C_1 is a positive constant depending on $\epsilon, n, \lambda, T, t_0$ and R_2/R_3.

Corollary 1.3 and Theorem 1.1 immediately imply that if $(u,p)\in H^1((-1,1);H^2_{loc}(\Omega))\times L^2((-1,1);H^1_{loc}(\Omega))$ satisfies (1.1) and

$$\iint_{Q_{0,r}^{0,T}}|u(t,x)|^2dxdt=O(r^N)\quad\forall\,N\in\mathbb{N},\tag{1.5}$$

then u is zero and p is a constant in $(-T,T)\times\Omega$. It is clear that our theorems imply the following unique continuation property of the Navier-Stokes equation. Assume that $(u,p)\in H^1((-1,1);H^2_{loc}(\Omega))\times L^2((-1,1);H^1_{loc}(\Omega))$ satisfies

$$\begin{cases}\partial_t u-\Delta u+u\cdot\nabla u+\nabla p=0\quad\text{in}\quad(-1,1)\times\Omega,\\ \nabla\cdot u=0\quad\text{in}\quad(-1,1)\times\Omega.\end{cases}$$

If the velocity field u satisfies the first condition of (1.2) and (1.5), then u is trivial and p is a constant.

The main tool in the derivation of (1.3) and (1.4) is Carleman type estimates. We derive such estimates with a weight function depending only on the spatial variable x. To take care of the time variable t, we introduce a cut-off function $\chi(t)$ into the Carleman estimates. The main difficulty comes from the integral over the set where $0<\chi(t)<1$. Fortunately, we can overcome the difficulty by carefully choosing the cut-function $\chi(t)$ (see the definition of χ in (4.1)).

For parabolic equations with time-independent or time-dependent coefficients, optimal three cylinder inequalities similar to (1.3) are derived in Vessella [13] and Escauriaza-Vessella [2]. These quantitative estimates are useful in the study of the stability for various types of inverse parabolic problems with unknown boundaries (see Vessella's review article [14]). Our approach share the same spirit as in Vessella's works. We refer readers to Yamamoto's review article [15] for other types of Carleman estimates for the parabolic equation and their applications to inverse problems. In the consideration of the inverse source problem for the Navier-Stokes equations [1], a Carleman estimate for the linearized Navier-Stokes equations is derived. We would like to point out that the weight function in our Carleman estimate is time-independent and singular at the origin, while the weight function used in the Carleman estimate of [1] is regular in the spatial variable x while blows up at end points of time. Other Carleman estimates for the Navier-Stokes equations can also be found in the references in [15]. For the stationary Stokes or Navier-Stokes equations, unique continuation and quantitative estimates (local or at infinity) are proved in [3, 7, 8, 12].

The singular behavior (1.2) of $A(t,x)$ and $B(t,x)$ is motivated by the study of the strong unique continuation property for the elliptic equations [4]. For the elliptic equation, the critical case (when $\epsilon=0$) are considered in [9, 11]. This paper is organized as follows. In Section 2, we derive suitable Carleman estimates. A technical interior estimate is proved in Section 3. Section 4 is devoted to the proofs of Theorem 1.1, Corollary 1.3, and Theorem 1.2.

2 Reduced system and Carleman estimates

To study the Stokes type equation, it is useful to consider the vorticity equation. Let us now define the vorticity q of the velocity u by

$$q = \operatorname{curl} u := \frac{1}{\sqrt{2}}(\partial_i u_j - \partial_j u_i)_{1\le i,j\le n}.$$

Note that here q is a matrix-valued function. The formal transpose of curl is given by

$$(\operatorname{curl}^\top v)_{1\le i\le n} := \frac{1}{\sqrt{2}}\sum_{1\le j\le n} \partial_j(v_{ij} - v_{ji}),$$

where $v = (v_{ij})_{1\le i,j\le n}$. It is easy to see that

$$\Delta u = \nabla(\nabla\cdot u) - \operatorname{curl}^\top \operatorname{curl} u$$

(see, for example, [10] for a proof), which implies

$$\Delta u + \operatorname{curl}^\top q = 0. \tag{2.1}$$

Since there is no equation for p in the Stokes system (1.1), we apply the curl operator $\nabla\times$ on the first equation and obtain

$$\partial_t q - \Delta q + \nabla\cdot F = 0, \tag{2.2}$$

where $q = \nabla\times u$ is the vorticity and $\nabla\cdot F$ is a vector function defined by $(\nabla\cdot F)_i = \sum_{j=1}^n \partial_j F_{ij},\ i = 1, 2, \cdots, n,$ where $F_{ij} = \sum_{k,\ell=1}^n \tilde{A}_{ijk\ell}(x,t)\partial_k u_\ell + \sum_{k=1}^n \tilde{B}_{ijk}(x,t)u_k$ with appropriate $\tilde{A}_{ijk\ell}(x)$ and $\tilde{B}_{ijk}(x)$ satisfying

$$|\tilde{A}_{ijk\ell}(x,t)| \le \lambda|x|^{-1+\epsilon}, |\tilde{B}_{ijk}(x,t)| \le \lambda|x|^{-2+\epsilon}, \quad \forall\ (x,t)\in\Omega\times(-1,1). \tag{2.3}$$

In summary, to study our problem it suffices to consider equations (2.1), (2.2), which is a system of elliptic and parabolic equations.

The proof of main theorem relies heavily on suitable Carleman estimates. In the rest of the section, we will derive two needed Carleman estimates. We begin with the delicate construction of the weight function $\varphi = \varphi(x) = \exp(\psi(y))$, where $\beta > 0$, $y = -\log|x|$ and

$$\psi(y) = \beta y + \frac{1}{16} y\tan^{-1} y - \frac{1}{32}\ln(1+y^2) + \psi_0(y).$$

To define $\psi_0(y)$, we let $\mu(y)\in C^\infty(\mathbb{R})$ satisfy $0\le\mu(y)\le 1$ and

$$\mu(y) = \begin{cases} 0, & y\le 0,\\ 1, & y\ge 1.\end{cases}$$

We now set
$$\psi_0''(y) = \frac{\beta}{128\sigma}\epsilon\mu(y)e^{-\epsilon y/2}$$
and
$$\psi_0'(y) = \int_0^y \psi_0''(s)ds - \int_0^1 \psi_0''(s)ds - \frac{\beta}{64\sigma}e^{-\epsilon/2},$$
$$\psi_0(y) = \int_0^y \psi_0'(s)ds - \int_0^1 \psi_0'(s)ds + \frac{\beta}{32\epsilon\sigma}e^{-\epsilon/2}$$
with
$$\sigma = \int_0^1 \frac{\epsilon}{2}\mu(y)e^{-\epsilon y/2} + e^{-\epsilon/2} > \frac{1}{2}.$$
It should be noted that
$$\psi_0'(y) = \begin{cases} -\dfrac{\beta}{64}, & y \le 0, \\ -\dfrac{\beta}{64\sigma}e^{-\epsilon y/2}, & y \ge 1 \end{cases}$$
and
$$\psi_0(y) = \begin{cases} -\dfrac{\beta}{64}y - \displaystyle\int_0^1 \psi_0'(s)ds + \dfrac{\beta}{32\epsilon\sigma}e^{-\epsilon/2}, & y \le 0, \\ \dfrac{\beta}{32\epsilon\sigma}e^{-\epsilon y/2}, & y \ge 1. \end{cases}$$
Moreover, the function $\psi(y)$ for $y > 0$ is a convex function satisfying for $\frac{\beta}{64} \in \mathbb{N} + \frac{1}{256}$ that
$$\begin{cases} \dfrac{1}{2}\beta \le \psi' \le 2\beta, \\ \mathrm{dist}(2\psi', \mathbb{Z}) + \psi'' \gtrsim 1 \end{cases} \tag{2.4}$$
and for any $C > 0$ there exists $R_\epsilon > 0$ such that
$$C|x|^\epsilon\beta \le (1 + \psi''(-\log|x|)) \tag{2.5}$$
for all β and $|x| \le R_\epsilon$. From now on, the notation $X \lesssim Y$ or $X \gtrsim Y$ means that $X \le CY$ or $X \ge CY$ with some constant C depending only on a priori given constants. To check the second inequality of (2.4), we note that
$$\begin{cases} \psi'' \ge \dfrac{\epsilon}{128}, & y \le \dfrac{2}{\epsilon}\log\dfrac{\beta}{\sigma}, \\ \mathrm{dist}(2\psi', \mathbb{Z}) \ge \dfrac{1}{4}, & y \ge \dfrac{2}{\epsilon}\log\dfrac{\beta}{\sigma}. \end{cases}$$

We now introduce polar coordinates in $\mathbb{R}^n\setminus\{0\}$ by setting $x = r\omega$, with $r = |x|$, $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. Using new coordinate $y = -\log r$, we obtain that
$$\frac{\partial}{\partial x_j} = e^y(-\omega_j\partial_y + \Omega_j), \quad 1 \le j \le n,$$

where Ω_j is a vector field in S^{n-1}. We could check that the vector fields Ω_j satisfy

$$\sum_{j=1}^{n} \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} \Omega_j \omega_j = n-1.$$

It is easy to see that

$$\frac{\partial^2}{\partial x_j \partial x_\ell} = e^{2y}(-\omega_j \partial_y - \omega_j + \Omega_j)(-\omega_\ell \partial_y + \Omega_\ell), \quad 1 \le j, \ell \le n.$$

and, therefore, the Laplacian becomes

$$e^{-2y}\Delta = \partial_y^2 - (n-2)\partial_y + \Delta_\omega,$$

where $\Delta_\omega = \Sigma_{j=1}^n \Omega_j^2$ denotes the Laplace-Beltrami operator on S^{n-1}. We recall that the eigenvalues of $-\Delta_\omega$ are $k(k+n-2)$, $k \in \mathbb{N} \cup \{0\}$, and the corresponding eigenspaces are E_k, where E_k is the space of spherical harmonics of degree k. We remark that

$$\sum_j \iint |\Omega_j v|^2 dy d\omega = \sum_{k \ge 0} k(k+n-2) \int |v_k|^2 dy, \tag{2.6}$$

where v_k is the projection of v onto E_k. Let

$$\Lambda = \sqrt{\frac{(n-2)^2}{4} - \Delta_\omega},$$

then Λ is an elliptic first-order positive pseudodifferential operator in $L^2(S^{n-1})$. The eigenvalues of Λ are $k+\frac{n-2}{2}$ and the corresponding eigenspaces are E_k which represents the space of spherical harmonics of degree k. Hence

$$\Lambda = \Sigma_{k \ge 0}(k + \frac{n-2}{2})\pi_k, \tag{2.7}$$

where π_k is the orthogonal projector on E_k. Let

$$L^{\pm} = \partial_y - \frac{n-2}{2} \pm \Lambda,$$

then it follows that

$$e^{-2y}\Delta = L^+ L^- = L^- L^+.$$

Denote $L_\psi^{\pm} = \partial_y - \frac{n-2}{2} \pm \Lambda - \psi'(y)$. Then we have that $L_\psi^{\pm} v = e^{\psi(y)} L^{\pm}(e^{-\psi(y)} v)$ and $e^{-2y} e^{\psi(y)} \Delta(e^{-\psi(y)} v) = L_\psi^+ L_\psi^- v = L_\psi^- L_\psi^+ v$.

Lemma 2.1 *Let $\chi(t) \in C_0^2(\mathbf{R})$. There exists a sufficiently large constant β_1, depending on n, such that for all $v(t,y,\omega) \in C^1(\mathbf{R}; C_0^\infty(\mathbf{R}\times S^{n-1}))$ and $\beta \geq \beta_1$ with $\frac{\beta}{64} \in \mathbf{N} + \frac{1}{256}$, we have that*

$$\begin{aligned} &\int |\chi(L_\psi^+ L_\psi^- v - 2L_\psi^+ v - e^{-2y}\partial_t v)|^2 + \int |\chi' e^{-2y} v|^2 \\ \gtrsim &\sum_{j+|\alpha|\leq 1} \beta^{2-2(j+|\alpha|)} \int (1+\psi'')|\partial_y^j \Omega^\alpha (\chi v)|^2, \end{aligned} \tag{2.8}$$

where we denote $\int f = \int f dt dy d\omega$ for any function $f(t,y,\omega)$.

Proof. From $v = \Sigma_k v_k$, we can compute

$$\begin{aligned} &\chi(L_\psi^+ L_\psi^- v - 2L_\psi^+ v) \\ =&\chi[\partial_y^2 v - 2\psi'\partial_y v - n\partial_y v + (n-2+(\psi')^2 + n\psi' - \psi'')v - 2\Lambda v + \Delta_\omega v] \\ =&\Sigma_{k\geq 0}\chi(\partial_y^2 v_k - \tilde{b}\partial_y v_k + \tilde{a} v_k), \end{aligned}$$

where

$$\begin{cases} \tilde{a} = (\psi' - k)(\psi' + k + n) - \psi'' \\ \tilde{b} = 2\psi' + n. \end{cases}$$

By abusing the notation $v = v_k$, it is enough to prove that

$$\begin{aligned} &\sum_{j\leq 1} \int (1+\psi'')|(\beta^{2-2j} + k^{2-2j})\partial_y^j(\chi v)|^2 \\ \lesssim &\int |\chi(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a} v - e^{-2t}\partial_t v)|^2 + \int |\chi' e^{-2y} v|^2. \end{aligned} \tag{2.9}$$

It is helpful to note that

$$\begin{aligned} &\psi' = \beta + \frac{1}{16}\tan^{-1} y + \psi_0'(y) \geq \frac{31\beta}{32}, \\ &0 < \psi'' = \frac{1}{16(1+y^2)} + \frac{\beta}{128\sigma}\epsilon\mu(y)e^{-\epsilon y/2} \leq \frac{1}{16} + \frac{\beta}{64} \leq \frac{\psi'}{32}, \quad \beta \geq 4, \\ &|\psi'''| = |\frac{-y}{8(1+y^2)^2} - \frac{\beta}{256\sigma}\epsilon^2\mu(y)e^{-\epsilon y/2} + \frac{\beta}{128\sigma}\epsilon\mu'(y)e^{-\epsilon y/2}| \leq 2\psi'', \end{aligned}$$

where we choose $|\mu'| \leq 4$ in the last inequality. Observe that

$$\begin{aligned} &2|\chi(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a} v - e^{-2y}\partial_t v)|^2 + 4|\chi' e^{-2y} v|^2 \\ \geq&|\chi(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a} v) - e^{-2y}\partial_t(\chi v)|^2 \\ =&|H(v)|^2 - 2\tilde{b}\partial_y(\chi v)H(v) - 2e^{-2y}\partial_t(\chi v)H(v) + |\tilde{b}\partial_y(\chi v) + e^{-2y}\partial_t(\chi v)|^2, \end{aligned} \tag{2.10}$$

where $H(v) := \chi(\partial_y^2 v + \tilde{a}v)$. Now we write

$$\begin{cases} -2\int \tilde{b}\partial_y(\chi v)H(v) = -2\int \tilde{b}\partial_y(\chi v)\partial_y^2(\chi v) - 2\int \tilde{a}\tilde{b}(\chi v)\partial_y(\chi v) \\ -2\int e^{-2y}\partial_t(\chi v)H(v) = -2\int e^{-2y}\partial_t(\chi v)\partial_y^2(\chi v) - 2\int \tilde{a}(\chi v)e^{-2y}\partial_t(\chi v). \end{cases} \tag{2.11}$$

Straightforward computations imply that

$$\begin{cases} -2\int \tilde{b}\partial_y(\chi v)\partial_y^2(\chi v) = 2\int \psi''|\partial_y(\chi v)|^2, \\ -2\int \tilde{a}\tilde{b}(\chi v)\partial_y(\chi v) = \int \partial_y(\tilde{a}\tilde{b})|\chi v|^2, \end{cases} \tag{2.12}$$

$$-2\int e^{-2y}\partial_t(\chi v)\partial_y^2(\chi v) = -4\int e^{-2y}\partial_t(\chi v)\partial_y(\chi v), \tag{2.13}$$

$$-2\int \tilde{a}(\chi v)e^{-2y}\partial_t(\chi v) = 0, \tag{2.14}$$

Note that here $\tilde{a}$ is independent of t. Combining (2.10) to (2.14) yields

$$\begin{aligned} &2\int |\chi(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a}v - e^{-2t}\partial_t v)|^2 + 4\int |\chi' e^{-2y} v|^2 \\ \geq &\int \left(|H(v)|^2 + |\tilde{b}\partial_y(\chi v) + e^{-2y}\partial_t(\chi v)|^2\right) + 2\int \psi''|\partial_y(\chi v)|^2 \\ &- 4\int e^{-2y}\partial_t(\chi v)\partial_y(\chi v) + \frac{17}{3}\int (\psi')^2\psi''|\chi v|^2 - \frac{5}{2}\int k^2\psi''|\chi v|^2 \end{aligned} \tag{2.15}$$

for $\beta \geq \beta_0$. In deriving (2.15), we have used estimates $(\psi'')^2 < \frac{1}{32}\psi'\psi'' < \frac{1}{\beta}(\psi')^2\psi''$ and $|\psi'\psi'''| < 2\psi'\psi'' < \frac{1}{\beta}(\psi')^2\psi''$.

Likewise, we write

$$\begin{cases} |\tilde{b}\partial_y(\chi v) + e^{-2y}\partial_t(\chi v)|^2 \\ \quad = |(2\psi' + n - 2)\partial_y(\chi v) + e^{-2y}\partial_t(\chi v)|^2 + 4(2\psi' + n - 1)|\partial_y(\chi v)|^2 \\ \qquad + 4e^{-2y}\partial_t(\chi v)\partial_y(\chi v). \\ \frac{1}{2}|H(v)|^2 = \frac{1}{2}|H(v) + 3\psi''\chi v|^2 - 3\psi''\chi v H(v) - \frac{9}{2}(\psi'')^2|\chi v|^2. \end{cases} \tag{2.16}$$

It is easy to check that

$$\begin{aligned} &-3\int \psi''\chi v H(v) \\ = &-3\int \psi''\chi^2 v(\partial_y^2 v + \tilde{a}v) \\ \geq &\, 3\int \psi''|\partial_y(\chi v)|^2 - \frac{10}{3}\int (\psi')^2\psi''|\chi v|^2 + 3\int k^2\psi''|(\chi v)|^2 \end{aligned} \tag{2.17}$$

for all $\beta \geq \beta_0$. From (2.15)-(2.17), we have that for $\beta \geq \beta_0$

$$\begin{aligned}
&2\int |\chi(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a}v - e^{-2t}\partial_t v)|^2 + 4\int |\chi' e^{-2y} v|^2 \\
\geq & 8\int \psi'|\partial_y(\chi v)|^2 + 2\int (\psi')^2\psi''|\chi v|^2 + \frac{1}{2}\int k^2\psi''|\chi v|^2 \\
&+ \frac{1}{2}\int |H(v)|^2. \qquad (2.18)
\end{aligned}$$

Our next strategy is to divide y into three region. Before doing so, we need to further simplify (2.18). Note that

$$\begin{aligned}
\frac{1}{4}\int |H(v)|^2 =& \frac{1}{4}\int |H(v) - \frac{\beta(\psi'-k)\chi v}{100|\beta-k|}|^2 + \int \frac{\beta(\psi'-k)}{200|\beta-k|}\chi v H(v) \\
&- \int \frac{\beta^2(\psi'-k)^2}{40000|\beta-k|^2}|\chi v|^2 \\
\geq & \int \frac{\beta(\psi'-k)}{200|\beta-k|}\chi v H(v) - \int \frac{\beta^2(\psi'-k)^2}{40000|\beta-k|^2}|\chi v|^2
\end{aligned}$$

and note

$$\begin{aligned}
\int \frac{\beta(\psi'-k)}{200|\beta-k|}\chi^2 v\partial_y^2 v =& -\int \frac{\beta(\psi'-k)}{200|\beta-k|}|\partial_y(\chi v)|^2 + \int \frac{\beta\psi'''}{400|\beta-k|}|\chi v|^2 \\
=& -\int \frac{\beta(\beta-k)}{200|\beta-k|}|\partial_y(\chi v)|^2 + \int \frac{\beta(-\psi_0'(y) - \frac{1}{16}\tan^{-1} y)}{200|\beta-k|}|\partial_y(\chi v)|^2 \\
&+ \int \frac{\beta\psi'''}{400|\beta-k|}|\chi v|^2
\end{aligned}$$

with

$$\int \frac{\beta(\psi'-k)}{200|\beta-k|}\chi^2 v\tilde{a}v = \int \frac{\beta(\psi'-k)^2(\psi'+k+n)}{200|\beta-k|}|\chi v|^2 - \int \frac{\beta(\psi'-k)\psi''}{200|\beta-k|}|\chi v|^2.$$

Combining (2.18), we have that

$$\begin{aligned}
&2\int |\chi(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a}v - e^{-2t}\partial_t v)|^2 + 4\int |\chi' e^{-2y} v|^2 \\
\geq & 7\int \psi'|\partial_y(\chi v)|^2 + \frac{3}{2}\int (\psi')^2\psi''|\chi v|^2 + \frac{1}{2}\int k^2\psi''|\chi v|^2 \\
&+ \int \frac{\beta(\psi'-k)^2(\psi'+k+n)}{400|\beta-k|}|\chi v|^2 + \frac{1}{4}\int |H(v)|^2. \qquad (2.19)
\end{aligned}$$

We need further simplification. Let $\tilde{\mu}(y) = \mu(y+1)$, then

$$1 - \tilde{\mu}(y) = \begin{cases} 1, & y \leq -1 \\ 0, & y \geq 0. \end{cases}$$

For a small positive constant $\rho < \frac{1}{10}$ which will be determined later, we can estimate

$$\begin{aligned}
\frac{1}{4}\int |H(v)|^2 =& \frac{1}{4}\int |H(v) - \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)\chi v}{|\frac{63\beta}{64}-k|}|^2 + \int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)}{2|\frac{63\beta}{64}-k|}\chi v H(v) \\
&- \int \frac{\rho^2\beta^2(1-\tilde{\mu})^2(\psi'-k)^2}{4|\frac{63\beta}{64}-k|^2}|\chi v|^2 \\
\geq& \int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)}{2|\frac{63\beta}{64}-k|}\chi v H(v) - \int \frac{\rho^2\beta^2(1-\tilde{\mu})^2(\psi'-k)^2}{4|\frac{63\beta}{64}-k|^2}|\chi v|^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)}{2|\frac{63\beta}{64}-k|}\chi^2 v\partial_y^2 v \\
=& -\int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)}{2|\frac{63\beta}{64}-k|}|\partial_y(\chi v)|^2 + \int \frac{\partial_y^2[\rho\beta(1-\tilde{\mu})(\psi'-k)]}{4|\frac{63\beta}{64}-k|}|\chi v|^2 \\
=& -\int \frac{\rho\beta(1-\tilde{\mu})(\frac{63\beta}{64}-k)}{2|\frac{63\beta}{64}-k|}|\partial_y(\chi v)|^2 - \int \frac{\rho\beta(1-\tilde{\mu})\tan^{-1}y}{32|\frac{63\beta}{64}-k|}|\partial_y(\chi v)|^2 \\
&+ \int \frac{\partial_y^2[\rho\beta(1-\tilde{\mu})(\psi'-k)]}{4|\frac{63\beta}{64}-k|}|\chi v|^2 \\
\geq& -\int \frac{\rho\beta(1-\tilde{\mu})(\frac{63\beta}{64}-k)}{2|\frac{63\beta}{64}-k|}|\partial_y(\chi v)|^2 + \int \frac{\partial_y^2[\rho\beta(1-\tilde{\mu})(\psi'-k)]}{4|\frac{63\beta}{64}-k|}|\chi v|^2
\end{aligned}$$

and

$$\begin{aligned}
&\int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)}{2|\frac{63\beta}{64}-k|}\chi^2 v\tilde{a}v \\
=& \int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)^2(\psi'+k+n)}{2|\frac{63\beta}{64}-k|}|\chi v|^2 - \int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)\psi''}{2|\frac{63\beta}{64}-k|}|\chi v|^2.
\end{aligned}$$

To estimate the term $\int \frac{\partial_y^2[\rho\beta(1-\tilde{\mu})(\psi'-k)]}{4|\frac{63\beta}{64}-k|}|\chi v|^2$, we compute that

$$\begin{aligned}
|\partial_y^2[(1-\tilde{\mu})(\psi'-k)]| &= |-\tilde{\mu}'(\psi'-k) - 2\tilde{\mu}'\psi'' + (1-\tilde{\mu})\psi'''| \\
&\leq |\tilde{\mu}''||\psi'-k| + 2|\tilde{\mu}'|\psi'' + (1-\tilde{\mu})|\psi'''|
\end{aligned}$$

It should be noted that the supports of $\tilde{\mu}'$ and $\tilde{\mu}''$ are $[-1,0]$. We assume $||\tilde{\mu}'||_\infty + ||\tilde{\mu}''||_\infty \leq C_2$, then we can choose $\rho < \frac{1}{20000C_2}$. Recall that for $y \leq 0$

$$\psi' = \frac{63}{64}\beta + \frac{1}{16}\tan^{-1}y, \quad \psi'' = \frac{1}{16(1+y^2)}, \quad \psi''' = -\frac{y}{8(1+y^2)^2}.$$

Therefore, combining (2.19), we have that

$$
\begin{aligned}
&2\int |\chi(\partial_y^2 v - \tilde{b}\partial_y v + \tilde{a}v - e^{-2t}\partial_t v)|^2 + 4\int |\chi' e^{-2y} v|^2 \\
&\geq 6\int \psi' |\partial_y(\chi v)|^2 + \int (\psi')^2 \psi'' |\chi v|^2 + \frac{1}{4}\int k^2 \psi'' |\chi v|^2 \\
&\quad + \int \frac{\beta(\psi'-k)^2(\psi'+k+n)}{400|\beta-k|}|\chi v|^2 + \int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)^2(\psi'+k+n)}{4|\frac{63\beta}{64}-k|}|\chi v|^2.
\end{aligned}
\tag{2.20}
$$

We now use (2.20) to deduce (2.9). The trick is to divide the domain of integration into three regions.

(i) For $-1 \leq y \leq \frac{2}{\epsilon}\log\frac{\beta}{\sigma}$, we have $\psi'' \geq \frac{\epsilon}{256}(1+\psi'')$ and (2.9) follows immediately. Recall that $\psi'' \leq \frac{\psi'}{32}$ for all $y \in \mathbf{R}$ provided $\beta \geq 4$.

(ii) For $y \geq \frac{2}{\epsilon}\log\frac{\beta}{\sigma}$ and $\frac{\beta}{64} \in \mathbf{N} + \frac{1}{256}$, we can see that

$$|\psi' - k| \geq |\beta - k| - \frac{1}{16},$$

which implies

$$|\psi' - k|^2 \geq |\beta - k|^2 - \frac{1}{8}|\beta - k| \geq \frac{1}{2}|\beta - k|^2.$$

The fourth term on the right hand side of (2.20) now satisfies

$$\int \frac{\beta(\psi'-k)^2(\psi'+k+n)}{400|\beta-k|}|\chi v|^2 \geq \int \frac{\beta|\beta-k|(\psi'+k+n)}{800}|\chi v|^2 \gtrsim \int (\beta^2+k^2)|\chi v|^2.$$

Consequently, (2.9) holds over this domain of integration.

(iii) For $y \leq -1$, we obtain from the last term on the right hand side of (2.20)

$$\int \frac{\rho\beta(1-\tilde{\mu})(\psi'-k)^2(\psi'+k+n)}{4|\frac{63\beta}{64}-k|}|\chi v|^2 \geq \int_{\{y\leq -1\}} \frac{\rho\beta(\psi'-k)^2(\psi'+k+n)}{4|\frac{63\beta}{64}-k|}|\chi v|^2.$$

That the estimate (2.9) is satisfied over $y \leq -1$ follows from the same arguments as in (ii).

□

We need another Carleman estimate to handle the divergence terms in the reduced system.

Lemma 2.2 *Let $\chi(t) \in C_0^2(\mathbf{R})$. There exists a sufficiently large number β_1' depending on n such that for all $u(t,y,\omega) \in C^1(\mathbf{R}; C_0^\infty(\mathbf{R}\times S^{n-1}))$, $g = (g_0, g_1, \cdots, g_n) \in (C_0(\mathbf{R}; C_0^\infty(\mathbf{R}\times S^{n-1})))^{n+1}$ and $\beta > \beta_1'$ with $\frac{\beta}{64} \in \mathbf{N} + \frac{1}{256}$, we have that*

$$\begin{aligned}\beta^2 \int (1+\psi'')|\chi u|^2 \lesssim \int |\chi(L_\psi^- L_\psi^+ u - e^{-2y}\partial_t u) + \partial_y g_0 + \sum_{j=1}^n \Omega_j g_j|^2 \\ + \beta^2\|g\|^2 + \int |\chi' e^{-2y} u|^2.\end{aligned} \tag{2.21}$$

Proof. For fixed $t \in \mathbf{R}$, we consider $v(t,y,\omega) \in C_0^\infty(\mathbf{R}\times S^{n-1})$. We denote $\hat{v}(t,\eta,\omega)$ its Fourier transformation with respect to y, and define

$$T_\psi v(t,y,\omega) = (2\pi)^{-1/2} \sum_{k\ge 0} \int_{-\infty}^{\infty} e^{i\eta y}(i\eta - \psi' - k - (n-2))\pi_k \hat{v}(t,\eta,\omega)d\eta. \tag{2.22}$$

Similarly, we define

$$\tilde{T}_\psi v(t,y,\omega) = (2\pi)^{-1/2} \sum_{k\ge 0} \int_{-\infty}^{\infty} e^{i\eta y}(i\eta - \beta - k - (n-2))\pi_k \hat{v}(t,\eta,\omega)d\eta. \tag{2.23}$$

Note that we have $T_\psi v = L_\psi^- v = \partial_y v - \psi' v - \frac{(n-2)}{2} v - \Lambda v = \tilde{T}_\psi v + (\beta - \psi')v$. It is clear that $\tilde{T}_\psi$ is invertible whose inverse is given by

$$\tilde{T}_\psi^{-1} v(t,y,\omega) = (2\pi)^{-1/2} \sum_{k\ge 0} \int_{-\infty}^{\infty} e^{i\eta y}(i\eta - \beta - k - (n-2))^{-1}\pi_k \hat{v}(t,\eta,\omega)d\eta. \tag{2.24}$$

From (2.23), (2.24), Plancherel's theorem, and integrating in t, we have for $v \in L^2(\mathbf{R}; C_0^\infty(\mathbf{R}\times S^{n-1}))$ that

$$\begin{cases} \|\tilde{T}_\psi v\|_{L^2(\mathbf{R}^2\times S^{n-1})} \ge \beta \|v\|_{L^2(\mathbf{R}^2\times S^{n-1})}, \\ \beta \|\tilde{T}_\psi^{-1} v\|_{L^2(\mathbf{R}^2\times S^{n-1})} \le \|v\|_{L^2(\mathbf{R}^2\times S^{n-1})}, \\ \|\tilde{T}_\psi^{-1}(\sum_{j+|\alpha|\le 1} \partial_y^j \Omega^\alpha v)\|_{L^2(\mathbf{R}^2\times S^{n-1})} \lesssim \|v\|_{L^2(\mathbf{R}^2\times S^{n-1})} \end{cases} \tag{2.25}$$

(see [12, p.1896] for the proofs). From (2.22), (2.23) and (2.24), we get that

$$T_\psi^{-1} = (I + \tilde{T}_\psi^{-1}(\beta - \psi')I)^{-1}\tilde{T}_\psi^{-1}.$$

Consequently, it follows from (2.25) that for $v \in L^2(\mathbf{R}; C_0^\infty(\mathbf{R}\times S^{n-1}))$ we have

$$\begin{cases} \|T_\psi v\|_{L^2(\mathbf{R}^2\times S^{n-1})} \ge \frac{\beta}{2} \|v\|_{L^2(\mathbf{R}^2\times S^{n-1})}, \\ \beta \|T_\psi^{-1} v\|_{L^2(\mathbf{R}^2\times S^{n-1})} \lesssim \|v\|_{L^2(\mathbf{R}^2\times S^{n-1})}, \\ \|T_\psi^{-1}(\sum_{j+|\alpha|\le 1} \partial_y^j \Omega^\alpha v)\|_{L^2(\mathbf{R}^2\times S^{n-1})} \lesssim \|v\|_{L^2(\mathbf{R}^2\times S^{n-1})}. \end{cases} \tag{2.26}$$

Now for $u(t,y,\omega) \in C_0^1(\mathbf{R}; C_0^\infty((y_0,y_1)\times S^{n-1}))$, we define $v(t,y,\omega) = (T_\psi^{-1}u)(t,y,\omega)$, i.e., $u(t,y,\omega) = T_\psi v(t,y,\omega)$. Let $u_k(t,y,\omega) = \pi_k u(t,y,\omega)$ and $v_k(t,y,\omega) = \pi_k v(t,y,\omega)$. From the definition of T_β, we have that

$$\begin{aligned} \sum_{k\geq 0} u_k &= u = T_\psi v \\ &= (2\pi)^{-1/2} \sum_{k\geq 0} \int_{-\infty}^{\infty} e^{i\eta y}(i\eta - \psi' - k - (n-2))\pi_k \hat{v}(t,\eta,\omega)d\eta \\ &= \sum_{k\geq 0} (\partial_y v_k - (\psi' + k + n - 2)v_k). \end{aligned} \tag{2.27}$$

We would like to find v_k satisfying $\lim_{|y|\to\infty} v_k(t,y,\omega) = 0$. Thus, from (2.27), we have that

$$v_k(t,y,\omega) = \begin{cases} 0, & y \geq y_1, \\ -\displaystyle\int_y^{y_1} e^{-\psi(\xi)-k\xi-(n-2)\xi} u_k(t,\xi,\omega)d\xi \cdot e^{\psi(y)+ky+(n-2)y}, & y < y_1. \end{cases} \tag{2.28}$$

Note that for $y < y_0$, v_k of (2.28) is equivalent to

$$v_k(t,y,\omega) = \begin{cases} 0, & y \geq y_1, \\ -e^{\psi(y)+ky+(n-2)y} \displaystyle\int_{y_0}^{y_1} e^{-\psi(\xi)-k\xi-(n-2)\xi} u_k(t,\xi,\omega)d\xi, & y < y_0. \end{cases}$$

We remark that although estimate (2.8) is proved for $v \in C^1(\mathbf{R}; C_0^\infty(\mathbf{R}\times S^{n-1}))$, it remains valid for $e^{2y}v(t,y,\omega) = \sum_{k\geq 0} e^{2y} v_k(t,y,\omega)$ with v_k given in (2.28).

Using $T_\psi = L_\psi^-$ and estimates in (2.26), we obtain that

$$\begin{aligned} & \|\chi L_\psi^- L_\psi^+ u - \chi e^{-2y}\partial_t u + \partial_y g_0 + \sum_{j=1}^n \Omega_j g_j\|_{L^2(\mathbf{R}^2\times S^{n-1})} \\ = \ & \|T_\psi\big(L_\psi^+(\chi u) + T_\psi^{-1}\big(-\chi e^{-2y}\partial_t u + \partial_y g_0 + \sum_{j=1}^n \Omega_j g_j\big)\big)\|_{L^2(\mathbf{R}^2\times S^{n-1})} \\ \geq \ & \beta\|L_\psi^+(\chi u) + T_\psi^{-1}\big(-\chi e^{-2y}\partial_t u + \partial_y g_0 + \sum_{j=1}^n \Omega_j g_j\big)\|_{L^2(\mathbf{R}^2\times S^{n-1})} \\ \geq \ & \beta\|L_\psi^+(\chi u) - \partial_t(T_\psi^{-1}(\chi e^{-2y}u)) + T_\psi^{-1}(\chi' e^{-2y}u)\|_{L^2(\mathbf{R}^2\times S^{n-1})} \\ & -\beta\|T_\psi^{-1}\big(\partial_y g_0 + \sum_{j=1}^n \Omega_j g_j\big)\|_{L^2(\mathbf{R}^2\times S^{n-1})} \\ \gtrsim \ & \beta\|L_\psi^+(\chi u) - \partial_t(T_\psi^{-1}(\chi e^{-2y}u)) + T_\psi^{-1}(\chi' e^{-2y}u)\|_{L^2(\mathbf{R}^2\times S^{n-1})} \\ & -\beta\|g\|_{L^2(\mathbf{R}^2\times S^{n-1})}. \end{aligned} \tag{2.29}$$

Now, we let $v = T_\psi^{-1}(e^{-2y}u)$ and hence $T_\psi v = e^{-2y}u$ and $\chi v = T_\psi^{-1}(e^{-2y}\chi u)$. We immediately obtain from (2.8) that

$$\begin{aligned}
&\|L_\psi^+(\chi u) - \partial_t(T_\psi^{-1}(\chi e^{-2y}u)) + T_\psi^{-1}(\chi' e^{-2y}u)\|_{L^2(\mathbf{R}^2\times S^{n-1})}\\
=&\|\chi\big(L_\psi^+ L_\psi^-(e^{2y}v) - 2L_\psi^+(e^{2y}v) - e^{-2y}\partial_t(e^{2y}v)\big)\|_{L^2(\mathbf{R}^2\times S^{n-1})}\\
\gtrsim& \sum_{j+|\alpha|\le 1} \beta^{1-(j+|\alpha|)}\|(1+\psi'')^{1/2}\partial_y^j\Omega^\alpha(\chi e^{2y}v)\|_{L^2(\mathbf{R}^2\times S^{n-1})} - \big\|(\chi' v)\big\|_{L^2(\mathbf{R}^2\times S^{n-1})}.
\end{aligned} \tag{2.30}$$

From the substitution $u = e^{2y}T_\beta v$, the definitions of T_β, Λ, and the first inequality of (2.26), it is not difficult to show that

$$\begin{cases}
\|(1+\psi'')^{1/2}\chi u\|_{L^2(\mathbf{R}^2\times S^{n-1})}\\
\lesssim \displaystyle\sum_{j+|\alpha|\le 1} \beta^{1-(j+|\alpha|)}\|(1+\psi'')^{1/2}\partial_y^j\Omega^\alpha(\chi e^{2y}v)\|_{L^2(\mathbf{R}^2\times S^{n-1})},\\
\beta\big\|\chi' v\big\|_{L^2(\mathbf{R}^2\times S^{n-1})} \lesssim \big\|\chi' e^{-2y}u\big\|_{L^2(\mathbf{R}^2\times S^{n-1})}.
\end{cases} \tag{2.31}$$

By (2.30) and (2.31), we obtain that

$$\begin{aligned}
&\beta\|L_\psi^+(\chi u) - \partial_t(T_\psi^{-1}(\chi e^{-2y}u)) + T_\psi^{-1}(\chi' e^{-2y}u)\|_{L^2(\mathbf{R}^2\times S^{n-1})}\\
&\gtrsim \beta\|(1+\psi'')^{1/2}\chi u\|_{L^2(\mathbf{R}^2\times S^{n-1})} - \big\|\chi' e^{-2y}u\big\|_{L^2(\mathbf{R}^2\times S^{n-1})}.
\end{aligned} \tag{2.32}$$

Finally, combining (2.29) and (2.32) yields (2.21). □

Now we are ready to prove our main Carleman estimate.

Lemma 2.3 *Let $\chi(t) \in C_0^2(\mathbf{R})$. There exists a sufficiently large number β_2 depending on n such that for all $w(t,x) \in C^1(\mathbf{R}; C_0^\infty(\mathbf{R}^n \setminus \{0\}))$, $f = (f_1, \cdots, f_n) \in (C_0(\mathbf{R}; C_0^\infty(\mathbf{R}^n \setminus \{0\})))^n$ and $\beta \ge \beta_2$ with $\frac{\beta}{64} \in \mathbf{N} + \frac{1}{256}$, we have that*

$$\begin{aligned}
&\iint \varphi^2(1+\psi'')(|x|^4|\nabla(\chi w)|^2 + \beta^2|x|^2|\chi w|^2)dxdt\\
\lesssim\ &\iint \varphi^2|x|^2(\chi|x|^2\Delta w - \chi|x|^2\partial_t w + |x|\mathrm{div} f)^2 dxdt\\
&+\beta^2\iint \varphi^2|x|^2\|f\|^2dxdt + \iint \varphi^2|x|^6|\chi' w|^2dxdt.
\end{aligned} \tag{2.33}$$

Proof. The estimate (2.33) remains valid if we replace ψ by $\psi + \frac{n}{2}y - y$. We first set $w = e^{-\psi}u$ and $g = e^{-\psi}f$. Working in polar coordinates and using the relation $e^{-2y}\Delta = L^+L^-$, it suffices to prove that

$$\begin{aligned}
&\sum_{j+|\alpha|\le 1} \beta^{2-2(j+|\alpha|)}\int(1+\psi'')|\partial_y^j\Omega^\alpha(\chi u)|^2\\
\lesssim &\int |\chi L_\psi^- L_\psi^+ u - e^{-2y}\partial_t(\chi u) + \partial_y g_0 + \sum_{j=1}^n \Omega_j g_j|^2\\
&+\beta^2\|g\|^2_{L^2(\mathbf{R}^2\times S^{n-1})} + \int |e^{-2y}\chi' u|^2,
\end{aligned} \tag{2.34}$$

where $g_0 = \langle \omega, e^{-\psi} f \rangle$, $g_j = e^{-\psi} f_j$, $j = 1, 2, \cdots, n$. Denote $J(u) = \chi L^-_\psi L^+_\psi u - e^{-2y}\partial_t(\chi u) + \partial_y g_0 + \sum_{j=1}^n \Omega_j g_j$. We now write

$$|J(u)|^2 = |J(u) + (1+\psi'')\chi u|^2 - 2(1+\psi'')\chi u J(u) - (1+\psi'')^2|\chi u|^2. \tag{2.35}$$

Denote that

$$\begin{cases} \tilde{A} = (\psi')^2 + (n-2)\psi' - \psi'' \\ \tilde{b} = -2\psi' - (n-2). \end{cases}$$

We obtain that for $\beta \geq \beta_2$

$$\begin{aligned}
&\int |J(u)|^2 + (1+\psi'')^2|\chi u|^2 \\
&\geq -2\int (1+\psi'')\chi u J(u) \\
&\geq -2\int (1+\psi'')\chi u(\chi L^-_\psi L^+_\psi u - e^{-2y}\partial_t(\chi u) + \partial_y g_0 + \sum_{j=1}^n \Omega_j g_j) \\
&= -2\int (1+\psi'')\chi u\big(\partial_y^2(\chi u) + \tilde{b}\partial_y(\chi u) + \tilde{A}(\chi u) + \Delta_\omega(\chi u)\big) \\
&\quad + 2\int (1+\psi'')\chi u e^{-2y}\partial_t(\chi u) - 2\int (1+\psi'')\chi u(\partial_y g_0 + \sum_{j=1}^n \Omega_j g_j) \\
&\geq \int (1+\psi'')|\partial_y(\chi u)|^2 - C\beta^2\int (1+\psi'')|\chi u|^2 + \int (1+\psi'')\sum_{j=1}^n |\Omega_j(\chi u)|^2 \\
&\quad - C\beta\|g\|^2_{L^2(\mathbf{R}^2\times S^{n-1})},
\end{aligned} \tag{2.36}$$

where C is an absolute constant. By multiplying a large constant K on both side of (2.21), if necessary, and adding this new estimate to (2.36), we obtain the desired estimate (2.34). □

We will apply Carleman estimate (2.33) to the parabolic equation (2.2). To treat the elliptic equation (2.1), we use the Carleman estimate below, which can be proved by following the same lines above while ignoring the t derivative. Alternatively, we can simply recall Theorem 3.1 of [6]. By replacing ψ by $\psi + y + \frac{1}{2}\log(1+\psi'')$, which also satisfies (2.4), we can derive the following Carleman estimate.

Lemma 2.4 *Let $\chi(t) \in C_0^2(\mathbf{R})$. There exists a sufficiently large number β_3 with $\frac{\beta}{64} \in \mathbf{N} + \frac{1}{256}$ depending on n such that for all $w(t,x) \in C^1(\mathbf{R}; C_0^\infty(\mathbf{R}^n \setminus \{0\}))$ and $\beta \geq \beta_2$, we have that*

$$\begin{aligned}
&\int\int (1+\psi'')^2\varphi^2(|x|^2|\nabla(\chi w)|^2 + \beta^2|\chi w|^2)dxdt \\
\lesssim\ &\int\int (1+\psi'')\varphi^2|x|^4|\Delta(\chi w)|^2 dxdt.
\end{aligned} \tag{2.37}$$

3 Interior estimate

Due to the use of cut-off functions, in addition to Carleman estimates, we also need the following interior estimate.

Lemma 3.1 *Let* $\mathcal{X} = \{(t,x) : t_1 < t < t_2, x \in \omega \subset \Omega\}$ *and* (u,p) *be a solution of* (1.1). *Assume that*

$$\operatorname{diam} \mathcal{X} < 2$$

Denote $d(t,x)$ *the distance from* $(t,x) \in \mathcal{X}$ *to* $\mathbb{R}^{n+1} \backslash \mathcal{X}$. *Then for any* $0 < a_1 < a_2$ *such that* $\omega = B_{a_2 r} \setminus \bar{B}_{a_1 r}$, *we have that*

$$\begin{aligned} & \iint_{\mathcal{X}} d(t,x)^2 |\nabla u|^2 dxdt + \iint_{\mathcal{X}} d(t,x)^2 |q|^2 dxdt + \iint_{\mathcal{X}} d(t,x)^4 |\nabla q|^2 dxdt \\ \lesssim & \iint_{\mathcal{X}} |u|^2 dxdt. \end{aligned} \tag{3.1}$$

Proof. The method used here is motivated by the proof of Theorem 17.1.3 in [5]. We apply a suitable cut-off function on u. Take $\xi(X) \in C_0^\infty(\mathbb{R}^{n+1})$ satisfying $0 \le \xi(X) \le 1$ and

$$\xi(X) = \begin{cases} 1, & |X| < 1/4, \\ 0, & |X| \ge 1/2. \end{cases}$$

Let us denote $\xi_Y(Z) = \xi_Y(t,x) = \xi((Z-Y)/d(Y))$ for $Y \in \mathcal{X}$. Multiplying $\xi_Y^2(Z)u(Z)$ on (2.1) yields that

$$\begin{aligned} & \iint_{|Z-Y| \le d(Y)/4} |\nabla u|^2 dZ \\ \le & \ \delta \iint_{|Z-Y| \le d(Y)/2} d(Y)^2 |\nabla q|^2 dZ + \delta \iint_{|Z-Y| \le d(Y)/2} |\nabla u|^2 dZ \\ & + \frac{C_1}{\delta} \iint_{|Z-Y| \le d(Y)/2} d(Y)^{-2} |u|^2 dZ \end{aligned} \tag{3.2}$$

for some absolute constant C_1. Now multiplying $d(Y)^2$ on both sides of (3.2), we obtain

$$\begin{aligned} & \iint_{|Z-Y| \le d(Y)/4} d(Y)^2 |\nabla u|^2 dZ \\ \le & \ \delta \iint_{|Z-Y| \le d(Y)/2} d(Y)^4 |\nabla q|^2 dZ + \delta \iint_{|Z-Y| \le d(Y)/2} d(Y)^2 |\nabla u|^2 dZ \\ & + \frac{C_1}{\delta} \iint_{|Z-Y| \le d(Y)/2} |u|^2 dZ. \end{aligned} \tag{3.3}$$

Integrating $d(Y)^{-n-1}dY$ over $\mathcal{X}$ on both sides of (3.3) and using Fubini's Theorem, we get that

$$\begin{aligned}
&\iint_{\mathcal{X}}\iint_{|Z-Y|\le d(Y)/4} d(Y)^{1-n}|\nabla u|^2 dY dZ \\
\le\ & \delta\iint_{\mathcal{X}}\iint_{|Z-Y|\le d(Y)/2} d(Y)^{3-n}|\nabla q|^2 dY dZ \\
&+\delta\iint_{\mathcal{X}}\iint_{|Z-Y|\le d(Y)/2} d(Y)^{1-n}|\nabla u|^2 dY dZ \\
&+\frac{C_1}{\delta}\iint_{\mathcal{X}}\iint_{|Z-Y|\le d(Y)/2} d(Y)^{-1-n}|u|^2 dY dZ. \qquad (3.4)
\end{aligned}$$

Note that $|d(Z)-d(Y)|\le|Z-Y|$. If $|Z-Y|\le d(Z)/3$, then

$$2d(Z)/3\le d(Y)\le 4d(Z)/3. \qquad (3.5)$$

If $|Z-Y|\le d(Y)/2$, then

$$d(Z)/2\le d(Y)\le 3d(Z)/2. \qquad (3.6)$$

By (3.5) and (3.6), we have

$$\begin{cases}
\int_{|Z-Y|\le d(Y)/4} d(Y)^{-n-1}dY \ge (3/4)^{n+1}\int_{|Z-Y|\le d(Z)/6} d(Z)^{-n-1}dy \ge 8^{-n-1}\int_{|Y|\le 1} dY, \\
\int_{|Z-Y|\le d(Y)/2} d(Y)^{-n-1}dY \le 2^{n+1}\int_{|Z-Y|\le 3d(Z)/4} d(Z)^{-n-1}dy \le (3/2)^{n+1}\int_{|Y|\le 1} dY.
\end{cases} \qquad (3.7)$$

Combining (3.4)–(3.7) and taking δ small, we obtain

$$\iint_{\mathcal{X}} d(Z)^2|\nabla u|^2 dZ \le C_2\delta\iint_{\mathcal{X}} d(Z)^4|\nabla q|^2 dZ + \frac{C_2}{\delta}\iint_{\mathcal{X}}|u|^2 dZ, \qquad (3.8)$$

where C_2 is another absolute constant.

On the other hand, if we multiply $\xi_Y^2(Z)q(Z)$ on both sides of (2.2), we have that

$$\begin{aligned}
\iint_{|Z-Y|\le d(Y)/4}|\nabla q|^2 dZ \le\ & C_3\iint_{|Z-Y|\le d(Y)/2}|F|^2 dZ \\
&+\frac{1}{2}\iint_{|Z-Y|\le d(Y)/2}|\nabla q|^2 dZ \qquad (3.9)\\
&+C_3\iint_{|Z-Y|\le d(Y)/2} d(Y)^{-2}|q|^2 dZ
\end{aligned}$$

since $d(Y) < 1$ for all $Y \in \mathcal{X}$. Now multiplying $d(Y)^4$ on both sides of (3.9) and repeating the argument (3.4)–(3.8), we have for ν small that

$$\begin{aligned}
&\nu \iint_{\mathcal{X}} d(Z)^4 |\nabla q|^2 dZ \\
\le & C_4 \nu \iint_{\mathcal{X}} d(Z)^2 |\nabla u|^2 dZ + C_4 \nu \iint_{\mathcal{X}} |u|^2 dZ + C_4 \nu \iint_{\mathcal{X}} d(Z)^2 |q|^2 dZ \\
\le & C_5 \nu \iint_{\mathcal{X}} d(Z)^2 |\nabla u|^2 dZ + C_5 \nu \iint_{\mathcal{X}} |u|^2 dZ.
\end{aligned} \tag{3.10}$$

Taking $\nu = \frac{1}{2C_5}$ and $\delta = \frac{1}{4C_2C_5}$, we obtain

$$\begin{aligned}
&\iint_{\mathcal{X}} d(Z)^2 |\nabla u|^2 dZ + \iint_{\mathcal{X}} d(Z)^2 |q|^2 dZ + \iint_{\mathcal{X}} d(Z)^4 |\nabla q|^2 dZ \\
\le & C_6 \iint_{\mathcal{X}} |u|^2 dZ
\end{aligned}$$

by adding (3.8) and (3.10). □

4 Proofs of main results

This section is devoted to the proofs of Theorem 1.1, Corollary 1.3, and Theorem 1.2. We choose the cut-off function χ as follows

$$\chi(t) = \begin{cases} 1, & |t| \le T_4, \\ 0, & |t| \ge T_3, \\ \exp\left(-(\frac{T}{T_3 - |t|})^3 (\frac{|t| - T_4}{T_3 - T_4})^4 \right), & T_4 < |t| < T_3, \end{cases} \tag{4.1}$$

where $T_3 = T - \frac{t_0}{2}$, $T_4 = T - t_0$. Moreover, we let $\theta(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \theta(x) \le 1$ and

$$\theta(x) = \begin{cases} 0, & |x| \le \frac{R_1}{e}, \\ 1, & \frac{R_1}{2} < |x| < 2\tilde{R}_3, \\ 0, & |x| \ge 3\tilde{R}_3 \end{cases}$$

with $\tilde{R}_3 < 1$.

Applying (2.37) to θu gives

$$\begin{aligned}
&\int\int (1 + \psi'')^2 \varphi^2 (|x|^2 |\nabla(\chi\theta u)|^2 + \beta^2 |\chi\theta u|^2) dx dt \\
\lesssim & \int\int (1 + \psi'') \varphi^2 |x|^4 |\Delta(\chi\theta u)|^2 dx dt.
\end{aligned} \tag{4.2}$$

Here and after, C and $\tilde{C}$ denote general constants whose value may vary from line to line. The dependence of C and $\tilde{C}$ will be specified whenever necessary. Next applying (2.33) to $w = \theta q$ and $f = \chi|x|\theta F$ yields that

$$\begin{aligned} &k\tilde{C}\iint (1+\psi'')\varphi^2(|x|^4|\nabla(\chi\theta q)|^2 + \beta^2|x|^2|\chi\theta q|^2)dxdt \\ \le &k\iint \varphi^2|x|^2\big[\chi|x|^2\Delta(\theta q) - \chi|x|^2\partial_t(\theta q) + |x|\mathrm{div}(\chi|x|\theta F)\big]^2 dxdt \qquad (4.3) \\ &+ k\beta^2\iint \varphi^2|x|^2\big\||x|\chi\theta F\big\|^2 dxdt + k\iint \varphi^2|x|^6|\chi'\theta q|^2 dxdt, \end{aligned}$$

where k is chosen such that $k\tilde{C}\|\nabla q\| > 2C\|\mathrm{curl}^\top q\|$. Note that the choice of k is independent of q.

Observe from (2.3) that

$$|x||F_{ij}| \le n^2\lambda|x|^\epsilon|\nabla u| + n\lambda|x|^{-1+\epsilon}|u|.$$

Combining (4.2), (4.3) and using (2.1), (2.2), (2.5) (by choosing $\tilde{R}_3$ sufficiently small, if necessary) we obtain that

$$\begin{aligned} &\iint_W (1+\psi'')^2\varphi^2(|x|^2|\nabla(\chi u)|^2 + \beta^2|\chi u|^2)dxdt \\ &+ \iint_W (1+\psi'')\varphi^2(|x|^4|\nabla(\chi q)|^2 + \beta^2|x|^2|\chi q|^2)dxdt \qquad (4.4) \\ \lesssim &\iint_W \varphi^2|x|^6|\chi' q|^2 dxdt + C\beta^2\iint_{\tilde{Y}\cup\tilde{Z}} \varphi^2|U|^2 dxdt, \end{aligned}$$

where $W = \{(t,x) : |t| < T_3, \quad \frac{R_1}{2} < |x| < 2\tilde{R}_3\}$, $\tilde{Y} = \{(t,x) : |t| < T_3, \quad \frac{R_1}{e} \le |x| \le \frac{R_1}{2}\}$, $\tilde{Z} = \{(t,x) : |t| < T_3, \quad 2\tilde{R}_3 \le |x| \le 3\tilde{R}_3\}$ and $|U(x)|^2 = |x|^4|\nabla q|^2 + |x|^2|q|^2 + |x|^2|\nabla u|^2 + |u|^2$. We then obtain from (4.4) that

$$\begin{aligned} &\iint_W (1+\psi'')^2\varphi^2\beta^2|\chi u|^2 dxdt + \iint_W (1+\psi'')\varphi^2\beta^2|x|^2|\chi q|^2 dxdt \\ \le \quad &\tilde{J}_1 + C\beta^2\iint_{\tilde{Y}\cup\tilde{Z}} \varphi^2|U|^2 dxdt, \qquad (4.5) \end{aligned}$$

where

$$\tilde{J}_1 = C\iint_W \varphi^2|x|^6\left|\frac{\chi'}{\chi}\right|^2|\chi q|^2 dxdt.$$

Here we define $\frac{\chi'}{\chi} = 0$ whenever $\chi = 0$. To estimate $\tilde{J}_1$, we only need to consider the integrals over $W_1 = \{(t,x) : T_4 < |t| < T_3, \quad \frac{R_1}{2} < |x| < 2\tilde{R}_3\}$. To this end, we consider two cases. Firstly, with the help of (2.5), we have

$$C\left|\frac{\chi'}{\chi}\right|^2 \le \frac{\beta^3}{4}|x|^{-3} \le \frac{(1+\psi'')\beta^2}{4}|x|^{-4}.$$

In this case, $\tilde{J}_1$ can be absorbed by the left hand side. On the other hand, we consider the case

$$C|\frac{\chi'}{\chi}|^2 \geq \frac{\beta^3}{4}|x|^{-3}.$$

Since

$$\sqrt{C}|\frac{\chi'}{\chi}| \leq \tilde{C}\frac{T^3}{(T_3-|t|)^4},$$

it suffices to consider the set

$$\tilde{C}\frac{T^3}{(T_3-|t|)^4} \geq \left(\frac{\beta^3}{4|x|^3}\right)^{1/2}. \tag{4.6}$$

Consequently, taking $\beta \geq \beta_3$ with $\beta_3 = \left(\frac{\tilde{C}^2 4^9 T^6}{t_0^8}\right)^{1/3} \geq \left(\frac{\tilde{C}^2 4^9 \tilde{R}_3^3 T^6}{t_0^8}\right)^{1/3}$, we can get from (4.6) that

$$|T_3 - |t|| \leq t_0/4,$$

which implies that

$$|t| - T_4 \geq \frac{T_3 - T_4}{2}. \tag{4.7}$$

Combining (4.1), (4.6) and (4.7), we get for $(t,x) \in W_1$ that

$$\chi(t) \leq \exp\left(-\frac{1}{16}\left(\frac{\beta^3}{4|x|^3\tilde{C}^2T^2}\right)^{3/8}\right). \tag{4.8}$$

Thus, from (4.8) and (3.1) we have that for $\beta \geq \beta_3$

$$\tilde{J}_1 \leq C\varphi^2(2\tilde{R}_3)\iint_{W_1}|x|^2|q|^2dxdt \leq C_1\varphi^2(2\tilde{R}_3)\iint_{Q^{0,T}_{0,3\tilde{R}_3}}|u|^2dxdt, \tag{4.9}$$

where C_1 is a positive constant depending on $\varepsilon, \lambda, n, T, t_0$. Putting together (4.5) and (4.9) implies that for $R_1 < R_2 < \tilde{R}_3$

$$\begin{aligned} &\beta^2\varphi^2(R_2)\iint_{W_2}|u|^2dxdt \leq \iint_W (1+\psi'')\varphi^2\beta^2|u|^2dxdt \\ &\lesssim \tilde{J}_1 + \beta^2\iint_{\tilde{Y}\cup\tilde{Z}}\varphi^2|U|^2dxdt \\ &\lesssim \beta^2\varphi^2(\frac{R_1}{e})\iint_{Q^{0,T}_{0,R_1}}|u|^2dxdt + \beta^2\varphi^2(2\tilde{R}_3)\iint_{Q^{0,T}_{0,3\tilde{R}_3}}|u|^2dxdt, \end{aligned} \tag{4.10}$$

where $W_2 = \{(t,x) : |t| < T - t_0, \quad \frac{R_1}{2} < |x| < R_2\}$.

Dividing $\beta^2\varphi_\beta^2(R_2)$ on both sides of (4.10) and by (2.4), we have that

$$\begin{aligned}\iint_{W_2}|u|^2dxdt &\lesssim e^{2\psi(-\log(\frac{R_1}{e}))-2\psi(-\log R_2)}\iint_{Q^{0,T}_{0,R_1}}|u|^2dxdt\\ &\quad + e^{2\psi(-\log(2\tilde{R}_3)-2\psi(-\log R_2)}\iint_{Q^{0,T}_{0,3\tilde{R}_3}}|u|^2dxdt\\ &\lesssim e^{4\beta(\log 4R_2-\log R_1)}\iint_{Q^{0,T}_{0,R_1}}|u|^2dxdt\\ &\quad + e^{-\beta(\log(2\tilde{R}_3)-\log R_2)}\iint_{Q^{0,T}_{0,3\tilde{R}_3}}|u|^2dxdt.\end{aligned} \tag{4.11}$$

Adding $\iint_{Q^{0,T-t_0}_{0,R_1/2}}|w|^2dxdt$ to both sides of (4.11) leads to

$$\begin{aligned}\iint_{Q^{0,T-t_0}_{0,R_2}}|u|^2dxdt &\lesssim e^{4\beta\log(\frac{4R_2}{R_1})}\iint_{Q^{0,T}_{0,R_1}}|u|^2dxdt\\ &\quad + e^{-\beta\log(\frac{2R_3}{3R_2})}\iint_{Q^{0,T}_{0,R_3}}|u|^2dxdt,\end{aligned} \tag{4.12}$$

where $R_3 = 3\tilde{R}_3$. Standard arguments give

$$\iint_{Q^{0,T-t_0}_{0,R_2}}|u|^2dxdt \le \tilde{C}\left(\iint_{Q^{0,T}_{0,R_1}}|u|^2dxdt\right)^\kappa\left(\iint_{Q^{0,T}_{0,R_3}}|u|^2dxdt\right)^{1-\kappa}, \tag{4.13}$$

where $\tilde{C}$ depends on $\lambda, \epsilon, n, T, t_0, \frac{R_2}{R_3}$, and

$$\kappa = \frac{\log(\frac{2R_3}{3R_2})}{4\log(\frac{4R_2}{R_1})+\log(\frac{2R_3}{3R_2})}.$$

Hence, Theorem 1.1 is proved.

We now turn to the proof of Theorem 1.2. Let us recall (4.12), i.e.,

$$\begin{aligned}\iint_{Q^{0,T-t_0}_{0,R_2}}|u|^2dxdt &\lesssim e^{4\beta\log(\frac{4R_2}{R_1})}\iint_{Q^{0,T}_{0,R_1}}|u|^2dxdt\\ &\quad + e^{-\beta\log(\frac{2R_3}{3R_2})}\iint_{Q^{0,T}_{0,R_3}}|u|^2dxdt.\end{aligned}$$

We choose β large enough such that

$$e^{\beta\log(\frac{2R_3}{3R_2})} \ge \frac{2\iint_{Q^{0,T}_{0,R_3}}|u|^2dxdt}{\iint_{Q^{0,T-t_0}_{0,R_2}}|u|^2dxdt}. \tag{4.14}$$

Indeed, if $3R_2 \le R_3$, we can take

$$\beta = m_1 := 64 + \frac{1}{2} + 64\left[\log\Big(\frac{2\iint_{Q^{0,T}_{0,R_3}} |u|^2 dxdt}{\iint_{Q^{0,T-t_0}_{0,\frac{R_3}{2}}} |u|^2 dxdt}\Big)\right],$$

where $[x]$ is the largest integer small than x. Combining (4.12) and (4.14), we immediately obtain

$$\iint_{Q^{0,T-t_0}_{0,R_2}} |u|^2 dxdt \lesssim 2e^{4(\log \frac{4R_2}{R_1})m_1} \iint_{Q^{0,T}_{0,R_1}} |u|^2 dxdt,$$

which is (1.4). This ends the proof of Theorem 1.2.

Finally, we would like to prove Corollary 1.3. We fix R_2, T, t_0 and R_3 in Theorem 1.1 and define

$$\widetilde{u}(x) := u(x)\Big/\sqrt{\iint_{Q^{0,T-t_0}_{0,R_2}} |u|^2 dxdt}.$$

From (4.13), we have that

$$1 \le \tilde{C}\left(\iint_{Q^{0,T}_{0,R_1}} |\widetilde{u}|^2 dxdt\right)^{\kappa}\left(\iint_{Q^{0,T}_{0,R_3}} |\widetilde{u}|^2 dxdt\right)^{1-\kappa}. \tag{4.15}$$

Raising both sides of (4.15) by $1/\kappa$ yields that

$$\iint_{Q^{0,T}_{0,R_3}} |\widetilde{u}|^2 dxdt \le \left(\iint_{Q^{0,T}_{0,R_1}} |\widetilde{u}|^2 dxdt\right)\left(\tilde{C}\iint_{Q^{0,T}_{0,R_3}} |\widetilde{u}|^2 dxdt\right)^{1/\kappa}. \tag{4.16}$$

In view of the formula for κ, it follows from (4.16) that

$$\iint_{Q^{0,T}_{0,R_3}} |\widetilde{u}|^2 dxdt \le \left(\iint_{Q^{0,T}_{0,R_1}} |\widetilde{u}|^2 dxdt\right)\left(\frac{1}{R_1}\right)^{C_1 + C_1\log(\iint_{Q^{0,T}_{0,R_3}} |\widetilde{u}|^2 dxdt)}, \tag{4.17}$$

where C_1 is a positive constant depending on $n, \epsilon, \lambda, T, t_0$ and R_2/R_3. Consequently, (4.17) is equivalent to

$$R_1^m \iint_{Q^{0,T}_{0,R_3}} |u|^2 dxdt \le \iint_{Q^{0,T}_{0,R_1}} |u|^2 dxdt$$

for all R_1 sufficiently small, where

$$m = C_1 + C_1 \log\left(\frac{\iint_{Q^{0,T}_{0,R_3}} |u|^2 dxdt}{\iint_{Q^{0,T-t_0}_{0,R_2}} |u|^2 dxdt}\right).$$

The proof of Corollary 1.3 is now completed. $\square$

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References

[1] M. Choulli, O.Y. Imanuvilov, and M. Yamamoto, *Inverse Source Problem for the Navier-Stokes Equations* UTMS 2006-3 (Tokyo, Gradute School of Mathematical Sciences, The University of Tokyo), http://kyokan.ms.u-tokyo.ac.jp/users/preprint/pdf/2006-3.pdf

[2] L. Escauriaza and S. Vessella, *Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients*, (Inverse Problems Theory and Applications Contemporary Mathematics vol 333, 2003), ed. G Alessandrini and G Uhlmann (Providence, RI: American Mathematical Society), 79-87.

[3] C. Fabre and G. Lebeau *Prolongement unique des solutions de l'equation de Stokes*, Comm. Part. Diff. Eqns., **21** (1996), 573-596.

[4] L. Hörmander, *Uniqueness theorems for second order elliptic differential equations*, Comm. Part. Diff. Eqns. **8** (1983), 21-64.

[5] L. Hörmander, "The Analysis of Linear Partial Differential Operators", Vol. 3, Springer-Verlag, Berlin/New York, 1985.

[6] H. Koch, C.-L. Lin and J.-N. Wang, *Doubling inequalities for the Lame system with rough coefficients*, Proceedings of the A.M.S., **144**, No.12 (2016), 5309-5318.

[7] C.-L. Lin, G. Uhlmann, and J.-N. Wang, *Optimal three-ball inequalities and quantitative uniqueness for the Stokes system*, Discrete Contin. Dyn. Syst., **28** (2010), 1273-1290.

[8] C.-L. Lin, G. Uhlmann, and J.-N. Wang, *Asymptotic behavior of solutions of the stationary Navier-Stokes equations in an exterior domain.* Indiana Univ. Math. J., **60** (2011), 2093-2106.

[9] C.-L. Lin, G. Nakamura, and J.-N. Wang, *Quantitative uniqueness for second order elliptic operators with strongly singular coefficients*, Rev. Mat. Iberoam., **27** (2011), 475-491.

[10] M. Mitrea and S. Monniaux, *Maximal regularity for the Lamé system in certain classes of non-smooth domains*, J. Evol. Equ., DOI 10.1007/s00028-010-0071-1.

[11] R. Regbaoui, *Strong uniqueness for second order differential operators*, J. Differential Equations, **141** (1997), 201-217.

[12] R. Regbaoui, *Strong unique continuation for stokes equations*, Comm. Part. Diff. Eqns., **24** (1999), 1891-1902.

[13] S. Vessella, *Carleman estimates, optimal three cylinder inequality and unique continuation properties for solutions to parabolic equations*, Commun. Part. Diff. Eqns., **28**(2003), 637-676.

[14] S. Vessella, *Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates*, Inverse Problems, **24**(2008) 023001 (81pp).

[15] M. Yamamoto, *Carleman estimates for parabolic equations and applications*, Inverse Problems, **25** (2009) 123013 (75pp).