SIZE ESTIMATES FOR THE WEIGHTED p-LAPLACE EQUATION WITH ONE MEASUREMENT

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Abstract. In this work, we are concerned with the problem of estimating the size of an inclusion embedded in an object laying in the two dimensional domain. We assume that the object is occupied by an exotic material which obeys a nonlinear Ohms' law. In view of the assumption of the power law, we thus consider the weighted p-Laplace equation as a model problem in this case. Using only one voltage-current measurement, we give upper and lower bounds of the size of the inclusion.

1. Introduction

We study the size estimate problem for the non-linear p-Laplace type equation in the plane. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega$. The regularity of $\partial\Omega$ will be specified later. Suppose that Ω is occupied by a special material which obeys a nonlinear Ohms' law. The usual Ohms' law states that the current density $I(x) = -a(x)\nabla v$, where v is the electric potential and $a(x)$ is the conductivity of the material. Here we assume that the conductivity a also depends on ∇v according to the following power law

$$a(x) = \gamma(x)|\nabla v|^{p-2},$$

i.e.,

$$I(x) = -\gamma(x)|\nabla v|^{p-2}\nabla v$$

for $1 < p < \infty$, where $\gamma(x) \in L^\infty_+(\Omega)$, an essentially bounded function that is positive almost everywhere. Such power laws can occur in dielectrics, plastic moulding, electro-rheological and thermo-rheological fluids, viscous flows in glaciology and plasticity phenomena, etc. We refer to [13] for related references. In the absence of sources in Ω, the potential v satisfies the equilibrium equation

$$\operatorname{div}(\gamma(x)|\nabla v|^{p-2}\nabla v) = 0 \quad \text{in} \quad \Omega.$$

This is the well known p-Laplace equation with weight γ for all $1 < p < \infty$.

Wang is supported in part by MOST 105-2115-M-002-014-MY3.

Let $D \subset \Omega$ be a subdomain of Ω, where $\Omega \setminus \bar{D}$ is connected. The region D represents an inclusion whose medium parameter is different from the background one. In other words, we can assume that the medium parameter $\gamma(x)$ is distributed as follows

$$\gamma(x) = \begin{cases} \sigma(x) & \text{when } x \in \Omega \setminus \overline{D}, \\ \tilde{\sigma}(x) & \text{when } x \in D. \end{cases}$$

In this work we are concerned with the estimate of the size of D by one voltage-current pair on the boundary $\partial\Omega$. Namely, we would like to derive upper and lower bounds of $|D|$ using $\{v|_{\partial\Omega}, \gamma(x)|\nabla v|^{p-2}\nabla v \cdot \nu|_{\partial\Omega}\}$, where ν is the unit outer normal of $\partial\Omega$. In practice, this problem can be considered as a preliminary assessment of the size of the abnormality inside of Ω.

The size estimate problem for linear equations or systems has been extensively studied. We refer to the nice survey article [5] for some early results and to the recent paper [17] for the case where the background medium is discontinuous. The idea is to use the power gap to derive upper and lower bounds of $|D|$. To do so, let us consider the Dirichlet boundary value problem

$$\begin{cases} \operatorname{div}(\gamma(x)|\nabla v|^{p-2}\nabla v) = 0 & \text{in} \quad \Omega, \\ v = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Under the assumption γ, for a given Dirichlet data $f \in W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$, the problem (1.1) is well posed in $W^{1,p}(\Omega)$. The power

$$W = \int_{\partial\Omega} f(x)\gamma(x)|\nabla v|^{p-2}\nabla v \cdot \nu ds = \int_{\Omega} \gamma(x)|\nabla v|^p dx$$

is the energy that needs to maintain the voltage f on $\partial\Omega$. Likewise, we also consider the unperturbed equation, i.e., D is empty,

$$\begin{cases} \operatorname{div}(\sigma(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.2}$$

and the associated power

$$W_0 = \int_{\partial\Omega} f(x)\sigma(x)|\nabla u|^{p-2}\nabla u \cdot \nu ds = \int_{\Omega} \sigma(x)|\nabla u|^p dx.$$

Our main result in this paper is to estimate the size of D in terms of the "normalized power gap". Precisely, we plan to prove the following estimate

$$C_1 \left| \frac{W - W_0}{W_0} \right| \le |D| \le C_2 \left| \frac{W - W_0}{W_0} \right|^{1/q}, \tag{1.3}$$

where $q > 1$, which appears from the fact that $|\nabla u|^p$ is an A_q weight. The constants C_1 and C_2 depend on the apriori data. If we assume that D satisfies the fatness condition (see (4.8)), then we can obtain the following estimate

$$C_1 \left| \frac{W - W_0}{W_0} \right| \leq |D| \leq \widetilde{C}_2 \left| \frac{W - W_0}{W_0} \right|. \tag{1.4}$$

To prove (1.3) and (1.4), we first derive energy inequalities connecting the power gap and the energy of the free solution u (see (4.3)). With the help of the energy inequalities, lower bounds in (1.3) and (1.4) are consequences of the interior estimate for the solution of (1.2). Derivations of upper bounds in (1.3) and (1.4) rely on some quantitative unique continuation estimates for (1.2) with constants depending on a priori data.

The unique continuation property for the p-Laplace equation in higher dimensions ($n \geq 3$) is largely an open problem (see [19] for some partial results). On the contrary, when $n = 2$, the problem is much more manageable due to the intimate connection between the solutions of the p-Laplace equation and quasiregular mappings, see [2], [9], [25] for some qualitative results and [20] for related quantitative estimates. On the other hand, we refer to [11], [12], [13], [14], [21], [28] for some recent results on inverse problems for the p-Laplace equation.

This paper is organized as follows. In Section 2, we list the assumptions used throughout the paper. We also recall some useful estimates of quasiregular mappings. Based on these estimates, we derive doubling and three-ball inequalities for quasiregular mappings. In Section 3, we prove the Lipschitz propagation of smallness and doubling inequalities for solutions of the weighted p-Laplacian. Finally, in Section 4, we state and prove lower and upper bounds of $|D|$ in terms of the normalized power gap with or without the fatness condition.

2. Assumptions and preliminaries

2.1. **Assumptions.** The following assumptions will be needed throughout the paper. We first assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with $C^{1,\alpha}$ boundary for some $\alpha \in (0, 1)$ with parameters r_0, M_0. We say a domain is $C^{1,\alpha}$ regular if for any $x \in \partial\Omega$, there exists a $C^{1,\alpha}$-regular transformation ψ satisfying $\psi(0) = 0, \nabla\psi(0) = 0$ and $\|\psi\|_{C^{1,\alpha}(-r_0,r_0)} \leq M_0$ so that

$$\Omega \cap B_{r_0}(0) = \{x = (x_1, x_2) \in B_{r_0}(0) : x_2 > \psi(x_1)\}.$$

Unless otherwise stated, we denote $z = x_1 + ix_2, x = (x_1, x_2)$ and $B_r(x)$, the disc of radius r centered at x. We assume that σ is Lipschitz, i.e., there exists $M > 0$ such that

$$\|\sigma\|_{C^{0,1}(\overline{\Omega})} \leq M$$

and for $\lambda \geq 1$

$$\frac{1}{\lambda} \leq \sigma(x) \leq \lambda, \quad \forall\, x \in \Omega. \tag{2.1}$$

Also, we use the usual differential operators

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

We assume that the Dirichlet condition f is zero on some part of the boundary $\partial\Omega$. Precisely, let Γ be a subset of $\partial\Omega$ with positive measure. Suppose that

$$f = 0 \quad \text{on} \quad \Gamma. \tag{2.2}$$

Finally, we assume there

$$\operatorname{dist}(D, \partial\Omega) > d$$

with $0 < d < 1$. Throughout the paper, we denote for $\rho > 0$

$$\Omega_\rho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \rho\}.$$

2.2. **Quasiregular mapping and some properties.** A mapping $\phi \in W^{1,2}_{loc}(\Omega; \mathbb{R}^2)$ is said to be a K-quasiregular mapping if

$$\|D\phi(x)\|^2 \leq K|J_\phi(x)|, \ \forall\, x \in \Omega,$$

where, $K \geq 1$ is a constant, J_ϕ is the Jacobian determinant, $D\phi$ is the derivative of ϕ and $\|\cdot\|$ denotes the usual Euclidean norm of the matrix. For more detailed discussion on quasiregular mappings, we refer to [6].

In this subsection, we will recollect some properties of the quasiregular mappings and give sketchy proofs of some of the results if necessary. We mainly recall Hadamard's three circle theorem, Harnack type inequality and doubling inequality for the quasiregular mapping. We begin with Hadamard's three circle theorem whose proof can be found in [3, Theorem 3.9] or [20, (3.8)]. From now on, we identify $\mathbb{R}^2 = \mathbb{C}$ and write ϕ as a complex-valued function.

Lemma 2.1. (Hadamard's three circle theorem) *Let $\phi : B_{r_3}(x) \to \mathbb{C}$ be a K-quasiregular mapping. Then for all $0 < r_1 < r_2 < r_3$, there exists a constant $0 < \theta < 1$, depending only on K, r_3/r_1, r_3/r_2, such that*

$$\|\phi\|_{L^\infty(B_{r_2}(x))} \leq \|\phi\|^{\theta}_{L^\infty(B_{r_1}(x))} \|\phi\|^{1-\theta}_{L^\infty(B_{r_3}(x))}.$$

We also recall a version of Harnack's inequality for the quasiregular mapping from [20, Lemma 4.1].

Lemma 2.2. (Harnack's inequality) *If $\phi : B_1(x) \to \mathbb{C}$ is a K-quasiregular mapping, then for each $r \in (0,1)$, we have*

$$\|\phi\|_{L^\infty(B_{r/2})} \le C\frac{1}{|B_r|}\int_{B_r} |\phi| dx,$$

where the constant $C > 0$ depends on K.

Lemma 2.3. (Doubling inequality) *Let $\phi : \Omega \to \mathbb{C}$ be a K-quasiregular mapping. Given any $\rho > 0$, there exist positive constants $\delta = \delta(\rho) \in (0,\rho)$ and $C = C(\rho)$ such that, for all $x \in \Omega_\rho$ and $r \in (0,\delta)$, we have*

$$\frac{\|\phi\|_{L^\infty(B_{4r}(x))}}{\|\phi\|_{L^\infty(B_{r/2}(x))}} \le C\frac{\|\phi\|_{L^\infty(B_{\rho_2}(x))}}{\|\phi\|_{L^\infty(B_{\rho_1}(x))}}$$

where $\rho_1, \rho_2 > 0$ explicitly depend on ρ and δ but not on r.

Proof. The proof follows the similar arguments used in [27]. Since $\phi : \Omega \to \mathbb{C}$ is K-quasiregular mapping, by the Ahlfors-Bers representation [1] (see also [8]), we have that

$$\phi = h \circ \chi,$$

where $h : \chi(\Omega) \to \mathbb{C}$ is holomorphic, and $\chi : \Omega \to \chi(\Omega)$ is a K-quasiconformal homeomorphism. Moreover, χ satisfies the bi-Lipschitz property, i.e., there exist $\tilde{M}, \beta > 1$ depending on K such that

$$\tilde{M}^{-1}|x-y|^\beta \le |\chi(x) - \chi(y)| \le \tilde{M}|x-y|^{1/\beta}, \ \forall\, x, y \in \Omega.$$

Choosing $\delta = (10\tilde{M}^2)^{-\beta}\rho^{\beta^2}$ and $R = (10\tilde{M})^{-1}\rho^\beta$, we have $\chi(B_\delta) \subset B_R(\chi(x))$ and $B_{10R}(\chi(x)) \subset \Omega$.

We note that, since χ is K-quasiconformal in Ω, so due to Theorem 3.6.2, [6], there exists an increasing function η depending only on K with $\eta(0) = 0$ such that if $x_1, x_2, x_3 \in B_\delta(x)$ then

$$\frac{|\chi(x_1) - \chi(x_2)|}{|\chi(x_1) - \chi(x_3)|} \le \eta\left(\frac{|x_1 - x_2|}{|x_1 - x_3|}\right).$$

Letting $c = \eta(8) > 1$, then for any $x \in \Omega_\rho$ and $r \in (0,\delta)$, there exists $s \in (0, R/c)$ such that if $y = \chi(x)$ then

$$B_s(y) \subset \chi(B_{r/2}(x)) \text{ and } \chi(B_{4r}(x)) \subset B_{cs}(y). \tag{2.3}$$

By the Hadamard's three circles theorem for the holomorphic function, see [26], there exists an absolute positive constant C such that

$$\frac{\|h\|_{L^\infty(B_{cs}(y))}}{\|h\|_{L^\infty(B_s(y))}} \le C\frac{\|h\|_{L^\infty(B_{4R}(y))}}{\|h\|_{L^\infty(B_{3R}(y))}}. \tag{2.4}$$

Finally, using the fact that, measure zero sets map to a measure zero set via the quasiconformal mapping, see (Theorem 3.1.2, [6]) and combining (2.3) and (2.4), we obtain

$$\frac{\|\phi\|_{L^\infty(B_{4r}(x))}}{\|\phi\|_{L^\infty(B_{r/2}(x))}} \leq C\frac{\|\phi\|_{L^\infty(B_{\rho_2}(x))}}{\|\phi\|_{L^\infty(B_{\rho_1}(x))}},$$

where

$$\rho_1 = (3R/\tilde{M})^\beta = 3^\beta \delta > \delta,$$

and

$$\rho_2 = (4\tilde{M}R)^{1/\beta} = (2/5)^{1/\beta}\rho < \rho.$$

□

Lemma 2.4. *Let $\phi : B_{r_3} \to \mathbb{C}$ be a K-quasiregular mapping. Then for all $0 < r_1 < r_2 < r_3$, there exists $C > 0$ depending on $\frac{r_1}{r_3}, \frac{r_2}{r_3}, K$ such that,*

$$\|\phi\|_{L^\infty(B_{r_2})} \leq C r_3^{-1} \|\phi\|^\theta_{L^2(B_{2r_1})} \|\phi\|^{1-\theta}_{L^2(B_{2r_3})},$$

where θ is given in Lemma 2.1.

Proof. From Lemma 2.1, we have

$$\|\phi\|_{L^\infty(B_{r_2})} \leq \|\phi\|^\theta_{L^\infty(B_{r_1})} \|\phi\|^{1-\theta}_{L^\infty(B_{r_3})}. \tag{2.5}$$

Applying Harnack's inequality (see Lemma 2.2) and Hölder's inequality, we deduce that

$$\begin{aligned}\|\phi\|_{L^\infty(B_{r_1})} &\leq C\frac{1}{|B_{2r_1}|}\int_{B_{2r_1}} |\phi|\,dx \\ &\leq C\,|B_{2r_1}|^{-1/2}\,\|\phi\|_{L^2(B_{2r_1})}\end{aligned} \tag{2.6}$$

Hence combining (2.5) and (2.6) we obtain our required result. □

2.3. **Quasilinear Beltrami equation.** In this section, we are aiming at transforming the weighted p-Laplace equation into the Beltrami equation in the planar domain. To be more precise, the nonlinear counterpart of the gradient of the solution of p-Laplace satisfies a certain kind of quasilinear Beltrami equation. Here, we mainly follow some computations from [21, Appendix A3] to deduce the following Beltrami equation in the complex plane. Let $G = \sigma u_{x_1} - i\sigma u_{x_2}$, where u solves (1.2) and define $F = |G|^{\frac{p-2}{2}} G$. Then F satisfies

$$\frac{\partial F}{\partial \bar{z}} = q_1 \frac{\partial F}{\partial z} + q_2 \overline{\frac{\partial F}{\partial z}} + q_3 F, \text{ in } \Omega, \tag{2.7}$$

where

$$q_1 = -\frac{1}{2}\left(\frac{p-2}{p+2} + \frac{p-2}{3p-2}\right)\frac{\overline{F}}{F},$$
$$q_2 = -\frac{1}{2}\left(\frac{p-2}{3p-2} - \frac{p-2}{p+2}\right)\frac{F}{\overline{F}}$$

and

$$\begin{aligned} q_3 =& \sigma\frac{p}{p+2}\left[\frac{\overline{F}}{F}\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right) - \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma}\right)\right] \\ &- \sigma^{p-2}\frac{p}{3p-2}\left[\frac{\overline{F}}{F}\frac{\partial}{\partial z}\left(\frac{1}{\sigma^{p-2}}\right) + \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma^{p-2}}\right)\right]. \end{aligned}$$

Since $\sigma \geq \frac{1}{\lambda}$ and σ is Lipschitz in Ω, it is easy to check that $\kappa = \|q_1\|_{L^\infty(\Omega)} + \|q_2\|_{L^\infty(\Omega)} < 1$ and $\|q_3\|_{L^\infty(\Omega)} \leq M$.

We now state the following proposition, which allows one to represent the solution of the Beltrami equation (2.7) in terms of the quasiconformal maps and holomorphic functions. The proof can be found in ([20], Theorem 3.1). See also ([10], Theorem 4.4).

Proposition 2.5. *Under the above stated assumptions on q_1, q_2, q_3 and σ, the solution of the Beltrami equation (2.7) has the following representation:*

$$F(z) = (h \circ \chi)(x)e^{\omega(x)} \quad in\ \Omega$$

where $\chi : \Omega \to \chi(\Omega)$ is K-quasiconformal (with K depending only on p), $h : \chi(\Omega) \to \mathbb{C}$ is holomorphic, and $\omega(x) = (Tg)(x)$ is the Cauchy transform of g for some $g \in L^\delta(\Omega)$ with $2 < \delta < (1 + \frac{1}{\kappa})$.

Remark 2.6. Here the function g satisfies the integral equation

$$g - qSg = \chi_\Omega q_3 \quad \text{in} \quad \mathbb{C},$$

where $\|q\|_{L^\infty(\mathbb{C})} \leq \kappa$ and S is the Beurling transform. It follows from [7, Theorem 1] that $I - qS$ is invertible in $L^\delta(\mathbb{C})$ if $\delta \in (2, 1 + \frac{1}{\kappa})$. In view of (2.7), we then obtain that

$$\|g\|_{L^\delta(\Omega)} \leq C(\delta, \lambda, p, \Omega)M. \tag{2.8}$$

3. Propagation of smallness and doubling inequality

We begin this section with the following three-ball inequality.

Lemma 3.1. *(three-ball inequality) Let u be the solution of the weighted p-Laplace equation (1.2) in B_{R_0}. For all $0 < 2r_1 < r_2 < 2r_3 < R_0$, there exist constant C, depending on $\frac{r_1}{r_3}, \frac{r_2}{r_3}, p, R_0, \lambda, M$, such that*

$$\|\nabla u\|_{L^p(B_{r_2})} \leq Cr_3^{-1} \|\nabla u\|^{\theta}_{L^p(B_{2r_1})} \|\nabla u\|^{1-\theta}_{L^p(B_{2r_3})},$$

where θ is given in Lemma 2.1.

Proof. As above, we define $F=|G|^{\frac{p-2}{2}}G$, with $G=\sigma u_{x_1}-i\sigma u_{x_2}$. Then Proposition 2.5 asserts that

$$F(x)=(h\circ\chi)(x)e^{T(g(x))}, \tag{3.1}$$

where $\chi : B_{R_0}(x)\to\chi(B_{R_0}(x))$ is a K-quasiconformal map, $h:\chi(B_{R_0}(x))\to\mathbb{C}$ is holomorphic and $T(g)$ is the Cauchy transform of g for some $g\in L^\delta(B_{R_0}(x))$ with $2<\delta<1+\frac{1}{\kappa}$ satisfying $\|g\|_{L^\delta(B_{R_0})}\le CM$, where $C=C(R_0)$ with any fixed δ. Recall that the Cauchy transform $T:L^\delta(B_{R_0})\to L^\infty(B_{R_0})$. Therefore, we have that

$$\|e^{T(g)}\|_{L^\infty(B_{R_0})}\le Ce^{CM}. \tag{3.2}$$

$$\begin{aligned}
\int_{B_{r_2}(x)}|\nabla u|^p &\le C\int_{B_{r_2}(x)}|F|^2\\
&\le C\int_{B_{r_2}(x)}|h\circ\chi|^2\,e^{2|T(g(x))|}\\
&\le C\,\|h\circ\chi\|^2_{L^\infty(B_{r_2}(x))}\int_{B_{r_2}(x)}e^{2|T(g(x))|}\\
&\le Ce^{CM}\,\|h\circ\chi\|^2_{L^\infty(B_{r_2}(x))}\,.
\end{aligned}$$

Replacing ϕ by $h\circ\chi$ in Lemma 2.4, we obtain that

$$\|\nabla u\|^p_{L^p(B_{r_2}(x))}\le Ce^{CM}r_3^{-1}\,\|h\circ\chi\|^{2\theta}_{L^2(B_{2r_1}(x))}\,\|h\circ\chi\|^{2(1-\theta)}_{L^2(B_{2r_3}(x))}\,. \tag{3.3}$$

Now, we compute

$$\begin{aligned}
\|h\circ\chi\|^2_{L^2(B_{2r_1(x)})} &= \int_{B_{2r_1}}|h\circ\chi|^2=\int_{B_{2r_1}}|F|^2\,e^{-2T(g(x))}\\
&\le C\int_{B_{2r_1}}|\nabla u|^p\,e^{-2T(g(x))}\\
&\le Ce^{CM}\int_{B_{2r_1}}|\nabla u|^p\,.
\end{aligned}\tag{3.4}$$

Similar computations hold for $\|h\circ\chi\|^2_{L^2(B_{2r_3}(x))}$ and Lemma follows. $\square$

We now will derive two key estimates. The first important estimate is the Lipschitz propagation of smallness, where we show that the L^p norm of the gradient of the solution for p-Laplace over the whole domain can be controlled by the L^p norm of the gradient over a smaller subdomain in a Lipschitz manner. Precisely, we will prove the following lemma.

Lemma 3.2. *Assume that the assumptions in* Section 2 *hold. Let* $u \in W^{1,p}(\Omega)$ *be the solution for the p-Laplace equation* (1.2) *with* $2 < p < \infty$. *Then there exists* ρ_0, *depending on* $|\Omega|, \alpha, p, r_0, M_0, \Gamma$ *and* $C(\alpha, \lambda, \Omega, p, \|f\|_{C^{1,\alpha}(\partial\Omega)})/\|f\|_{W^{1-1/p,p}(\partial\Omega)}$, *such that for all* $\rho \le \rho_0$ *and for every* $x \in \Omega_{8\rho}$, *we have*

$$\int_\Omega |\nabla u|^p dx \le C_\rho \int_{B_\rho(x)} |\nabla u|^p dx,$$

where the constant $C_\rho > 0$ *depends on* $\lambda, M, p, |\Omega|$ *and* ρ.

Proof. The proof of this lemma goes along the same line as [5]. For any $x \in \Omega_{8\rho}$, choosing $r_1 = \frac{\rho}{2}, r_2 = 3\rho, r_3 = 4\rho$ in the Lemma 3.1, we have

$$\|\nabla u\|_{L^p(B_{3\rho}(x))} \le C \|\nabla u\|^\theta_{L^p(B_\rho(x))} \|\nabla u\|^{1-\theta}_{L^p(B_{8\rho}(x))}. \tag{3.5}$$

Here C depends on ρ, p, λ, M. Let us first take two points x, y in $B_{8\rho}$. Then join x and y with a curve $\tilde\gamma$ as follows: Let $x_1 = x$. For $k > 1$, let $x_k = \tilde\gamma(t_k)$, where $t_k = \max\{t : |\tilde\gamma(t) - x_{k-1}| = 2\rho\}$ if $|x_k - y| > 2\rho$; otherwise let $x_k = y$, $N = k$ and stop the process. Remark that the balls $B_\rho(x_k)$ are disjoint and $N \le N_0 = \frac{|\Omega|}{\pi\rho^2}$. Also note that

$$B_\rho(x_{k+1}) \subset B_{3\rho}(x_k)$$

since $|x_{k+1} - x_k| \le 2\rho$. Therefore, using (3.5), we deduce that

$$\frac{\|\nabla u\|_{L^p(B_\rho(x_{k+1}))}}{\|\nabla u\|_{L^p(\Omega)}} \le C \left(\frac{\|\nabla u\|_{L^p(B_\rho(x_k))}}{\|\nabla u\|_{L^p(\Omega)}} \right)^\theta.$$

By induction we obtain

$$\frac{\|\nabla u\|_{L^p(B_\rho(y))}}{\|\nabla u\|_{L^p(\Omega)}} \le C^{\frac{1}{1-\theta}} \left(\frac{\|\nabla u\|_{L^p(B_\rho(x))}}{\|\nabla u\|_{L^p(\Omega)}} \right)^{\theta^N}.$$

Since, we can cover $\Omega_{8\rho}$ by no more than $\frac{|\Omega|}{\pi\rho^2}$ balls of radius ρ, so we obtain,

$$\frac{\|\nabla u\|_{L^p(\Omega_{8\rho})}}{\|\nabla u\|_{L^p(\Omega)}} \le C \left(\frac{\|\nabla u\|_{L^p(B_\rho(x))}}{\|\nabla u\|_{L^p(\Omega)}} \right)^{\theta^{N_0}}. \tag{3.6}$$

where C depends on $\lambda, M, p, |\Omega|$ and ρ. Note that,

$$\int_{\Omega\setminus\Omega_{8\rho}} |\nabla u|^p \le C\,|\Omega \setminus \Omega_{8\rho}|\, \|\nabla u\|^p_{L^\infty(\Omega)}. \tag{3.7}$$

Now we want to estimate $\|\nabla u\|_{L^\infty(\Omega)}$. First of all, combining the estimates for the Dirichlet problem (1.2) and the trace map, we have that

$$\|u\|_{W^{1,p}(\Omega)} \leq C\|f\|_{W^{1-1/p,p}(\partial\Omega)}$$

(see for example [24, 28]). From the Sobolev embedding theorem, if $p > 2$, we have

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{W^{1-1/p,p}(\partial\Omega)}. \tag{3.8}$$

Having bounded the solution u, we can apply the regularity estimate of the Dirichlet problem for the degenerate elliptic equation in [23] to (1.2) and derive that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(\alpha, \lambda, \Omega, p, \|f\|_{C^{1,\alpha}(\partial\Omega)}). \tag{3.9}$$

Note that we have used the fact that the embedding $C^{1,\alpha}(\partial\Omega) \hookrightarrow W^{1-1/p,p}(\partial\Omega)$ is continuous. On the other hand, using the Poincaré inequality, recalling $f|_\Gamma = 0$ (see (2.2)), and the trace theorem, we have

$$\|f\|_{W^{1-1/p,p}(\partial\Omega)} \leq C\,\|u\|_{W^{1,p}(\Omega)} \leq C\,\|\nabla u\|_{L^p(\Omega)}\,. \tag{3.10}$$

Recall from [4, Lemma 2.8] there exists a constant $C' = C'(|\Omega|, \alpha, r_0, M_0)$ such that

$$|\Omega \setminus \Omega_{8\rho}| \leq C'\rho. \tag{3.11}$$

Combining (3.9) and (3.10) implies

$$\frac{\|\nabla u\|^p_{L^p(\Omega_{8\rho})}}{\|\nabla u\|^p_{L^p(\Omega)}} = 1 - \frac{\|\nabla u\|^p_{L^p(\Omega\setminus\Omega_{8\rho})}}{\|\nabla u\|^p_{L^p(\Omega)}} \geq 1 - C'\rho\frac{(C(\alpha, \lambda, \Omega, p, \|f\|_{C^{1,\alpha}(\partial\Omega)}))^p}{\|f\|^p_{W^{1-1/p,p}(\partial\Omega)}} \geq \frac{1}{2} \tag{3.12}$$

for all $\rho \leq \rho_0$, where ρ_0 depends on $|\Omega|, \alpha, p, r_0, M_0, \Gamma$ and $C(\alpha, \lambda, \Omega, p, \|f\|_{C^{1,\alpha}(\partial\Omega)})/\|f\|_{W^{1-1/p,p}(\partial\Omega)}$. Applying (3.12) to the left-hand side of (3.6) finishes the proof of this lemma. ☐

Now we derive a version of doubling inequality for the solution of p-Laplace equation (1.2).

Lemma 3.3. *Let u be a solution of the p-Laplace equation* (1.2)*. Given any $\rho > 0$, there exist positive constants $\delta = \delta(\rho) \in (0, \rho)$ and $C > 0$ such that, for all $x \in \Omega_{2\rho}$ and $r \in (0, \delta)$, we have*

$$\frac{\|\nabla u\|_{L^p(B_{4r}(x))}}{\|\nabla u\|_{L^p(B_r(x))}} \leq C\frac{\|\nabla u\|_{L^p(B_{2\rho_2}(x))}}{\|\nabla u\|_{L^p(B_{\rho_1}(x))}} \tag{3.13}$$

where ρ_1, ρ_2, δ are constants given in the proof of Lemma 2.3 and $C > 0$ depends only on $\rho_1, \rho_2, p, \lambda, M$ but not on r.

Proof. We recall the following estimate from Lemma 2.3.

$$\frac{\|h \circ \chi\|_{L^\infty(B_{4r}(x))}}{\|h \circ \chi\|_{L^\infty(B_{r/2}(x))}} \leq C\frac{\|h \circ \chi\|_{L^\infty(B_{\rho_2}(x))}}{\|h \circ \chi\|_{L^\infty(B_{\rho_1}(x))}}, \tag{3.14}$$

where $r \in (0,\delta), \delta = (10\tilde{M}^2)^{-\beta}\rho^{\beta^2}$ (see the explicit forms of these quantities in the proof of Lemma 2.3).

Applying Harnack inequality for the quasiregular mapping in Lemma 2.2 implies

$$\|h\circ\chi\|_{L^\infty(B_{r/2}(x))} \le \frac{C}{|B_r(x)|^{1/2}} \|h\circ\chi\|_{L^2(B_r(x))}.$$

On the other hand, a simple estimate gives

$$\|h\circ\chi\|_{L^2(B_{4r}(x))} \le |B_{4r}(x)|^{1/2} \|h\circ\chi\|_{L^\infty(B_{4r}(x))}.$$

Therefore, using above estimates and (3.14), we obtain

$$\frac{\|h\circ\chi\|_{L^2(B_{4r}(x))}}{\|h\circ\chi\|_{L^2(B_r(x))}} \le C\frac{\|h\circ\chi\|_{L^2(B_{2\rho_2}(x))}}{\|h\circ\chi\|_{L^2(B_{\rho_1}(x))}}, \tag{3.15}$$

where, $C>0$ is independent of r. We recall the form $F(x) = (h\circ\chi)(x)e^{\omega(x)}$, where $F(x) = \sigma^{\frac{p}{2}}|\nabla u|^{\frac{p-2}{2}}(u_{x_1} - iu_{x_2})$ and $\omega(x) = T(g(x))$ as described in (3.1). Now replacing $h\circ\chi$ in (3.15) by $F(x)e^{-\omega(x)}$ and using the estimate for $\omega(x)$ as in (3.2), we derive that

$$\begin{aligned}
&\|\nabla u\|^p_{L^p(B_{4r}(x))}\|\nabla u\|^p_{L^p(B_{\rho_1}(x))}\\
&\le C\int_{B_{4r}(x)}|h\circ\chi|^2\left|e^{2\omega(z)}\right|dz\int_{B_{\rho_1}(x)}|h\circ\chi|^2\left|e^{2\omega(z)}\right|dz\\
&\le \left\|e^{2\omega(z)}\right\|_{L^\infty(B_{4r}(x))}\left\|e^{2\omega(z)}\right\|_{L^\infty(B_{\rho_1}(x))}\|h\circ\chi\|^2_{L^2(B_{4r}(x))}\|h\circ\chi\|^2_{L^2(B_{\rho_1}(x))}\\
&\le Ce^{CM}\|h\circ\chi\|^2_{L^2(B_r(x))}\|h\circ\chi\|^2_{L^2(B_{2\rho_2}(x))}\\
&\le Ce^{CM}\int_{B_r(x)}|F|^2e^{-2\omega(z)}dz\int_{B_{2\rho_2}(x)}|F|^2e^{-2\omega(z)}dz\\
&\le Ce^{C'M}\|\nabla u\|^p_{L^p(B_r(x))}\|\nabla u\|^p_{L^p(B_{2\rho_2}(x))},
\end{aligned}$$

i.e.,

$$\frac{\|\nabla u\|_{L^p(B_{4r}(x))}}{\|\nabla u\|_{L^p(B_r(x))}} \le C\frac{\|\nabla u\|_{L^p(B_{2\rho_2}(x))}}{\|\nabla u\|_{L^p(B_{\rho_1}(x))}},$$

where $C>0$ is constant independent of r. □

4. Size estimate for p-Laplace

To begin, we need to impose some suitable jump conditions. Assume that for some constants $\eta,\zeta>0$ we have either

$$(1+\eta)\sigma \le \tilde{\sigma} \le \zeta\sigma \quad a.e. \text{ in } D \tag{4.1}$$

or

$$\zeta\sigma \le \tilde{\sigma} \le (1-\eta)\sigma \quad a.e. \text{ in } D. \tag{4.2}$$

The following energy estimate is key to our approach.

Lemma 4.1. *Assume that the background conductivity σ satisfies the ellipticity condition* (2.1)*. If either* (4.1) *or* (4.2) *holds, then,*

$$C_1 \int_D |\nabla u|^p dx \le |W_0 - W| \le C_2 \int_D |\nabla u|^p dx, \tag{4.3}$$

where C_1, C_2 are positive constants depending only on p, λ, η and ζ.

Proof. Note that,

$$W = \langle \Lambda_\gamma(f), f \rangle = \int_\Omega \gamma(x) |\nabla v|^p dx,$$

where v solves the equation (1.1) and

$$W_0 = \langle \Lambda_\sigma(f), f \rangle = \int_\Omega \sigma(x) |\nabla u|^p dx,$$

where u solves the equation (1.2). Now, applying the monotonicity inequality, see Lemma 2.1, [14], we obtain

$$(p-1) \int_\Omega \frac{\sigma}{\gamma^{1/(p-1)}} \left(\gamma^{\frac{1}{p-1}} - \sigma^{\frac{1}{p-1}} \right) |\nabla u|^p \; dx \tag{4.4}$$

$$\le ((\Lambda_\gamma - \Lambda_\sigma) f, f) \le \int_\Omega (\gamma - \sigma) |\nabla u|^p \; dx. \tag{4.5}$$

Using the assumptions either (4.1) or (4.2) and the ellipticity condition on σ, the above monotonicity inequality becomes

$$C_1 \int_D |\nabla u|^p dx \le |W_0 - W| \le C_2 \int_D |\nabla u|^p dx,$$

where C_1, C_2 positive constants depending only on p, λ, η and ζ. □

We also need an interior estimate.

Lemma 4.2. *Let u satisfy the first equation of* (1.2)*. Then for any $B_r(x) \subset \Omega$, we have*

$$\|\nabla u\|_{L^\infty(B_{r/2}(x))} \le \frac{C}{r^{2/p}} \|\nabla u\|_{L^p(B_r(x))}, \tag{4.6}$$

where $C > 0$ depends on λ, M, p.

Proof. We apply the Harnack inequality in Lemma 2.2 to Fe^{-Tg}. Thus, we have

$$\begin{aligned} \|Fe^{-Tg}\|_{L^\infty(B_{r/2}(x))} &\le C \frac{1}{|B_r|} \int_{B_r(x)} |Fe^{-Tg}| dx \\ &\le C \frac{1}{|B_r|^{1/2}} \left(\int_{B_r(x)} |Fe^{-Tg}|^2 dx \right)^{1/2}. \end{aligned}$$

In view of the form of F and the mapping property of the Cauchy transform, we immediately obtain (4.6) from the above inequality. □

Now we are in a position to state and prove our main result. From now on we consider the case $2 < p < \infty$.

Theorem 4.3. (i) *We assume that all the assumptions in* Section 2 *hold. Then there exists $C_1 > 0$, depending on $\lambda, p, M, d, \eta, \xi$, $C_2 > 0$ and $q > 1$, depending on $\Omega, \Gamma, \lambda, \alpha, M_0, M, p, d, \eta, \zeta, \rho$ and $C(\alpha, \lambda, \Omega, p, \|f\|_{C^{1,\alpha}(\partial\Omega)})/\|f\|_{W^{1-1/p,p}(\partial\Omega)}$, such that*

$$C_1 \left|\frac{W - W_0}{W_0}\right| \le |D| \le C_2 \left|\frac{W - W_0}{W_0}\right|^{1/q}. \tag{4.7}$$

(ii) *If moreover, there exists $h > 0$ such that*

$$|D_h| \ge \frac{1}{2}|D| \quad \text{(fatness condition)}, \tag{4.8}$$

then

$$C_1 \left|\frac{W - W_0}{W_0}\right| \le |D| \le \widetilde{C}_2 \left|\frac{W - W_0}{W_0}\right|, \tag{4.9}$$

where C_1 is given in (i) *and $\widetilde{C}_2$ depend on various constants as C_2 in* (i) *and h.*

Proof. We follow the approach of [5] and [27]. In order to proceed that we begin with the derivation of the lower bound estimate. We note

$$\|\nabla u\|^p_{L^p(D)} \le |D| \|\nabla u\|^p_{L^\infty(D)}. \tag{4.10}$$

Recall that $\mathrm{dist}(D, \partial\Omega) > d$. By covering D with balls of radius $d/4$ and using the interior estimate (4.6), we obtain

$$\|\nabla u\|^p_{L^\infty(D)} \le C\|\nabla u\|^p_{L^p(\Omega_{d/2})} \le C\|\nabla u\|^p_{L^p(\Omega)} \le CW_0, \tag{4.11}$$

where C depends on λ, p, M, d. Combining (4.10), (4.11), and the second inequality of (4.3) leads to the lower bound of (4.7) and (4.9) with C_1 depending on $\lambda, p, M, d, \eta, \xi$.

The upper bound estimate is little bit tricky, which involves propagation of smallness estimate and the doubling inequality for the solutions of p-Laplace equation. We consider this part in two cases.

Case 1. Without assuming the fatness condition (4.8).

We will show that $|\nabla u|^p$ is an A_q-weight, for some $q > 1$. To prove this argument, we follow the proof of Theorem 1.1 in [18]. Let $\rho = \min\{\frac{d}{8}, \rho_0\}$ and $\delta, \rho_1 > \delta$ be defined in Lemma 2.3. For any $x \in \Omega_{8\rho}$, we can derive from the propagation of smallness, Lemma 3.2, the Poincaré inequality, and the trace estimate, that

$$\begin{aligned}\|\nabla u\|_{L^p(B_{\rho_1}(x))} &\geq \|\nabla u\|_{L^p(B_\delta(x))} \geq C\|\nabla u\|_{L^p(\Omega)} \\ &\geq C\|u\|_{W^{1,p}(\Omega)} \geq C\|f\|_{W^{1-1/p,p}(\partial\Omega)},\end{aligned} \tag{4.12}$$

where $C>0$ depends on $|\Omega|, \alpha, p, r_0, M_0, \Gamma, d, M$, and $C(\alpha,\lambda,\Omega,p,\|f\|_{C^{1,\alpha}(\partial\Omega)})/\|f\|_{W^{1-1/p,p}(\partial\Omega)}$. On the other hand, we can estimate

$$\|\nabla u\|_{L^p(B_{2\rho_2}(x))} \leq \|u\|_{W^{1,p}(\Omega)} \leq C\|f\|_{W^{1-1/p,p}(\partial\Omega)}. \tag{4.13}$$

Therefore, combining (4.12), (4.13), and the doubling inequality (3.13), we have that

$$\|\nabla u\|_{L^p(B_{4r}(x))} \leq C\,\|\nabla u\|_{L^p(B_r(x))}\,, \tag{4.14}$$

where C depends on $|\Omega|, \alpha, p, r_0, M_0, \Gamma, d, M$, and $C(\alpha,\lambda,\Omega,p,\|f\|_{C^{1,\alpha}(\partial\Omega)})/\|f\|_{W^{1-1/p,p}(\partial\Omega)}$.

To show that $|\nabla u|^p$ is an A_q-weight, it suffices to prove that $|\nabla u|^p$ satisfies a reverse Hölder inequality [15]. To do so, we will apply the Caccioppoli inequality for the p-Laplace (see [22, Lemma 3.32]) and the Poincaré inequality or the Poincaré-Sobolev inequality (see for example [6, Theorem A.6.3]) on both sides of (4.14). Denote $c_1 = |B_r|^{-1}\int_{B_r(x)} u$. We first apply the Caccioppoli inequality and then the Poincaré-Sobolev inequality to the right hand term of (4.14)

$$\|\nabla u\|_{L^p(B_r(x))} \leq Cr^{-1}\|u-c_1\|_{L^p(B_{2r}(x))} \leq C|B_{2r}|^{1/p}\left(\frac{1}{|B_{2r}|}\int_{B_{2r}}|\nabla u|^s\right)^{1/s}, \tag{4.15}$$

where $s=\frac{2p}{p+2}<2$ and $C=C(\lambda,p)$. Similarly, letting $c_2 = |B_{4r}|^{-1}\int_{B_{4r}(x)} u$, first applying the Poincaré inequality and then the Caccioppoli inequality, we can obtain

$$\|\nabla u\|_{L^p(B_{4r}(x))} \geq Cr^{-1}\,\|u-c_2\|_{L^p(B_{4r}(x))} \geq C\,\|\nabla u\|_{L^p(B_{2r}(x))}\,, \tag{4.16}$$

where $C=C(\lambda,p)$. Putting together (4.14), (4.15), (4.16) gives

$$\left(\frac{1}{|B_{2r}|}\int_{B_{2r}}|\nabla u|^p\right)^{1/p} \leq C\left(\frac{1}{|B_{2r}|}\int_{B_{2r}}|\nabla u|^s\right)^{1/s}$$

with C depending on $|\Omega|, \alpha, p, r_0, M_0, \Gamma, d, M$, and $C(\alpha,\lambda,\Omega,p,\|f\|_{C^{1,\alpha}(\partial\Omega)})/\|f\|_{W^{1-1/p,p}(\partial\Omega)}$. This reverse Hölder inequality shows that $|\nabla u|^p$ is an A_q-weight for some $q>1$. We now cover D internally by the sequence of disjoint closed squares Q_k with side length 2ρ. In view of [16, (7.2)], we can show that

$$\frac{|D\cap Q_k|}{|Q_k|}\le C\left(\frac{\int_{D\cap Q_k}|\nabla u|^p}{\int_{Q_k}|\nabla u|^p}\right)^{1/q}.$$

Summing over k and applying Lemma 3.2 again, we obtain

$$|D|\le C\left(\frac{\int_D|\nabla u|^p}{\min_k\int_{Q_k}|\nabla u|^p}\right)^{1/q}\le C\left(\frac{\int_D|\nabla u|^p}{\int_\Omega|\nabla u|^p}\right)^{1/q}.$$

The upper bound of $|D|$ then follows from the first inequality of (4.3).

Case 2. Assuming the fatness condition (4.8).

Let $\rho=\frac{1}{4}\min\{d,h,\rho_0\}$, $D_h=\cup_{k=1}^J Q_k$, for some indices J, where Q_k's are nonoverlapping closed squares of side length 2ρ. Therefore using Lemma 3.2 and (4.8), we have

$$\begin{aligned}\int_D|\nabla u|^p\,dx&\ge\int_{\cup_{k=1}^J Q_k}|\nabla u|^p\,dx\\&\ge\frac{|D_h|}{\rho^2}\min_k\int_{Q_k}|\nabla u|^p\,dx\\&\ge C\frac{|D|}{\rho^2}\int_\Omega|\nabla u|^p\,dx.\end{aligned}$$

Again, the upper bound of $|D|$ is then a consequence of the first inequality of (4.3). □

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