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Optimal stability estimate of the inverse boundary value problem by partial measurements

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We dedicate this work to Giovanni Alessandrini for his 60th birthday and for his pioneering contribution in the stability estimates of inverse problems.

Abstract. *This manuscript was originally uploaded to arXiv in 2007 (arXiv:0708.3289v1). In the current version, we expand the Introduction and the list of references which are related to the results of this paper after 2007. In this work we establish* log *type stability estimates for the inverse potential and conductivity problems with partial Dirichlet-to-Neumann map, where the Dirichlet data is homogeneous on the inaccessible part. The proof is based on the uniqueness result of the inverse boundary value problem in Isakov's work [16].*

1. Introduction

In this paper we study the stability question of the inverse boundary value problem for the Schrödinger equation with a potential and the conductivity equation by partial Cauchy data. This type of inverse problem with full data, i.e., Dirichlet-to-Neumann map, were first proposed by Calderón [6]. For three or higher dimensions, the uniqueness issue was settled by Sylvester and Uhlmann [27] and a reconstruction procedure was given by Nachman [25]. For two dimensions, Calderón's problem was solved by Nachman [26] for $W^{2,p}$ conductivities and by Astala and Päivärinta [3] for L^∞ conductivities. This inverse problem is known to be ill-posed. A log-type stability estimate was derived by Alessandrini [1]. On the other hand, it was shown by Mandache [24] that the log-type estimate is optimal.

All results mentioned above are concerned with the full data. Over the last decade, the inverse problems with partial data have received a lot of attention. We list several earlier results [14], [17], [5], [20], [12], [16], [23], [21], [18] [11] and refer the reader to the survey article [19] for its detailed development and for related references. After the uniqueness proof comes stability estimates. We summarize related results in the following.

- $\log\log$ type: [15], [29], [7], [9], [8], [10], [22].

- log type: [7], [13], [4], [2].

The method in [15] was based on [5] and a stability estimate for the analytic continuation proved in [30]. We believe that the log type estimate should be the right estimate for the inverse boundary problem, even with partial data. In this paper, motivated by the uniqueness proof in Isakov's work [16], we prove a *log* type estimate for the inverse boundary value problem under the same *a priori* assumption on the boundary as given in [16]. Precisely, the inaccessible part of the boundary is either a part of a sphere or a plane. Also, one is able to use zero data on the inaccessible part of the boundary. The strategy of the proof in [16] follows the framework in [27] where complex geometrical optics solutions are key elements. A key observation in [16] is that when Γ_0 is a part of a sphere or a plane, we are able to use a reflection argument to guarantee that complex geometrical optics solutions have homogeneous data on Γ_0. Caro in [7] also used Isakov's idea to derive a log type estimate for the Maxwell equations. The articles [13], [4], [2] have a common feature that the undetermined coefficients are known near the boundary.

Now we would like to describe the results in this work. Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be an open domain with smooth boundary $\partial\Omega$. Given $q \in L^\infty(\Omega)$, we consider the boundary value problem:

$$\begin{aligned} (\Delta - q)u &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $f \in H^{1/2}(\partial\Omega)$. Assume that 0 is not a Dirichlet eigenvalue of $\Delta - q$ on Ω. Then (1) has a unique solution $u \in H^1(\Omega)$. The usual definition of the Dirichlet-to-Neumann map is given by

$$\Lambda_q f = \partial_\nu u|_{\partial\Omega}$$

where $\partial_\nu u = \nabla u \cdot \nu$ and ν is the unit outer normal of $\partial\Omega$.

Let $\Gamma_0 \subset \partial\Omega$ be an open part of the boundary of Ω. We set $\Gamma = \partial\Omega \setminus \Gamma_0$. We further set $H_0^{1/2}(\Gamma) := \{f \in H^{1/2}(\partial\Omega) : \text{supp } f \subset \Gamma\}$ and $H^{-1/2}(\Gamma)$ the dual space of $H_0^{1/2}(\Gamma)$. Then the partial Dirichlet-to-Neumann map $\Lambda_{q,\Gamma}$ is defined as

$$\Lambda_{q,\Gamma} f := \partial_\nu u|_\Gamma \in H^{-1/2}(\Gamma)$$

where u is the unique weak solution of (1) with Dirichlet Data $f \in H_0^{1/2}(\Gamma)$. In what follows, we denote the operator norm by

$$\|\Lambda_{q,\Gamma}\|_* := \|\Lambda_{q,\Gamma}\|_{H_0^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)}$$

We consider two types of domains in this paper:

(a) Ω is a bounded domain in $\{x_n < 0\}$ and $\Gamma_0 = \partial\Omega \cap \{x_n = 0\}$;

(b) Ω is a subdomain of $B(a,R)$ and $\Gamma_0 = \partial B(a,R) \cap \partial\Omega$ with $\Gamma_0 \neq \partial B(a,R)$, where $B(a,R)$ is a ball centered at a with radius R. Denote by $\hat{q}$ the zero extension of the function q defined on Ω to $\mathbb{R}^n$.

The main result of the paper reads as follows:

THEOREM 1.1. *Assume that Ω is given as in either (a) or (b). Let $N > 0$, $s > \frac{n}{2}$ and $q_j \in H^s(\Omega)$ such that*

$$\|q_j\|_{H^s(\Omega)} \leq N \tag{2}$$

for $j = 1, 2$, and 0 is not a Dirichlet eigenvalue of $\Delta - q_j$ for $j = 1, 2$. Then there exist constants $C > 0$ and $\sigma > 0$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C \big| \log \|\Lambda_{q_1,\Gamma} - \Lambda_{q_2,\Gamma}\|_* \big|^{-\sigma} \tag{3}$$

where C depends on Ω, N, n, s and σ depends on n and s.

Theorem 1.1 can be generalized to the conductivity equation. Let $\gamma \in H^s(\Omega)$ with $s > 3 + \frac{n}{2}$ be a strictly positive function on $\overline{\Omega}$. The equation for the electrical potential in the interior without sinks or sources is

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) = 0 \quad &\text{in} \quad \Omega \\ u = f \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

As above, we take $f \in H_0^{1/2}(\Gamma)$. The partial Dirichlet-to-Neumann map defined in this case is

$$\Lambda_{\gamma,\Gamma} : f \mapsto \gamma \partial_\nu u|_\Gamma.$$

COROLLARY 1.2. *Let the domain Ω satisfy (a) or (b). Assume that $\gamma_j \geq N^{-1} > 0$, $s > \frac{n}{2}$, and*

$$\|\gamma_j\|_{H^{s+3}(\Omega)} \leq N \tag{4}$$

for $j = 1, 2$, and

$$\partial_\nu^\beta \gamma_1|_\Gamma = \partial_\nu^\beta \gamma_2|_\Gamma \quad on \quad \partial\Omega, \quad \forall \quad 0 \leq \beta \leq 1. \tag{5}$$

Then there exist constants $C > 0$ and $\sigma > 0$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C \big| \log \|\Lambda_{\gamma_1,\Gamma} - \Lambda_{\gamma_2,\Gamma}\|_* \big|^{-\sigma} \tag{6}$$

where C depend on Ω, N, n, s and σ depend on n, s.

REMARK 1.3. *For the sake of simplicity, we impose the boundary identification condition (5) on conductivities. However, using the arguments in [1] (also see [15]), this condition can be removed. The resulting estimate is still in the form of (6) with possible different constant C and σ.*

2. Preliminaries

We first prove an estimate of the Riemann-Lebesgue lemma for a certain class of functions. Let us define

$$g(y) = \|f(\cdot - y) - f(\cdot)\|_{L^1(\mathbb{R}^n)}$$

for any $f \in L^1(\mathbb{R}^n)$. It is known that $\lim_{|y|\to 0} g(y) = 0$.

LEMMA 2.1. *Assume that* $f \in L^1(\mathbb{R}^n)$ *and there exist* $\delta > 0$, $C_0 > 0$, *and* $\alpha \in (0,1)$ *such that*

$$g(y) \le C_0|y|^\alpha \tag{7}$$

whenever $|y| < \delta$. *Then there exists a constant* $C > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *the inequality*

$$|\mathcal{F}f(\xi)| \le C(\exp(-\pi\varepsilon^2|\xi|^2) + \varepsilon^\alpha) \tag{8}$$

holds with $C = C(C_0, \|f\|_{L^1}, n, \delta, \alpha)$.

Proof. Let $G(x) := \exp(-\pi|x|^2)$ and set $G_\varepsilon(x) := \varepsilon^{-n}G(\frac{x}{\varepsilon})$. Then we define $f_\varepsilon := f * G_\varepsilon$. Next we write

$$|\mathcal{F}f(\xi)| \le |\mathcal{F}f_\varepsilon(\xi)| + |\mathcal{F}(f_\varepsilon - f)(\xi)|.$$

For the first term on the right hand side we get

$$\begin{aligned} |\mathcal{F}f_\varepsilon(\xi)| &\le |\mathcal{F}f(\xi)| \cdot |\mathcal{F}G_\varepsilon(\xi)| \\ &\le \|f\|_1|\varepsilon^{-n}\varepsilon^n\mathcal{F}G(\varepsilon\xi)| \\ &\le \|f\|_1 \exp(-\pi\varepsilon^2|\xi|^2). \end{aligned} \tag{9}$$

To estimate the second term, we use the assumption (7) and derive

$$\begin{aligned} |\mathcal{F}(f_\varepsilon - f)(\xi)| &\le \|f_\varepsilon - f\|_1 \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y) - f(x)|G_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{|y|<\delta}\int_{\mathbb{R}^n} |f(x-y) - f(x)|G_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{|y|\ge\delta}\int_{\mathbb{R}^n} |f(x-y) - f(x)|G_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= I + II. \end{aligned}$$

In view of (7) we can estimate

$$\begin{aligned} I &= \int_{|y|<\delta} g(y)G_\varepsilon(y)\,\mathrm{d}y \\ &\le C_0 \int_{|y|<\delta} |y|^\alpha G_\varepsilon(y)\,\mathrm{d}y \\ &= C_0 \int_{S^{n-1}} \int_0^\delta r^\alpha \varepsilon^{-n} \exp(-\pi\varepsilon^{-2}r^2) r^{n-1}\,\mathrm{d}r\,\mathrm{d}\psi \\ &= C_1 \int_0^\delta \varepsilon^\alpha u^\alpha \varepsilon^{-n} \exp(-u^2)\varepsilon^{n-1}u^{u-1}\varepsilon\,\mathrm{d}u \\ &= C_2\varepsilon^\alpha \int_0^\delta u^{n+\alpha-1}\exp(-u^2)\,\mathrm{d}u = C_3\varepsilon^\alpha, \end{aligned}$$

where $C_3 = C_3(C_0, n, \delta, \alpha)$.

As for II, we obtain that for ε sufficiently small

$$\begin{aligned} II &= \int_{|y|\ge\delta} g(y)G_\varepsilon(y)\,\mathrm{d}y \\ &\le 2\|f\|_{L^1} \int_{|y|\ge\delta} G_\varepsilon(y)\,\mathrm{d}y \\ &\le C_4\|f\|_1 \int_\delta^\infty \varepsilon^{-n}\exp(-\pi\varepsilon^{-2}r^2)r^{n-1}\,\mathrm{d}r \\ &= C_4\|f\|_1 \int_{\delta\varepsilon^{-1}}^\infty u^{n-1}\exp(-\pi u^2)\,\mathrm{d}u \\ &\le C_4\|f\|_1 \int_{\delta\varepsilon^{-1}}^\infty \exp(-\pi u)\,\mathrm{d}u \\ &\le C_4\|f\|_1 \frac{1}{\pi}\exp(-\pi\delta\varepsilon^{-1}) \le C_5\varepsilon^\alpha, \end{aligned}$$

where $C_5 = C_5(\|f\|_{L^1}, n, \delta, \alpha)$. Combining the estimates for I, II, and (9), we immediately get (8). □

We now provide a sufficient condition on f, defined on Ω, such that (7) in the previous lemma holds.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Let $f \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ and denote by $\hat{f}$ the zero extension of f to $\mathbb{R}^n$. Then there exists $\delta > 0$ and $C > 0$ such that*

$$\|\hat{f}(\cdot - y) - \hat{f}(\cdot)\|_{L^1(\mathbb{R}^n)} \le C|y|^\alpha$$

for any $y \in \mathbb{R}^n$ with $|y| \le \delta$.

Proof. Since Ω is bounded and of class C^1, there exist a finite number of balls, say $m \in \mathbb{N}$, $B_i(x_i)$ with center $x_i \in \partial\Omega$, $i = 1, \ldots, m$ and associated C^1-diffeomorphisms $\varphi_i : B_i(x_i) \to Q$ where $Q = \{x' \in \mathbb{R}^{n-1} : \|x'\| \leq 1\} \times (-1, 1)$. Set $d = \text{dist}\ (\partial\Omega, \partial(\bigcup_{i=1}^m B_i(x_i))) > 0$ and $\tilde{\Omega}_\varepsilon = \bigcup_{x\in\partial\Omega} B(x, \varepsilon)$, where $B(x, \varepsilon)$ denotes the ball with center x and radius $\varepsilon > 0$. Obviously, for $\varepsilon < d$, it holds that $\tilde{\Omega}_\varepsilon \subset \bigcup_{i=1}^m B_i(x_i)$. Let $x \in \partial\Omega$ and $0 < |y| < \delta \leq d$, then for any $z_1, z_2 \in B(x, |y|) \cap B_i(x_i)$ we get that

$$|\varphi_i(z_1) - \varphi_i(z_2)| \leq \|\nabla\varphi_i\|_{L^\infty}|z_1 - z_2| \leq C|y|$$

for some constant $C > 0$. Therefore, $\varphi_i(\tilde{\Omega}_{|y|} \cap B_i(x_i)) \subset \{x' \in \mathbb{R}^{n-1} : \|x'\| \leq 1\} \times (-C|y|, C|y|)$. By the transformation formula this yields $\text{vol}(\tilde{\Omega}_{|y|}) \leq C|y|$.

Since $|y| < \delta$ we have $\hat{f}(x - y) - \hat{f}(x) = 0$ for $x \notin \Omega \cup \tilde{\Omega}_{|y|}$. Now we write

$$\begin{aligned}
\|\hat{f}(\cdot - y) - \hat{f}\|_{L^1(\mathbb{R}^n)} &= \int_{\Omega\setminus\tilde{\Omega}_{|y|}} |\hat{f}(x - y) - \hat{f}(x)|\,\mathrm{d}x \\
&\quad + \int_{\tilde{\Omega}_{|y|}} |\hat{f}(x - y) - \hat{f}(x)|\,\mathrm{d}x \\
&\leq C\,\text{vol}(\Omega)|y|^\alpha + 2\|f\|_{L^\infty}\,\text{vol}(\tilde{\Omega}_{|y|}) \\
&\leq C(|y|^\alpha + |y|) \leq C|y|^\alpha
\end{aligned}$$

for $\delta \leq 1$. □

Now let q_1 and q_2 be two potentials and their corresponding partial Dirichlet-to-Neumann maps are denoted by $\Lambda_{1,\Gamma}$ and $\Lambda_{2,\Gamma}$, respectively. The following identity plays a key role in the derivation of the stability estimate.

LEMMA 2.3. *Let v_j solve* (1) *with $q = q_j$ for $j = 1, 2$. Further assume that $v_1 = v_2 = 0$ on Γ_0. Then*

$$\int_\Omega (q_1 - q_2)v_1\overline{v_2}\,\mathrm{d}x = \langle(\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma})v_1, v_2\rangle$$

Proof. Let u_2 denote the solution of (1) with $q = q_2$ and $u_2 = v_1$ on $\partial\Omega$. Therefore

$$\int_\Omega \nabla v_1 \cdot \overline{\nabla v_2} + q_1 v_1\overline{v_2}\,\mathrm{d}x = \langle\partial_\nu v_1, v_2\rangle$$
$$\int_\Omega \nabla u_2 \cdot \overline{\nabla v_2} + q_2 u_2\overline{v_2}\,\mathrm{d}x = \langle\partial_\nu u_2, v_2\rangle.$$

Setting $v := v_1 - u_2$ and $q_0 = q_1 - q_2$ we get after subtracting these identities

$$\int_\Omega \nabla v \cdot \overline{\nabla v_2} + q_2 v\overline{v_2} + q_0 v_1\overline{v_2} = \langle(\Lambda_1 - \Lambda_2)v_1, v_2\rangle.$$

Since v_2 solves $(\Delta - q_2)v_2 = 0$, $v = 0$ on $\partial\Omega$ and $v_2 = 0$ on Γ_0, we have

$$\int_\Omega \nabla v \cdot \overline{\nabla v_2} + q_2 v \overline{v_2} = 0,$$

$$\langle(\Lambda_1 - \Lambda_2)v_1, v_2\rangle = \langle(\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma})v_1, v_2\rangle,$$

and the assertion follows. □

In treating inverse boundary value problems, complex geometrical optics solutions play a very important role. We now describe the complex geometrical optics solutions we are going to use in our proofs. We will follow the idea in [16]. Assume that q_1, $q_2 \in L^\infty(\mathbb{R}^n)$ are compactly supported and are even in x_n, i.e.

$$q_1^*(x_1, \cdots, x_{n-1}, x_n) = q_1(x_1, \cdots, x_{n-1}, x_n)$$

and

$$q_2^*(x_1, \cdots, x_{n-1}, x_n) = q_2(x_1, \cdots, x_{n-1}, x_n).$$

Hereafter, we denote

$$h^*(x_1, \cdots, x_{n-1}, x_n) = h(x_1, \cdots, x_{n-1}, -x_n).$$

Given $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. Let us first introduce new coordinates obtained by rotating the standard Euclidean coordinates around the x_n axis such that the representation of ξ in the new coordinates, denoted by $\tilde{\xi}$, satisfies $\tilde{\xi} = (\tilde{\xi}_1, 0, \cdots, 0, \tilde{\xi}_n)$ with $\tilde{\xi}_1 = \sqrt{\xi_1^2 + \cdots + \xi_{n-1}^2}$ and $\tilde{\xi}_n = \xi_n$. In the following we also denote by $\tilde{x}$ the representation of x in the new coordinates. Then we define for $\tau > 0$

$$\begin{aligned} \tilde{\rho}_1 &:= (\frac{\tilde{\xi}_1}{2} - \tau\tilde{\xi}_n, i|\tilde{\xi}|(\frac{1}{4} + \tau^2)^{1/2}, 0, \cdots, 0, \frac{\tilde{\xi}_n}{2} + \tau\tilde{\xi}_1), \\ \tilde{\rho}_2 &:= (\frac{\tilde{\xi}_1}{2} + \tau\tilde{\xi}_n, -i|\tilde{\xi}|(\frac{1}{4} + \tau^2)^{1/2}, 0, \cdots, 0, \frac{\tilde{\xi}_n}{2} - \tau\tilde{\xi}_1), \end{aligned} \tag{10}$$

and let ρ_1 and ρ_2 be representations of $\tilde{\rho}_1$ and $\tilde{\rho}_2$ in the original coordinates. Note that $x_n = \tilde{x}_n$ and $\sum_{i=1}^n x_i y_i = \sum_{i=1}^n \tilde{x}_i \tilde{y}_i$. It is clear that, for $j = 1, 2$, $\rho_j \cdot \rho_j = 0$ as well as $\rho_j^* \cdot \rho_j^* = 0$ hold.

The construction given in [27] ensures that there are complex geometrical optics solutions $u_j = e^{i\rho_j \cdot x}(1 + w_j)$ of $(\Delta - q_j)u_j = 0$ in $\mathbb{R}^n$, $j = 1, 2$, and the functions w_j satisfy $\|w_j\|_{L^2(K)} \le C_K \tau^{-1}$ for any compact set $K \subset \mathbb{R}^n$. We then set

$$\begin{aligned} v_1(x) &= e^{i\rho_1 \cdot x}(1 + w_1) - e^{i\rho_1^* \cdot x}(1 + w_1^*) \\ v_2(x) &= e^{-i\rho_2 \cdot x}(1 + w_2) - e^{-i\rho_2^* \cdot x}(1 + w_2^*). \end{aligned} \tag{11}$$

From this definition it is clear that these functions are solutions of $(\Delta - q_j)v_j = 0$ in $\mathbb{R}^n_+$ with $v_j = 0$ on $x_n = 0$.

3. Stability estimate for the potential

Now we are in the position to prove Theorem 1.1. We first consider the case (a) where Γ_0 is a part of a hyperplane. To construct the special solutions described in the previous section, we first perform zero extension of q_1 and q_2 to $\mathbb{R}^+_n$ and then even extension to the whole $\mathbb{R}^n$. As in the last section, we can construct special geometrical optics solutions v_j of the form (11) to $(\Delta - q_j)v_j = 0$ in Ω for $j = 1, 2$. Note that $v_1 = v_2 = 0$ on Γ_0. We now plug in these solutions into the identity (2.3) and write $q_0 = q_1 - q_2$. This gives

$$\begin{aligned}
&\langle(\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma})v_1, v_2\rangle \\
&= \int_\Omega q_0 v_1 \overline{v_2}\,\mathrm{d}x \\
&= \int_\Omega q_0(x)\Big(e^{i(\rho_1+\rho_2)\cdot x}(1+w_1)(1+\overline{w_2}) + e^{i(\rho_1^*+\rho_2^*)\cdot x}(1+w_1^*)(1+\overline{w_2^*}) \\
&\qquad - e^{i(\rho_1+\rho_2^*)\cdot x}(1+w_1)(1+\overline{w_2^*}) - e^{i(\rho_1^*+\rho_2)\cdot x}(1+w_1^*)(1+\overline{w_2})\Big)\,\mathrm{d}x \\
&= \int_\Omega q_0(x)(e^{i\xi\cdot x} + e^{i\xi^*\cdot x})\,\mathrm{d}x + \int_\Omega q_0(x) f(x, w_1, w_2, w_1^*, w_2^*)\,\mathrm{d}x \\
&\qquad - \int_\Omega q_0(x)\big(e^{i(\rho_1+\rho_2^*)\cdot x} + e^{i(\rho_1^*+\rho_2)\cdot x}\big)\,\mathrm{d}x,
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
f \ &= e^{i\xi\cdot x}(w_1 + \overline{w_2} + w_1\overline{w_2}) + e^{i\xi^*\cdot x}(w_1^* + \overline{w_2^*} + w_1^*\overline{w_2^*}) \\
&\quad - e^{i(\rho_1^*+\rho_2)\cdot x}(w_1^* + \overline{w_2} + w_1^*\overline{w_2}) - e^{i(\rho_1+\rho_2^*)\cdot x}(w_1 + \overline{w_2^*} + w_1\overline{w_2^*}).
\end{aligned}$$

The first term on the right hand side of (12) is equal to

$$\int_{\mathbb{R}^n} q_0(x)e^{i\xi\cdot x}\,\mathrm{d}x = \mathcal{F}q_0(\xi)$$

because q_0 is even in x_n. For the second term, we use the estimate

$$\|w_1\|_2 + \|w_1^*\|_2 + \|\overline{w_2}\|_2 + \|\overline{w_2^*}\|_2 \le C\tau^{-1}$$

to obtain

$$\Big|\int_\Omega q_0 f(x, w_1, w_2, w_1^*, w_2^*)\,\mathrm{d}x\Big| \le C\|q_0\|_2\tau^{-1}. \tag{13}$$

As for the last term on the right hand side of (12), we first observe that

$$(\rho_1 + \rho_2^*)\cdot x = (\tilde\rho_1 + \tilde\rho_2^*)\cdot\tilde x = \tilde\xi_1\tilde x_1 + 2\tau\tilde\xi_1\tilde x_n = \xi'\cdot x' + 2\tau|\xi'|x_n$$

and

$$(\rho_1^* + \rho_2)\cdot x = (\tilde\rho_1^* + \tilde\rho_2)\cdot\tilde x = \tilde\xi_1\tilde x_1 - 2\tau\tilde\xi_1\tilde x_n = \xi'\cdot x' - 2\tau|\xi'|x_n,$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1})$ and $x' = (x_1, \cdots, x_{n-1})$. Therefore, we can write

$$\int_\Omega q_0(x) e^{i(\rho_1+\rho_2^*)\cdot x}\, \mathrm{d}x = \mathcal{F}q_0(\xi', 2\tau|\xi'|)$$

as well as

$$\int_\Omega q_0(x) e^{i(\rho_1^*+\rho_2)\cdot x}\, \mathrm{d}x = \mathcal{F}q_0(\xi', -2\tau|\xi'|).$$

The Sobolev embedding and the assumptions on q_j ensure that $q_0 \in C^\alpha(\overline{\Omega})$ for $\alpha = s - \frac{n}{2}$ and therefore q_0 satisfies the assumption of Lemma 2.2. Applying Lemma 2.1 to q_0 yields that for $\varepsilon < \varepsilon_0$

$$|\mathcal{F}q_0(\xi', 2\tau|\xi'|)| + |\mathcal{F}q_0(\xi', -2\tau|\xi'|)| \le C(\exp(-\pi\varepsilon^2(1+4\tau^2)|\xi'|^2) + \varepsilon^\alpha). \tag{14}$$

Finally, we estimate the boundary integral

$$\begin{aligned}
&\left|\int_\Gamma (\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma}) v_1 \cdot v_2 \,\mathrm{d}\sigma\right| \\
&\le \|\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma}\|_* \|v_1\|_{H^{\frac{1}{2}}(\Gamma)} \|v_2\|_{H^{\frac{1}{2}}(\Gamma)} \\
&\le \|\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma}\|_* \|v_1\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} \\
&\le C\exp(|\xi|\tau) \|\Lambda_1 - \Lambda_2\|_*.
\end{aligned} \tag{15}$$

Combining (12), (13), (14), and (15) leads to the inequality

$$|\mathcal{F}q_0(\xi)| \le C\{\exp(|\xi|\tau)\|\Lambda_1 - \Lambda_2\|_* + \exp(-\pi\varepsilon^2(1+4\tau^2)|\xi'|^2) + \varepsilon^\alpha + \frac{1}{\tau}\} \tag{16}$$

for all $\xi \in \mathbb{R}^n$ and $\varepsilon < \varepsilon_0$, where C only depends on a priori data on the potentials.

Next we would like to estimate the norm of q_0 in H^{-1}. As usual, other estimates of q_0 in more regular norms can be obtained by interpolation. To begin, we set $Z_R = \{\xi \in \mathbb{R}^n : \ |\xi_n| < R \text{ and } |\xi'| < R\}$. Note that $B(0,R) \subset Z_R \subset B(0,cR)$ for some $c > 0$. Now we use the a priori assumption on potentials and (16) and calculate

$$\begin{aligned}
\|q_0\|_{H^{-1}}^2 &\le \int_{Z_R} |\mathcal{F}q_0(\xi)|^2 (1+|\xi|^2)^{-1}\,\mathrm{d}\xi + \int_{{Z_R}^c} |\mathcal{F}q_0(\xi)|^2 (1+|\xi|^2)^{-1}\,\mathrm{d}\xi \\
&\le \int_{Z_R} |\mathcal{F}q_0(\xi)|^2 (1+|\xi|^2)^{-1}\,\mathrm{d}\xi + CR^{-2} \\
&\le C\{R^n \exp(cR\tau)\|\Lambda_1 - \Lambda_2\|_*^2 + R^n\varepsilon^{2\alpha} + R^n\tau^{-2} + R^{-2} \\
&\quad + \int_{-R}^{R}\int_{B'(0,R)} \exp(-2\pi\varepsilon^2(1+4\tau^2)|\xi'|^2)\,\mathrm{d}\xi'\,\mathrm{d}\xi_n\},
\end{aligned} \tag{17}$$

here $B'(x',R)$ denotes the ball in $\mathbb{R}^{n-1}$ with center x' and radius $R>0$. For the second term on the right hand side of (17), we choose $\varepsilon=(1+4\tau^2)^{-1/4}$ with $\tau\geq\tau_0\gg 1$ and integrate

$$\begin{aligned}
&\int_{-R}^{R}\int_{B'(0,R)}\exp(-2\pi\varepsilon^2(1+4\tau^2)|\xi'|^2)\,\mathrm{d}\xi'\,\mathrm{d}\xi_n\\
&=2R\int_{B'(0,R)}\exp(-2\pi(1+4\tau^2)^{1/2}|\xi'|^2)\,\mathrm{d}\xi'\\
&=2R\int_{S^{n-2}}\int_0^R r^{n-2}\exp(-2\pi((1+4\tau^2)^{1/4}r)^2)\,\mathrm{d}r\,\mathrm{d}\omega\\
&\leq CR(1+4\tau^2)^{-(n-1)/4}\int_0^\infty u^{n-2}\exp(-2\pi u^2)\,\mathrm{d}u\\
&\leq CR\tau^{-(n-1)/2}.
\end{aligned}\tag{18}$$

Plugging (18) into (17) with the choice of $\varepsilon=(1+4\tau^2)^{-1/4}$ we get for $R>1$

$$\begin{aligned}
\|q_0\|_{H^{-1}}^2&\leq C\{R^n\exp(cR\tau)\|\Lambda_1-\Lambda_2\|_*^2+R^n\tau^{-\alpha}+R\tau^{-(n-1)/2}+R^{-2}\}\\
&\leq C\{R^n\exp(cR\tau)\|\Lambda_1-\Lambda_2\|_*^2+R^n\tau^{-\tilde{\alpha}}+R^{-2}\},
\end{aligned}\tag{19}$$

where $\tilde{\alpha}=\min\{\alpha,(n-1)/2\}$.

Observing from (19), we now choose τ such that $R^n\tau^{-\tilde{\alpha}}=R^{-2}$, namely, $\tau=R^{(n+2)/\tilde{\alpha}}$. Substituting such τ back to (19) yields

$$\|q_0\|_{H^{-1}}^2\leq C\{R^n\exp(cR^{\frac{n+2}{\tilde{\alpha}}+1})\|\Lambda_1-\Lambda_2\|_*^2+R^{-2}\}.\tag{20}$$

Finally, we choose a suitable R so that

$$R^n\exp(cR^{\frac{n+2}{\tilde{\alpha}}+1})\|\Lambda_1-\Lambda_2\|_*^2=R^{-2},$$

i.e., $R=\big|\log\|\Lambda_1-\Lambda_2\|_*\big|^\gamma$ for some $0<\gamma=\gamma(n,\tilde{\alpha})$. Thus, we obtain from (20) that

$$\|q_1-q_2\|_{H^{-1}(\Omega)}\leq C\big|\log\|\Lambda_1-\Lambda_2\|_*\big|^{-\gamma}.\tag{21}$$

The derivation of (21) is legitimate under the assumption that τ is large. To make sure that it is true, we need to take R sufficiently large, i.e. $R>R_0$ for some large R_0. Consequently, there exists $\tilde{\delta}>0$ such that if $\|\Lambda_1-\Lambda_2\|_*<\tilde{\delta}$ then (21) holds. For $\|\Lambda_1-\Lambda_2\|_*\geq\tilde{\delta}$, (21) is automatically true with a suitable constant C when we take into account the a priori bound (2).

The estimate (3) is now an easy consequence of the interpolation theorem. Precisely, let $\epsilon>0$ such that $s=\frac{n}{2}+2\epsilon$. Using that $[H^{t_0}(\Omega),H^{t_1}(\Omega)]_\beta=H^t(\Omega)$ with $t=(1-\beta)t_0+\beta t_1$ (see e.g. [28, Theorem 1 in 4.3.1]) and the

Sobolev embedding theorem, we get $\|q_1 - q_2\|_{L^\infty} \le C\|q_1 - q_2\|_{H^{\frac{n}{2}+\epsilon}} \le C\|q_1 - q_2\|_{H^{t_0}}^{(1-\beta)}\|q_1 - q_2\|_{H^{t_1}}^{\beta}$. Setting $t_0 = -1$ and $t_1 = s$ we end up with

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \le C\|q_1 - q_2\|_{H^{-1}(\Omega)}^{\frac{\epsilon}{s+1}}$$

which yields the desired estimate (3) with $\sigma = \gamma\frac{\epsilon}{s+1}$.

We now turn to case (b). With a suitable translation and rotation, it suffices to assume $a = (0, \cdots, 0, R)$ and $0 \notin \overline{\Omega}$. As in [16], we shall use Kelvin's transform:

$$y = \left(\frac{2R}{|x|}\right)^2 x \quad \text{and} \quad x = \left(\frac{2R}{|y|}\right)^2 y. \tag{22}$$

Let

$$\tilde{u}(y) = \left(\frac{2R}{|y|}\right)^{n-2} u(x(y)),$$

then

$$\left(\frac{|y|}{2R}\right)^{n+2} \Delta_y \tilde{u}(y) = \Delta_x u(x).$$

Denote by $\tilde{\Omega}$ the transformed domain of Ω. In view of this transform, Γ_0 now becomes $\tilde{\Gamma}_0 \subset \{y_n = 2R\}$ and Γ is transformed to $\tilde{\Gamma}$ and $\tilde{\Gamma} = \partial\tilde{\Omega} \cap \{y_n > 2R\}$. On the other hand, if $u(x)$ satisfies $\Delta u - q(x)u = 0$ in Ω, then $\tilde{u}$ satisfies

$$\Delta\tilde{u} - \tilde{q}\tilde{u} = 0 \quad \text{in} \quad \tilde{\Omega}, \tag{23}$$

where

$$\tilde{q}(y) = \left(\frac{2R}{|y|}\right)^4 q(x(y)).$$

Therefore, for (23) we can define the partial Dirichlet-to-Neumann map $\tilde{\Lambda}_{\tilde{q},\tilde{\Gamma}}$ acting boundary functions with homogeneous data on $\tilde{\Gamma}_0$.

We now want to find the relation between $\Lambda_{q,\Gamma}$ and $\tilde{\Lambda}_{\tilde{q},\tilde{\Gamma}}$. It is easy to see that for $f, g \in H_0^{1/2}(\Gamma)$

$$\langle \Lambda_{q,\Gamma} f, g\rangle = \int_\Omega (\nabla u \cdot \nabla \overline{v} + qu\overline{v})\, \mathrm{d}x,$$

where u solves

$$\begin{aligned} \Delta u - qu &= 0 \quad \text{in} \quad \Omega, \\ u &= f \quad \text{on} \quad \partial\Gamma \end{aligned}$$

and $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Defining

$$\tilde{f} = \left(\frac{2R}{|y|}\right)^{n-2}\Big|_{\partial\tilde{\Omega}} f, \quad \tilde{g} = \left(\frac{2R}{|y|}\right)^{n-2}\Big|_{\partial\tilde{\Omega}} g,$$

and

$$\tilde{v}(y) = \left(\frac{2R}{|y|}\right)^{n-2} v(x(y)).$$

Then we have $\tilde{f}, \tilde{g} \in H_0^{1/2}(\tilde{\Gamma})$ and

$$\langle \Lambda_{q,\Gamma} f, g\rangle = \langle \tilde{\Lambda}_{\tilde{q},\tilde{\Gamma}} \tilde{f}, \tilde{g}\rangle,$$

in particular,

$$\langle (\Lambda_{q_1,\Gamma} - \Lambda_{q_2,\Gamma}) f, g\rangle = \langle (\tilde{\Lambda}_{\tilde{q}_1,\tilde{\Gamma}} - \tilde{\Lambda}_{\tilde{q}_2,\tilde{\Gamma}}) \tilde{f}, \tilde{g}\rangle. \tag{24}$$

With the assumption $0 \notin \overline{\Omega}$, the change of coordinates $x \to y$ by (22) is a diffeomorphism from $\overline{\Omega}$ onto $\overline{\tilde{\Omega}}$. Note that $(2R/|y|)^{n-2}$ is a positive smooth function on $\partial\tilde{\Omega}$. Recall a fundamental fact from Functional Analysis:

$$\|\Lambda_{q_1,\Gamma} - \Lambda_{q_2,\Gamma}\|_* = \sup\left\{ \frac{|\langle (\Lambda_{q_1,\Gamma} - \Lambda_{q_2,\Gamma}) f, g\rangle|}{\|f\|_{H_0^{1/2}(\Gamma)} \|g\|_{H_0^{1/2}(\Gamma)}} \;:\; f, g \in H_0^{1/2}(\Gamma) \right\}. \tag{25}$$

The same formula holds for $\|\tilde{\Lambda}_{\tilde{q}_1,\tilde{\Gamma}} - \tilde{\Lambda}_{\tilde{q}_2,\tilde{\Gamma}}\|_*$. On the other hand, it is not difficult to check that $\|f\|_{H_0^{1/2}(\Gamma)}$ and $\|\tilde{f}\|_{H_0^{1/2}(\tilde{\Gamma})}$, $\|g\|_{H_0^{1/2}(\Gamma)}$ and $\|\tilde{g}\|_{H_0^{1/2}(\tilde{\Gamma})}$ are equivalent, namely, there exists C depending on $\partial\Omega$ such that

$$\begin{aligned} \frac{1}{C}\|f\|_{H_0^{1/2}(\Gamma)} &\le \|\tilde{f}\|_{H_0^{1/2}(\tilde{\Gamma})} \le C\|f\|_{H_0^{1/2}(\Gamma)}, \\ \frac{1}{C}\|g\|_{H_0^{1/2}(\Gamma)} &\le \|\tilde{g}\|_{H_0^{1/2}(\tilde{\Gamma})} \le C\|g\|_{H_0^{1/2}(\Gamma)}. \end{aligned} \tag{26}$$

Putting together (24), (25), and (26) leads to

$$\|\tilde{\Lambda}_{\tilde{q}_1,\tilde{\Gamma}} - \tilde{\Lambda}_{\tilde{q}_2,\tilde{\Gamma}}\|_* \le C\|\Lambda_{q_1,\Gamma} - \Lambda_{q_2,\Gamma}\|_* \tag{27}$$

with C only depending on $\partial\Omega$.

With all the preparations described above, we use case (a) for the domain $\tilde{\Omega}$ with the partial Dirichlet-to-Neumann map $\tilde{\Lambda}_{\tilde{q},\tilde{\Gamma}}$. Therefore, we immediately obtain the estimate:

$$\|\tilde{q}_1 - \tilde{q}_2\|_{L^\infty(\tilde{\Omega})} \le C\big|\log\|\tilde{\Lambda}_{\tilde{q}_1,\tilde{\Gamma}} - \tilde{\Lambda}_{\tilde{q}_2,\tilde{\Gamma}}\|_*\big|^{-\sigma}.$$

Finally, rewinding $\tilde{q}$ and using (27) yields the estimate (3).

4. Stability estimate for the conductivity

We aim to prove Corollary 1.2 in this section. We recall the following well-known relation: let $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ then

$$\Lambda_{q,\Gamma}(f) = \gamma^{-1/2}|_\Gamma \Lambda_{\gamma,\Gamma}(\gamma^{-1/2}|_\Gamma f) + \frac{1}{2}(\gamma^{-1}\partial_\nu\gamma)|_\Gamma f.$$

In view of the a priori assumption (5), we have that

$$(\Lambda_{q_1,\Gamma} - \Lambda_{q_2,\Gamma})(f) = \gamma^{-1/2}|_\Gamma(\Lambda_{\gamma_1,\Gamma} - \Lambda_{\gamma_2,\Gamma})(\gamma^{-1/2}|_\Gamma f)$$

where $\gamma^{-1/2}|_\Gamma := \gamma_1^{-1/2}|_\Gamma = \gamma_2^{-1/2}|_\Gamma$, which implies

$$\|\Lambda_{q_1,\Gamma} - \Lambda_{q_2,\Gamma}\|_* \leq C\|\Lambda_{\gamma_1,\Gamma} - \Lambda_{\gamma_2,\Gamma}\|_* \tag{28}$$

for some $C = C(N) > 0$. Hereafter, we set $q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}}$, $j = 1, 2$. The regularity assumption (4) and Sobolev's embedding theorem imply that $q_1, q_2 \in C^1(\overline{\Omega})$. Using this and (5), we conclude that $\hat{q}_1 - \hat{q}_2$ satisfies the assumptions of Lemma 2.2 with $\alpha = 1$. Therefore, Theorem 1.1 and (28) imply that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C\Big|\log\|\Lambda_{\gamma_1,\Gamma} - \Lambda_{\gamma_2,\Gamma}\|_*\Big|^{-\sigma_1} \tag{29}$$

where C depend on Ω, N, n, s and σ_1 depend on n, s. Next, we recall from [1, (26) on page 168] that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\|q_1 - q_2\|_{L^\infty(\Omega)}^{\sigma_2} \tag{30}$$

for some $0 < \sigma_2 < 1$, where $C = C(N, \Omega)$ and $\sigma_2 = \sigma_2(n, s)$. Finally, putting together (29) and (30) yields (6) with $\sigma = \sigma_1\sigma_2$ and the proof of Corollary 1.2 is complete.

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