Liouville-type theorem for the Lamé system with singular coefficients

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Abstract

In this paper, we study a Liouville-type theorem for the Lamé system with rough coefficients in the plane. Let u be a real-valued two-vector in $\mathbb{R}^2$ satisfying $\nabla u \in L^p(\mathbb{R}^2)$ for some $p > 2$ and the equation $\operatorname{div}\left(\mu\left[\nabla u + (\nabla u)^T\right]\right) + \nabla(\lambda \operatorname{div} u) = 0$ in $\mathbb{R}^2$. When $\|\nabla \mu\|_{L^2(\mathbb{R}^2)}$ is not large, we show that $u \equiv$ constant in $\mathbb{R}^2$. As by-products, we prove the weak unique continuation property and the uniqueness of the Cauchy problem for the Lamé system with small $\|\mu\|_{W^{1,2}}$.

1 Introduction

The study in this work is motivated by the Liouville theorem for harmonic functions and the unique continuation property for the Lamé system with rough coefficients. Let u be a harmonic function in $\mathbb{R}^n$ with $n \geq 2$. The Liouville theorem states that if u is bounded, then u is a constant. Alternatively, we can also formulate the Liouville theorem in terms of an integrability condition. Precisely, if u is harmonic and $u \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$, then u is zero. This implication can be easily seen by the mean value property of u.

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The Lamé system in $\mathbb{R}^n$, which represents the displacement equation of equilibrium, is given by

$$\operatorname{div}\left(\mu\left[\nabla u+(\nabla u)^T\right]\right)+\nabla(\lambda \operatorname{div} u)=0 \quad \text{in} \quad \mathbb{R}^n, \tag{1.1}$$

where $u=(u_1,\cdots,u_n)^T$ is the displacement vector and $(\nabla u)_{jk}=\partial_k u_j$ for $j,k=1,\cdots,n$. The coefficients μ and λ are called Lamé parameters, which usually satisfy the ellipticity condition:

$$\lambda+2\mu>0 \quad \text{and} \quad \mu>0.$$

When both λ and μ are constant, the Lamé system can be written as the Navier equation

$$\mu\Delta u+(\lambda+\mu)\nabla \operatorname{div} u=0.$$

When $n=2$, if we define $f=\partial_1 u_1+\partial_2 u_2$ and $g=\partial_2 u_1-\partial_1 u_2$, then straightforward computations show that $h=(\lambda+2\mu)f-i\mu g$ is holomorphic. Thus, the Liouville theorem is valid for h. In particular, if $\nabla u\in L^p(\mathbb{R}^2)$ with $p\in[1,\infty)$, then $h\equiv 0$, and therefore, u must be a constant.

The aim of this paper is to extend this property to the case where μ and λ are not constants and may even be unbounded. Precisely, we prove the following.

Theorem 1.1 *Let $u\in W^{1,p}_{loc}(\mathbb{R}^2)$ be a real-valued 2-vector satisfying (1.1) with $p\in(2,\infty)$. Assume that $\nabla u\in L^p(\mathbb{R}^2)$, that the Lamé coefficients λ, μ are measurable functions satisfying*

$$0<c\le\mu(x), \quad c\le\lambda(x)+2\mu(x) \quad a.e.\ x\in\mathbb{R}^2 \tag{1.2}$$

for some $0<c<1$. Furthermore, suppose that linear maps T_λ and T_μ, defined by $T_\lambda w=\lambda w$ and $T_\mu w=\mu w$, satisfy

$$\begin{cases} T_\lambda: L^p(\mathbb{R}^2)\to L^{r_1}(\mathbb{R}^2) & \textit{is bounded, for some } r_1\in[1,\infty),\\ T_\mu: L^p(\mathbb{R}^2)\to L^{r_2}(\mathbb{R}^2) & \textit{is bounded, for some } r_2\in[1,\infty). \end{cases} \tag{1.3}$$

Then there exists a constant $\varepsilon=\varepsilon(p,c)$ such that if $\|\nabla\mu\|_{L^2(\mathbb{R}^2)}<\varepsilon$, then u is a constant.

Remark 1.2 *It can be seen that if $\lambda\in L^{s_1}(\mathbb{R}^2)$ and $\mu\in L^{s_2}(\mathbb{R}^2)$ with $s_1,s_2\ge p'$, where p' is the conjugate exponent of p, then T_λ and T_μ satisfy (1.3). In* *particular, if* $\mu \in W^{1,2}(\mathbb{R}^2)$*, then* T_μ *satisfies* (1.3) *and* $\|\nabla\mu\|_{L^2(\mathbb{R}^2)}$ *is finite. The regularity of* $\mu \in W^{1,2}(\mathbb{R}^2)$ *is most likely optimal. We also want to point out that* μ *could be unbounded since* $W^{1,2}(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2)$*. In view of* (1.2)*, the coefficient* λ *can be unbounded as well. That is, we do not require* $\lambda \in L^\infty(\mathbb{R}^2)$*.*

By Theorem 1.1, we can establish the following *weak unique continuation property* for (1.1).

Corollary 1.3 *Let* $u \in W^{1,p}_{loc}(\mathbb{R}^2)$ *(for* $p \in (2,\infty)$*) be a solution to* (1.1) *for which* ∇u *is supported on a compact set* $K \subset \mathbb{R}^2$*. Assume that* λ*,* μ *are measurable functions of* $\mathbb{R}^2$*,* $\mu \in W^{1,2}(K)$ *and* $\lambda \in L^{s_1}(K)$ *with* $s_1 \geq p'$*, and the ellipticity* (1.2) *holds for* $x \in K$*. Then there exists an* $\varepsilon > 0$*, depending on* c*,* K*, and* p*, such that if* $\|\mu\|_{W^{1,2}(K)} \leq \varepsilon$*, then* $u \equiv \text{constant}$*.*

Corollary 1.3 implies the uniqueness of the Cauchy problem for the Lamé system (1.1). As far as we know, this is the first uniqueness result in the Cauchy problem for (1.1) having the least regularity assumptions on the Lamé coefficients.

Corollary 1.4 *Let* Ω *be a bounded and connected domain in* $\mathbb{R}^2$ *with boundary* $\partial\Omega$*. Let* Γ *be an open segment of* $\partial\Omega$ *with* $\Gamma \in C^{1,1}$*. Assume that* $\lambda \in L^{s_1}(\Omega)$ *and* $\|\mu\|_{W^{1,2}(\Omega)} < \varepsilon$ *with the same* ε *as given in Corollary 1.3, and that* λ*,* μ *satisfy the ellipticity condition* (1.2)*. Moreover, suppose that* $\mu \in W^{1,\infty}(\Omega_\delta)$ *and* $\lambda \in L^\infty(\Omega_\delta)$*, where* $\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$ *with any fixed small* δ*. If* $u \in W^{1,p}(\Omega)$ *satisfies*

$$\text{div}\left(\mu\left[\nabla u + (\nabla u)^T\right]\right) + \nabla(\lambda \text{div} u) = 0 \quad in \quad \Omega$$

and

$$u|_\Gamma = 0, \quad [\mu(\nabla u + (\nabla u)^T) + \lambda \text{div} u]\nu|_\Gamma = 0,$$

where ν *is the unit outer normal of* Γ*, then* $u \equiv 0$ *in* Ω*.*

As mentioned above, our study is also motivated by the unique continuation property for (1.1). For $u \in W^{1,q}_{loc}(\Omega)$, where Ω is a connected domain of $\mathbb{R}^n$, we say that u is flat at 0 if

$$\int_{|x|<r} |u|^q = O(r^N)$$

for any $N \in \mathbb{N}$. We are interested in determining whether $u \in W^{1,q}_{loc}(\Omega)$ satisfying (1.1) is identically zero in Ω when u is flat at 0. This is the so-called strong unique continuation property (SUCP). Our main focus here is on the regularity assumption of the parameters. There are a lot of results concerning this problem, and we mention the article [8] for the best possible regularity assumption so far. In that article, the authors showed the SUCP holds when $\mu \in W^{1,\infty}(\Omega)$ and $\lambda \in L^\infty(\Omega)$ for dimension $n \geq 2$. Based on the qualitative SUCP of [8], quantitative SUCP (doubling inequalities) was recently derived in [7]. To give perspective to put our study, we recall that the unique continuation property may fail for a second order elliptic equation in $n \geq 3$ if the leading coefficients are only Hölder continuous (see counterexamples in [9] and [10]). On the other hand, for a scalar second-order elliptic equation in divergence or non-divergence form with $n = 2$, the SUCP holds even when the leading coefficients are essentially bounded, [1], [3], [4], and [11]. We also would like to point out that the SUCP for (1.1) may not hold when μ is essentially bounded or even continuous, see [5] and [6]. On the positive side, it was proved in [6] that if $\|\mu - 1\|_{L^\infty(\mathbb{R}^2)} + \|\lambda + 1\|_{L^\infty(\mathbb{R}^2)} < \varepsilon$ for some small ε and u is a Lipschitz function that vanishes in the lower half space, then u is trivial. An interesting open question here is to prove or disprove the strong unique continuation property or even unique continuation property for (1.1) with $n = 2$ when $(\mu, \lambda) \in (W^{1,p}(\Omega), L^\infty(\Omega))$ for $p < \infty$. Any attempt to derive an $L^p - L^q$ Carleman estimate for our system has not yet worked.

Our strategy in proving Theorem 1.1 is to derive a reduced system from (1.1). The derivation uses the idea in [8] and [7]. We then apply the Liouville theorem for holomorphic functions and the mapping property of the Cauchy transform to finish the proof. We want to emphasize that the proof does not rely on a Carleman estimate.

2 Reduced system

Here we derive a useful reduced system from (1.1). We rewrite (1.1) as

$$
\begin{aligned}
&\operatorname{div}\begin{bmatrix} 2\mu\partial_1 u_1 & \mu(\partial_1 u_2+\partial_2 u_1) \\ \mu(\partial_1 u_2+\partial_2 u_1) & 2\mu\partial_2 u_2 \end{bmatrix} + \begin{bmatrix} \partial_1(\lambda(\partial_1 u_1+\partial_2 u_2)) \\ \partial_2(\lambda(\partial_1 u_1+\partial_2 u_2)) \end{bmatrix} \\
=&\begin{bmatrix} \partial_1(2\mu\partial_1 u_1)+\partial_2(\mu(\partial_1 u_2+\partial_2 u_1)) \\ \partial_1(\mu(\partial_1 u_2+\partial_2 u_1))+\partial_2(2\mu\partial_2 u_2) \end{bmatrix} + \begin{bmatrix} \partial_1(\lambda(\partial_1 u_1+\partial_2 u_2)) \\ \partial_2(\lambda(\partial_1 u_1+\partial_2 u_2)) \end{bmatrix} \\
=&\begin{bmatrix} \partial_1((2\mu+\lambda)\partial_1 u_1+\lambda\partial_2 u_2)+\partial_2(\mu(\partial_1 u_2+\partial_2 u_1)) \\ \partial_1(\mu(\partial_1 u_2+\partial_2 u_1))+\partial_2(\lambda\partial_1 u_1+(2\mu+\lambda)\partial_2 u_2) \end{bmatrix} \\
=&\,0.
\end{aligned}
\tag{2.1}
$$

Defining

$$
v=\frac{\lambda+2\mu}{\mu}\operatorname{div} u \quad \text{and} \quad \operatorname{rot} u = w = \partial_2 u_1 - \partial_1 u_2,
$$

we compute

$$
\begin{aligned}
&(\partial_1 - i\partial_2)(\mu v + i\mu w) \\
=&\{\partial_1(\mu v)+\partial_2(\mu w)\} + i\{\partial_1(\mu w)-\partial_2(\mu v)\} \\
=&\partial_1\{(\lambda+2\mu)(\partial_1 u_1+\partial_2 u_2)\}+\partial_2\{\mu(\partial_2 u_1-\partial_1 u_2)\} \\
&+ i\{\partial_1[\mu(\partial_2 u_1-\partial_1 u_2)]-\partial_2[(\lambda+2\mu)(\partial_1 u_1+\partial_2 u_2)]\}.
\end{aligned}
\tag{2.2}
$$

Using (2.1), we calculate the real and imaginary parts on the right hand side of (2.2) explicitly,

$$
\begin{aligned}
&\partial_1\{(\lambda+2\mu)(\partial_1 u_1+\partial_2 u_2)\}+\partial_2\{\mu(\partial_2 u_1-\partial_1 u_2)\} \\
=&\partial_1[(2\mu+\lambda)\partial_1 u_1+\lambda\partial_2 u_2)]+\partial_1(2\mu\partial_2 u_2) \\
&+\partial_2[\mu(\partial_2 u_1+\partial_1 u_2)]-\partial_2(2\mu\partial_1 u_2) \\
=&2\partial_1(\mu\partial_2 u_2)-2\partial_2(\mu\partial_1 u_2) \\
=&2\partial_1\mu\partial_2 u_2-2\partial_2\mu\partial_1 u_2
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_1[\mu(\partial_2 u_1-\partial_1 u_2)]-\partial_2[(\lambda+2\mu)(\partial_1 u_1+\partial_2 u_2)] \\
=&\partial_1[\mu(\partial_2 u_1-\partial_1 u_2)]-\partial_2[\lambda\partial_1 u_1+(2\mu+\lambda)\partial_2 u_2]-\partial_2(2\mu\partial_1 u_1) \\
=&\partial_1[\mu(\partial_2 u_1-\partial_1 u_2)]+\partial_1[\mu(\partial_1 u_2+\partial_2 u_1)]-\partial_2(2\mu\partial_1 u_1) \\
=&2\partial_1(\mu\partial_2 u_1)-2\partial_2(\mu\partial_1 u_1) \\
=&2\partial_1\mu\partial_2 u_1-2\partial_2\mu\partial_1 u_1.
\end{aligned}
$$

In other words, (2.2) is equivalent to

$$\bar{\partial}(\mu v - i\mu w) = g_1 - ig_2 := g, \tag{2.3}$$

where $\bar{\partial} = (\partial_1 + i\partial_2)/2$ and

$$g_1 = \partial_1\mu\partial_2 u_2 - \partial_2\mu\partial_1 u_2,$$
$$g_2 = \partial_1\mu\partial_2 u_1 - \partial_2\mu\partial_1 u_1.$$

In view of Theorem 1.1, we can assume that $\|\nabla\mu\|_{L^2(\mathbb{R}^2)}$ is finite. Thus, we have that $g \in L^{2p/(p+2)}(\mathbb{C})$. From now on, we identify $\mathbb{R}^2 := \mathbb{C}$. Note that $1 < 2p/(p+2) < 2$ since $p > 2$. Equation (2.3) is a neat $\bar{\partial}$ equation to which we can find explicit solutions via the Cauchy transform.

3 The Cauchy transform

Any solution of (2.3) is explicitly given by

$$(\mu v - i\mu w)(z) = h(z) + \mathcal{C}g(z), \tag{3.1}$$

where h is holomorphic and

$$\mathcal{C}g(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\xi)}{\xi - z} d\xi$$

is the Cauchy transform of g. We write (3.1) as

$$(\mu v - i\mu w) - \mathcal{C}g = h(z).$$

Note that $\mu v - i\mu w = (\lambda + 2\mu)\mathrm{div}\, u - i\mu \,\mathrm{rot}\, u$. By the boundedness assumptions of T_λ and T_μ given in (1.3), we have $(\mu v - i\mu w) \in L^{r_1}(\mathbb{C}) + L^{r_2}(\mathbb{C})$. Recall the mapping property of $\mathcal{C}$:

$$\|\mathcal{C}f\|_{L^{2q/(2-q)}(\mathbb{C})} \leq C_q \|f\|_{L^q(\mathbb{C})} \tag{3.2}$$

for $1 < q < 2$ (see [2, Theorem 4.3.8]). Applying (3.2) with $q = 2p/(p+2)$ to g implies that

$$\|\mathcal{C}g\|_{L^p(\mathbb{C})} \leq C_p \|g\|_{L^{2p/(p+2)}(\mathbb{C})}. \tag{3.3}$$

Liouville's theorem for the holomorphic function gives $h \equiv 0$ and thus

$$(\mu v - i\mu w) = \mathcal{C}g. \tag{3.4}$$

Combining (3.3) and (3.4), we can estimate

$$\begin{aligned} c\|(\operatorname{div} u, \operatorname{rot} u)\|_{L^p(\mathbb{C})} &\le \|(\mu v - i\mu w)\|_{L^p(\mathbb{C})} = \|\mathcal{C} g\|_{L^p(\mathbb{C})} \\ &\le C_p \|g\|_{L^{2p/(p+2)}(\mathbb{C})} \le C_p \|\nabla \mu\|_{L^2(\mathbb{C})} \|\nabla u\|_{L^p(\mathbb{C})} \\ &\le C_p \varepsilon \|\nabla u\|_{L^p(\mathbb{C})}. \end{aligned} \tag{3.5}$$

We now show that $\|\nabla u\|_{L^p(\mathbb{C})}$ can be bounded by $\|(\operatorname{div} u, \operatorname{rot} u)\|_{L^p(\mathbb{C})}$. This fact is well-known when $p = 2$. For $p > 2$, we can proceed as follows. Note that

$$\Delta u = \nabla \operatorname{div} u + \operatorname{rot}^{\perp} \operatorname{rot} u,$$

where $\operatorname{rot}^{\perp} f = (\partial_2 f, -\partial_1 f)^T$ for any scalar function f. Observing that $\nabla(-\Delta)^{-1}\operatorname{div}$ is a Calderón-Zygmund operator, we obtain that

$$\|\nabla u\|_{L^p(\mathbb{C})} \le \tilde{C}_p \|(\operatorname{div} u, \operatorname{rot} u)\|_{L^p(\mathbb{C})} \tag{3.6}$$

with another p-dependent constant $\tilde{C}_p$. Combining (3.5) and (3.6) shows that if $\varepsilon < c/\left(C_p \tilde{C}_p\right)$, then $\|(\operatorname{div} u, \operatorname{rot} u)\|_{L^p(\mathbb{C})} = 0$. It follows from (3.6) that $\|\nabla u\|_{L^p(\mathbb{C})} = 0$ as well, and then we may conclude that u is constant, finishing the proof of Theorem 1.1. ☐

Finally, we would like to prove Corollaries 1.3 and 1.4.

Proof of Corollary 1.3. In view of the fact $\nabla u \equiv 0$ in $\mathbb{C} \setminus K$, any extensions of λ and μ to $\mathbb{R}^2 \setminus K$ will satisfy (1.3) for all L^p functions supported in K. Theorem 1.1 implies that if ε is sufficiently small, then ∇u also vanishes in K. ☐

Proof of Corollary 1.4. Let $x_0 \in \Gamma$ and $B(x_0)$ be a ball centered at x_0 with radius chosen so that $B(x_0) \cap \partial\Omega \subset \Gamma$. Define $\tilde{\Omega} = \Omega \cup B(x_0)$ and

$$\tilde{u} = \begin{cases} u & x \in \Omega \\ 0 & \tilde{\Omega} \setminus \Omega. \end{cases}$$

Let $\tilde{\lambda}$ and $\tilde{\mu}$ be extensions of λ and μ to $\tilde{\Omega}$ such that $\tilde{\lambda}|_{\tilde{\Omega}\setminus\Omega}$ is bounded and $\tilde{\mu} \in W^{1,\infty}(\Omega_\delta \cup (\tilde{\Omega} \setminus \Omega))$. Then $\tilde{u} \in W^{1,p}(\tilde{\Omega})$ is a weak solution to (1.1) in $\tilde{\Omega}$ with the coefficients $\tilde{\lambda}$, $\tilde{\mu}$. By the unique continuation property [8], we obtain that u is zero in Ω_δ. We now can extend u to $\mathbb{R}^2$ by setting $u = 0$ in $\mathbb{R}^2 \setminus \Omega$. By the weak unique continuation property, Corollary 1.3, the extended u is trivial in $\mathbb{R}^2$. In other words, $u \equiv 0$ in Ω. ☐

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