ENCLOSURE METHODS FOR THE HELMHOLTZ-TYPE EQUATIONS

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1. Introduction

This paper serves as a survey of enclosure-type methods used to determine the obstacles or inclusions embedded in the background medium from the near-field measurements of propagating waves. A type of complex geometric optics waves that exhibits exponential decay with distance from some critical level surfaces (hyperplanes, spheres or other types of level sets of phase functions) are sent to probe the medium. One can easily manipulate the speed of decay such that the waves can only detect the material feature that is close enough to the level surfaces. As a result of sending such waves with level surfaces moving along each direction, one should be able to pick out those that enclose the inclusion.

The problem that Calderón proposed in 80's [3] was whether one can determine the electrical conductivity by making voltage and current measurements at the boundary of the medium. Such electrical methods are also known as Electrical Impedance Tomography (EIT) and have broad applications in medical imaging, geophysics and so on. A breakthrough in solving the problem was due to Sylvester and Uhlmann. In [26], they constructed the complex geometric optics (CGO) solutions to the conductivity equation and proved the unique determination of C^∞ isotropic conductivity from the boundary measurements in three and higher dimensional spaces. The result has been extended to conductivities with $3/2$ derivatives in three dimensions and L^∞ conductivities in two dimensions.

The inverse problem in this paper concerns reconstructing an obstacle or a jump-type inclusion embedded in a known background medium, which is not included in the previous results when considering electrostatics. Several methods are proposed to solve the problem based on utilizing, generally speaking, two special types of solutions. The Green's type solutions were considered first by Isakov [13], and several sampling methods [4, 14, 1, 2] and probing methods [10, 24] were developed. On the other hand, with the CGO solutions at disposal, the enclosure method was introduced by Ikehata [8, 9] with the idea as described in the first paragraph. Another method worth mentioning uses the oscillating-decaying type of solutions and was

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proved valid for elasticity systems [20]. It is the enclosure type of methods that is of the presenting paper's interest.

Here we aim to discuss the enclosure method for the Helmholtz type equations. For the enclosure method in the static equations, we refer to [8], [9], [12], [29], [27] for the conductivity equation, to [28], [30] for the isotropic elasticity. The major difference between the static equations and the Helmholtz type ones is the loss of positivity in the latter equations. It turns out we have to analyze the effect of the reflected solution due to the existence of lower order term in the Helmholtz type equation. For the acoustic equation outside of a cavity having C^2 boundary, i.e., impenetrable obstacle. one can overcome the difficulty by the Sobolev embedding theorem, see [21] (also see [11] for similar idea). Such result can be generalized for Maxwell's equations to determine impenetrable electromagnetic obstacles [33]. However, in the inclusion case, i.e., penetrable obstacle, the coefficient is merely *piecewise* smooth. The Sobolev embedding theorem does not work because the solution is not smooth enough. To tackle the problem, a Hölder type estimate for the second order elliptic equation with coefficients having jump discontinuity based on the result of Li-Vogelius [17] was developed by Nagayasu, Uhlmann, and the first author in [19]. Later, the result of [19] was improved by Sini and Yoshida [25] using L^p estimate for the second order elliptic equation in divergence form developed by Meyers [18]. Recently, Kuan [16] extended Sini-Yoshida's method to the elastic wave equations.

The paper is organized as follows. In Section 2, we discuss the enclosure method for the acoustic and electromagnetic equations with cavity (impenetrable obstacle). In Section 3, we would like to survey results in the inclusion case (penetrable obstacle) for the acoustic and elastic waves. We will list some open problems in Section 4.

2. Enclosing obstacles using acoustic and electromagnetic waves

In this section, we give more precise descriptions of the enclosure methods to identify impenetrable obstacles of acoustic or electromagnetic characteristics. In particular, we are interested in the results in [8] and [21] for both convex and non-convex sound hard obstacles using complex geometrical optics (CGO) solutions for the Helmholtz equations and the result in [33] for perfect magnetic conducting obstacles using CGO solutions for Maxwell's equations.

2.1. Non-convex sound hard obstacles.

In [8] and [21], the authors consider the inverse scattering problem of identifying a sound hard obstacle $D \subset \mathbb{R}^n$, $n \geq 2$ in a homogeneous medium from the far field pattern. It can be reformulated as an equivalent inverse boundary value problem with near-field measurements described as follows. Given a bounded domain $\Omega \subset \mathbb{R}^n$ such that $\overline{D} \subset \Omega$ and $\Omega \backslash \overline{D}$ is connected, the underlying boundary value problem for acoustic wave propagation in the known homogeneous medium in $\Omega\backslash\overline{D}$ with no source is given by

$$\begin{cases} (\Delta+k^2)u = 0 \quad \text{in } \Omega\backslash\overline{D}, \\ u\big|_{\partial\Omega} = f, \\ \partial_\nu u\big|_{\partial D} = 0 \end{cases} \tag{2.1}$$

where $k > 0$ is the wave number and ν denotes the unit outer normal of ∂D. At this point, we assume that ∂D is C^2. Suppose k is not a Dirichlet eigenvalue of Laplacian. Given each prescribed boundary sound pressure $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution $u(x) \in H^1(\Omega\backslash\overline{D})$ to (2.1). The inverse boundary value problem is then to reconstruct the obstacle D from the full boundary data that can be encoded as the Dirichlet to Neumann (DN) map

$$\begin{aligned} \Lambda_D : \ H^{1/2}(\partial\Omega) \ &\to \ H^{-1/2}(\partial\Omega) \\ f \ &\mapsto \ \partial_\nu u\big|_{\partial\Omega}. \end{aligned} \tag{2.2}$$

In particular, the enclosure method utilizes the measurements (DN map) for those f taking the traces of CGO solutions to $(\Delta + k^2)u = 0$ in the background domain Ω

$$u_0 = e^{\tau(\varphi(x)-t)+i\psi(\tau;x))}\,(a(x) + r(x;\tau)) \tag{2.3}$$

where $r(x;\tau)$ and its first derivatives are uniformly bounded in τ. As $\tau \to \infty$, u_0 evolves vertical slope at the level set $\{x|\ \varphi(x) = t\}$ for $t \in \mathbb{R}$. Physically speaking, such evanescent waves couldn't detect the change of the material, namely the presence of D in Ω, happening relatively far from the level set. Hence, there is little gap between the associated energies of domains with and without D. On the other hand, if $\overline{D}$ ever intersects the level set, the energy gap is going to be significant for large τ. This implies that the geometric relation between D and the level set $\{x|\ \varphi(x) = t\}$ can be read from the following indicator function describing the energy gap associated to the input $f = u_0|_{\partial\Omega}$

$$I(\tau, t) := \int_{\partial\Omega} \big(\Lambda_D - \Lambda_\emptyset\big)(u_0|_{\partial\Omega})\ \overline{u_0}|_{\partial\Omega}\ dS \tag{2.4}$$

where $\Lambda_\emptyset$ represents the DN map associated to the background domain Ω without D, hence $\Lambda_\emptyset(u_0|_{\partial\Omega}) = \partial_\nu u_0|_{\partial\Omega}$. When the linear phase $\varphi(x) = x\cdot\omega$, $\omega \in \mathbb{S}^{n-1}$ is used, the CGO solution (2.3) is the exponential function

$$u_0(x) = e^{\tau(x\cdot\omega - t)+i\sqrt{\tau^2+k^2}x\cdot\omega^\perp}$$

where $\omega^\perp \in \mathbb{S}^{n-1}$ satisfies $\omega\cdot\omega^\perp = 0$. The physical discussion above is verified in the following result by Ikehata to enclose the convex hull of D by reconstructing the support function

$$h_D(\omega) := \sup_{x\in D} x\cdot\omega.$$

Theorem 2.1. [8] *Assume that the set* $\{x \in \mathbb{R}^n \mid x \cdot \omega = h_D(\omega)\} \bigcap \partial D$ *consists of one point and the Gaussian curvature of* ∂D *is not vanishing at that point. Then the support function* $h_D(\omega)$ *can be reconstructed by the formula*

$$h_D(\omega) = \inf\{t \in \mathbb{R} \mid \lim_{\tau\to\infty} I(\tau,t) = 0\}. \tag{2.5}$$

This result shows that a strictly convex obstacle can be identified by an envelope surface of planes. Geometrically, this appears as the planes are enclosing the obstacle from every direction, justifying the name "enclosure method".

It is natural to expect the method can be generalized to recover some non-convex part of the shape of D by using CGO solutions with non-linear phase. Based on a Carleman estimate approach, such solutions were constructed in [15] (or see [7]) for the Schrödinger operator (or the conductivity operator) in $\mathbb{R}^3$, with φ being one of a few limiting Carleman weights

$$\varphi(x) = \ln|x - x_0|, \quad x_0 \in \mathbb{R}^3 \backslash \overline{\Omega},$$

which bears spherical level sets, and therefore were called complex spherical waves (CSW). Then such CSW were used into the enclosure method in [12] to identify non-convex inclusions in a conductive medium. In $\mathbb{R}^2$, there are more candidates for the limiting carleman weights than in $\mathbb{R}^3$: all of the harmonic functions. Then the similar reconstruction scheme is available in [29] for more generalized two dimensional systems by using level curves of harmonic polynomials.

Here we present the result in [21] that adopts the CSW described in the following proposition to enclose a non-convex sound hard obstacle.

Proposition 2.2. [7] *Choose* $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$ *and let* $\omega_0 \in \mathbb{S}^{n-1}$ *be a vector such that*

$$\{x \in \mathbb{R}^n \mid x - x_0 = m\omega_0,\ m \in \mathbb{R}\} \bigcap \partial\Omega = \emptyset.$$

Then there exists a solution to the Helmholtz equation in Ω *of the form*

$$u_0(x;\tau,t,x_0,\omega_0) = e^{\tau(t - \ln|x-x_0|) - i\tau\psi(x)}\big(a(x) + r(x;\tau,t,x_0,\omega_0)\big) \tag{2.6}$$

where $\tau > 0$ *and* $t \in \mathbb{R}$ *are parameters,* $a(x)$ *is a smooth function on* $\overline{\Omega}$ *and* $\psi(x)$ *is a function defined by*

$$\psi(x) := d_{\mathbb{S}^{n-1}}\left(\frac{x - x_0}{|x - x_0|}, \omega_0\right)$$

with the metric function $d_{\mathbb{S}^{n-1}}(\cdot,\cdot)$ *on* $\mathbb{S}^{n-1}$*. Moreover, the remainder function* $r \in H^1(\Omega)$ *and satisfies*

$$\|r\|_{H^1(\Omega)} = O(\tau^{-1}), \qquad \text{as } \tau \to \infty.$$

The corresponding support function is given by

$$h_D(x_0) = \inf_{x\in D} \ln|x - x_0|, \quad x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$$

and can be reconstructed based on the following result.

Theorem 2.3. [21] *Let $x_0 \in \mathbb{R}^n \backslash \overline{\Omega}$. Assume that the set $\{x \in \mathbb{R}^n \mid |x - x_0| = e^{h_D(x_0)}\} \bigcap \partial D$ consists of finite points and the relative curvatures of ∂D at these points are positive. Then there are two characterizations of $h_D(x_0)$:*

$$h_D(x_0) = \sup\{t \in \mathbb{R} \mid \liminf_{\tau \to \infty} |I(\tau, t)| = 0\}, \tag{2.7}$$

and

$$t - h_D(x_0) = \lim_{\tau \to \infty} \frac{\ln |I(\tau, t)|}{2\tau} \tag{2.8}$$

where $I(\tau, t)$ is defined by (2.4) with u_0 by (2.6).

Remark 2.1. *The relative curvature in the theorem refers to the Gaussian curvature after the change of coordinates that stretches the sphere into flat. For a more rigorous definition, we refer to [21].*

For completeness, we provide briefly the steps of the proof. The proof of (2.7) involves showing the following statements:

$$\lim_{\tau \to \infty} |I(\tau, t)| = 0 \quad \text{ when } t < h_D(x_0), \tag{2.9}$$

that is, when the level sphere $S_{t,x_0} := \{x \in \mathbb{R}^n \mid |x - x_0| = e^t\}$ has no intersection with $\overline{D}$;

$$\liminf_{\tau \to \infty} |I(\tau, t)| > C > 0 \quad \text{ when } t \geq h_D(x_0), \tag{2.10}$$

namely, when S_{t,x_0} intersects $\overline{D}$. These two statements can be shown by establishing proper upper and lower bounds of $I(\tau, t)$ from the following key equality

$$-I(\tau, t) = \int_{\Omega \backslash \overline{D}} |\nabla w|^2 \, dx + \int_D |\nabla u_0|^2 \, dx - k^2 \int_{\Omega \backslash \overline{D}} |w|^2 \, dx - k^2 \int_D |u_0|^2 \, dx \tag{2.11}$$

where $w := u - u_0$ is the reflected solution and u is the solution to (2.1) with $f = u_0|_{\partial \Omega}$. Since w is a solution to

$$\begin{cases} (\Delta + k^2) w = 0 & \text{in } \Omega \backslash \overline{D}, \\ w|_{\partial \Omega} = 0, \\ \partial_\nu w|_{\partial D} = -\partial_\nu u_0|_{\partial D}, \end{cases} \tag{2.12}$$

and by (2.11), one has the upper bound

$$|I(\tau, t)| \leq C \|u_0\|^2_{H^1(D)}$$

for some constant $C > 0$ (through out the whole article we use the same C to denote the general constant). As a consequence of plugging in the CGO solution (2.6), the first statement (2.9) is obtained since

$$|I(\tau, t)| \leq C\tau^2 \int_D e^{2\tau(t - \ln|x - x_0|)} \, dx \quad (\tau \gg 1).$$

On the other hand, difficulty arises in dealing with the second statement (2.10). Due to the loss of positivity for the associated bilinear form, two negative terms present in (2.11), which implies that to find a non-vanishing (as $\tau \to \infty$) lower bound for $I(\tau, t)$ is not as easy as the case of conductivity equation, where $I(\tau, t) \geq C \int_D |\nabla u_0|^2 \, dx$ is close at hand. As a remedy, one needs to show that the two negative terms can be absorbed by the positive terms for τ large. To be more specific, first it is not hard to see

$$I(\tau, t) = e^{2\tau(t - h_D(x_0))} I(\tau, h_D(x_0)). \tag{2.13}$$

This implies that it is sufficient to show (2.10) for $t = h_D(x_0)$, which in turn can be derived from (2.11) and the following two inequalities when $t = h_D(x_0)$:

$$\liminf_{\tau \to \infty} \int_D |\nabla u_0|^2 \, dx > C > 0 \tag{2.14}$$

and

$$\frac{k^2 \int_{\Omega \setminus \overline{D}} |w|^2 \, dx + k^2 \int_D |u_0|^2 \, dx}{\int_D |\nabla u_0|^2 \, dx} < \delta < 1 \quad (\tau \gg 1). \tag{2.15}$$

(2.14) is true since

$$\begin{aligned} \int_D |\nabla u_0|^2 \, dx &\geq C\tau^2 \int_D e^{-2\tau(\ln|x - x_0| - h_D(x_0))} \, dx \\ &\geq \begin{cases} O(\tau^{1/2}) & n = 2 \\ O(1) & n = 3 \end{cases} \quad (\tau \gg 1) \end{aligned} \tag{2.16}$$

given the geometric assumption of the positive relative curvature of ∂D.

As for (2.15), the actually difficult part is to show

$$\liminf_{\tau \to \infty} \frac{k^2 \int_{\Omega \setminus \overline{D}} |w|^2 \, dx}{\int_D |\nabla u_0|^2 \, dx} = 0 \tag{2.17}$$

since the property of CGO solutions gives

$$\frac{k^2 \int_D |u_0|^2 \, dx}{\int_D |\nabla u_0|^2 \, dx} = O(\tau^{-2}) \quad (\tau \gg 1).$$

In both [8] and [21], (2.17) is proved by establishing the following estimate.

Lemma 2.4. *Let $S_{h_D(x_0), x_0} \bigcap \partial D = \{x_1, \ldots, x_N\}$ and define for $\alpha \in (0, 1)$*

$$I_{x_j, \alpha} := \int_{\partial D} |\partial_\nu u_0| \, |x - x_j|^\alpha \, dS, \quad j = 1, \ldots, N.$$

Then

$$\|w\|^2_{L^2(\Omega \setminus \overline{D})} \leq C \left(\sum_{j=1}^N I^2_{x_j, \alpha} + \|u_0\|^2_{L^2(D)} \right), \quad \alpha \in (0, 1) \tag{2.18}$$

Remark 2.2. *The proof of Lemma 2.4 is based on the H^2-regularity theory and the Sobolev embedding theorem for an auxiliary boundary value problem*

$$\begin{cases} (\Delta + k^2)p = \overline{w} & \text{in } \Omega\backslash\overline{D}, \\ p|_{\partial\Omega} = 0, \\ \partial_\nu p|_{\partial D} = 0. \end{cases}$$

Such estimates of the term $\|w\|_{L^2(\Omega\backslash\overline{D})}$ for the impenetrable obstacle case and $\|w\|_{L^2(\Omega)}$ for the penetrable inclusion case which will be reviewed in the next section, are usually crucial for the justification of the enclosure methods. Therefore, several improvements of the result and removing geometric assumptions are basically due to the development of different estimates, which we will see shortly.

In particular, choosing $\alpha = 1/2$ for $n = 3$ and $\alpha = 3/4$ when $n = 2$, one can show

$$I^2_{x_j,\alpha} \leq \begin{cases} \sqrt{\varepsilon}\, O(\tau^{1/2}) & n = 2, \\ O(\tau^{-1/2}) & n = 3. \end{cases}$$

for arbitrary small ε, again by the assumption that the relative curvature is positive. Combined with (2.16), one immediately obtains (2.15).

At last, the formula (2.8) is directly derived from (2.13) and the fact

$$|I(\tau, h_D(x_0))| \leq C\tau^2, \quad (\tau \gg 1).$$

Remark 2.3. *The result can be easily extended to the case with inhomogeneous background medium in $\Omega\backslash\overline{D}$, where the CSW in proposition 2.2 is available.*

2.2. **Electromagnetic PMC obstacles.** This part will be contributed to reviewing the enclosure method for the Maxwell's equations [33] to identify perfect magnetic conducting (PMC) obstacles. The same reconstruction scheme works for identifying perfect electric conducting (PEC) obstacles and more generalized impenetrable obstacles described with impedance conditions.

In a bounded domain $\Omega \subset \mathbb{R}^3$ with an obstacle D such that $\overline{D} \subset \Omega$ with ∂D being C^2 and $\Omega\backslash\overline{D}$ connected, the electric-magnetic field (E, H) satisfies the Maxwell's equations

$$\begin{aligned} &\nabla \times E = ik\mu H, \quad \nabla \times H = -ik\varepsilon E, \qquad \text{in } \Omega\backslash\overline{D}, \\ &\nu \times E|_{\partial\Omega} = f, \\ &\nu \times H|_{\partial D} = 0 \qquad \text{(PMC condition)} \end{aligned} \tag{2.19}$$

where k is the frequency and $\mu(x)$ and $\varepsilon(x)$ describe the isotropic (inhomogeneous) background electromagnetic medium and satisfy the following assumptions: there are positive constants $\varepsilon_m, \varepsilon_M, \mu_m, \mu_M$, ε_c and μ_c such that for all $x \in \Omega$

$$\varepsilon_m \leq \varepsilon(x) \leq \varepsilon_M, \ \ \mu_m \leq \mu(x) \leq \varepsilon_M, \ \ \sigma(x) = 0$$

and $\varepsilon-\varepsilon_c, \mu-\mu_c \in C_0^3(\Omega)$. Given that k is not a resonant frequency, we have a well defined boundary impedance map

$$\begin{aligned}\Lambda_D:\ TH^{1/2}(\partial\Omega) &\to TH^{1/2}(\partial\Omega)\\ f=\nu\times E|_{\partial\Omega} &\mapsto \nu\times H|_{\partial\Omega}.\end{aligned}$$

To show that D can be determined by the impedance map Λ_D using the enclosure method, we first notice an analogue of the identity (2.11) for the Maxwell's equations:

$$\begin{aligned}&i\omega\int_{\partial\Omega}(\nu\times E_0)\cdot\left[\overline{(\Lambda_D-\Lambda_\emptyset)(\nu\times E_0)}\times\nu\right]dS\\ &\qquad=\int_{\Omega\setminus\overline{D}}\mu|\tilde{H}|^2-\omega^2\varepsilon|\tilde{E}|^2dx+\int_D\mu|H_0|^2-\omega^2\varepsilon|E_0|^2dx.\end{aligned}\tag{2.20}$$

where $(\tilde{E},\tilde{H}):=(E-E_0,H-H_0)$ denotes the reflected solutions, (E,H) is the solution to (2.19), (E_0,H_0) is the solution to the Maxwell's equations

$$\nabla\times E_0=ik\mu H_0,\quad \nabla\times H_0=-ik\varepsilon E_0 \qquad \text{in } \Omega,\tag{2.21}$$

and $\nu\times E|_{\partial\Omega}=\nu\times E_0|_{\partial\Omega}$.

One would encounter the same difficulty as that for the Helmholtz equations due to the loss of positivity of the system. We recall that this was actually overcome by the property that the CGO solution u_0 shares different asymptotic speed (τ^2 slower) from ∇u_0. More specifically, this is because of the H^1 boundedness of the remainder $r(x;\tau)$ w.r.t τ in (2.3). The natural question to ask is then whether this key ingredient: such CGO type of solutions, can be constructed for the background Maxwell's system.

The construction of CGO solutions for the Maxwell's equations has been extensively studied in [22, 23] and [6]. The work in [33] adopts the construction approach in [23] by reducing the Maxwell's equations into a matrix Schrödinger equation. Finally, to guarantee that the CGO solution for the reduced matrix Schrödinger operator derives the CGO solution (E_0,H_0) for the Maxwell's equations and at the same time that the electric field E_0 and H_0 share different asymptotical speed as $\tau\to\infty$, the incoming constant field corresponding to $a(x)$ in (2.3) has to be chosen very carefully. To summarize, one has

Proposition 2.5. *Let $\omega,\omega^\perp\in\mathbb{S}^2$ with $\omega\cdot\omega^\perp=0$. Denote $\zeta=-i\tau\omega+\sqrt{\tau^2+k^2}\omega^\perp$ where $k_1=k(\varepsilon_0\mu_0)^{1/2}$. Choose $a\in\mathbb{R}^3$ such that*

$$a\perp\omega,\quad a\perp\omega^\perp \ \text{ and } \ b=\frac{1}{\sqrt{2}}\overline{(-i\omega+\omega^\perp)}.$$

Then given

$$\theta:=\frac{1}{|\zeta|}\left(-(\zeta\cdot a)\zeta-k_1\zeta\times b+k_1^2a\right),\quad \eta:=\frac{1}{|\zeta|}\left(k_1\zeta\times a-(\zeta\cdot b)\zeta+k_1^2b\right),$$

for $t\in\mathbb{R}$ and $\tau>0$ large enough, there exists a unique complex geometric optics solution $(E_0,H_0)\in H^1(\Omega)^3\times H^1(\Omega)^3$ of Maxwell's equations (2.21) *of the form*

$$E_0 = \varepsilon(x)^{-1/2} e^{\tau(x\cdot\omega - t) + i\sqrt{\tau^2+k^2}x\cdot\omega^\perp}(\eta + R(x))$$
$$H_0 = \mu(x)^{-1/2} e^{\tau(x\cdot\omega - t) + i\sqrt{\tau^2+k^2}x\cdot\omega^\perp}(\theta + Q(x)).$$

Moreover, we have

$$\eta = \mathcal{O}(1), \qquad \theta = \mathcal{O}(\tau) \quad \textit{for } \tau \gg 1,$$

and $R(x)$ and $Q(x)$ are bounded in $(L^2(\Omega))^3$ for $\tau \gg 1$.

Plugging in (E_0, H_0) into the indicator function defined by

$$I(\tau, t) := i\omega \int_{\partial\Omega} (\nu \times E_0) \cdot \left[\overline{(\Lambda_D - \Lambda_\emptyset)(\nu \times E_0)} \times \nu\right] dS,$$

a similar argument as for the Helmholtz equations follows using identity (2.20) and we have

Theorem 2.6. [33] *There is a subset $\Sigma \subset \mathbb{S}^2$ of measure zero such that when $\omega \in \mathbb{S}^2 \setminus \Sigma$, the support function*

$$h_D(\omega) := \sup_{x\in D} x \cdot \omega$$

can be recovered by

$$h_D(\omega) = \inf\{t \in \mathbb{R} \mid \lim_{\tau\to\infty} I(\tau, t) = 0\}.$$

Moreover, if D is strictly convex, one can reconstruct D.

On the other hand, the construction of a proper CGO solution with nonlinear weight for the Maxwell's equations has not been successful using the Carleman estimate. An alternative approach to reconstruct non-convex part of the shape of D would be introducing some transformation that is coordinate invariant. For example, one can utilize the Kelvin transformation

$$T_{x_0,R}:\ x \mapsto R^2 \frac{x - x_0}{|x - x_0|^2} + x_0 := y,$$

which is the inversion transformation with respect to the sphere $S(x_0, R)$ for $R > 0$ and $x_0 \in \mathbb{R}^3\backslash\overline{\Omega}$. $T_{x_0,R}$ maps generalized spheres (spheres and planes) into generalized spheres. Geometrically, fixing a reference circle $S(x_0, R)$, enclosing D with spheres passing through x_0 corresponds to enclosing $\hat{D}_{x_0,R} = T_{x_0,R}(D)$ with planes, where the reconstruction scheme in Theorem 2.6 applies. A rigorous proof consists of showing that the Maxwell's equations are invariant under the transformation and computing the impedance map $\hat{I}(\tau, t)$ associated to the image domain. It is worth mentioning the byproduct of this method is the complex spherical wave

$$\hat{E}(y) = \hat{E}_j dy^j = \Big((DT^{-1}_{x_0,R})^k_j(y) E_k(T^{-1}_{x_0,R}(y))\Big)\, dy^j, \qquad y = T_{x_0,R}(x)$$

with nonlinear limiting Carleman weight

$$\varphi(x) = \left(R^2 \frac{x - x_0}{|x - x_0|^2} + x_0\right) \cdot \omega, \quad \omega \in \mathbb{S}^2.$$

Therefore, the corresponding support function is given by

$$\hat{h}_D(x_0, R, \omega) = \sup_{x\in D}\left\{R^2\left(\frac{x-x_0}{|x-x_0|^2}\right)\cdot\omega + x_0\cdot\rho\right\}$$

Theorem 2.7. [33] *Given $x_0 \in \mathbb{R}^3\backslash\overline{\Omega}$ and $R > 0$ such that $\overline{\Omega} \subset B(x_0, R)$, there is a zero measure subset Σ of $\mathbb{S}^2$, s.t., when $\omega \in \mathbb{S}^2\backslash\Sigma$, we have*

$$\hat{h}_D(x_0, R, \omega) = \inf\{t \in \mathbb{R} \mid \lim_{\tau\to\infty} \hat{I}(\tau, t) = 0\}.$$

3. Enclosing inclusions using acoustic and elastic waves

In this section we will consider the enclosure method for the case where the unknown domain is an inclusion by using acoustic and elastic waves. In other words, the obstacle is a penetrable one. In this situation, the reflected solution will satisfy the elliptic equation with discontinuous coefficients. Unlike the case of impenetrable obstacle, the Sobolev embedding theorem is not sufficient to provide us estimates of the reflected solution we need. When the background is an acoustic wave, the difficulty was overcome in [19] using estimates obtained by Li and Vogelius in [17]. We only consider $n = 2$ in [19] and the extension to $n = 3$ was done by Yoshida in [32]. Later, Sini and Yoshida improved the result in [19] with the help of Meyers' L^p estimate and the sharp Freidrichs inequality [25]. Kuan [16] then extended Sini-Yoshida's result to elastic waves.

3.1. Acoustic penetrable obstacle.

Here we will review the result in [19] for $n = 2$. For $n = 3$, one simply replaces CGO solutions in $n = 2$ by complex spherical waves [32]. We assume $D \Subset \Omega \subset \mathbb{R}^2$. For technical simplicity, we suppose that both D and Ω have C^2 boundaries. Let $\gamma_D \in C^2(\overline{D})$ satisfy $\gamma_D \geq c_\gamma$ for some positive constant c_γ and dnote $\widetilde{\gamma} := 1 + \gamma_D\chi_D$, where χ_D is the characteristic function of D. Let $k > 0$ and consider the steady state acoustic wave equation in Ω with Dirichlet condition

$$\begin{cases} \nabla\cdot(\widetilde{\gamma}\nabla v) + k^2 v = 0 \text{ in } \Omega, \\ v = f \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

We assume that k^2 is not a Dirichlet eigenvalue of the operator $-\nabla\cdot(\widetilde{\gamma}\nabla\bullet)$. Let $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be the associated Dirichlet-to-Neumann map. As before, our aim is to reconstruct the shape of D by Λ_D. The key in the enclosure method is the CGO solutions. For the two dimensional case, we have a lot of choices of phases in the CGO solutions. When the background medium is homogeneous, we make use of the CGO solutions to the Helmholtz equation. To construct the CGO solutions to the Helmholtz equation for $n = 2$, we begin with the CGO solutions with polynomial phases to the Laplacian operator. We then obtain the CGO solutions to the Helmholtz equation by way of the Vekua transform [31, Page 58].

More precisely, let us define $\eta(x) := c_*\big((x_1 - x_{*,1}) + i(x_2 - x_{*,2})\big)^N$ as the phase function, where $c_* \in \mathbb{C}$ satisfies $|c_*| = 1$, N is a positive integer, and

$x_* = (x_{*,1}, x_{*,2}) \in \mathbb{R}^2 \setminus \overline{\Omega}$. Without loss of generality we may assume that $x_* = 0$ using an appropriate translation. Denote $\eta_{\mathrm{R}}(x) := \operatorname{Re}\eta(x)$ and note that

$$\eta_{\mathrm{R}}(x) = r^N \cos N(\theta - \theta_*) \text{ for } x = r(\cos\theta, \sin\theta) \in \mathbb{R}^2.$$

It is readily seen that $\eta_{\mathrm{R}}(x) > 0$ for all $x \in \Gamma$, where

$$\Gamma := \left\{ r(\cos\theta, \sin\theta) : |\theta - \theta_*| < \frac{\pi}{2N} \right\},$$

i.e., a cone with opening angle π/N.

Given any $h > 0$, $\check{V}_\tau(x) := \exp\bigl(\tau\eta(x)\bigr)$ is a harmonic function. Following Vekua [31], we define a map T_k on any harmonic function $\check{V}(x)$ by

$$\begin{aligned} T_k \check{V}(x) &:= \check{V}(x) - \int_0^1 \check{V}(tx)\, \frac{\partial}{\partial t}\Bigl\{ J_0\bigl(k|x|\sqrt{1-t}\bigr)\Bigr\} dt \\ &= \check{V}(x) - k|x| \int_0^1 \check{V}\bigl((1-s^2)x\bigr)\, J_1\bigl(k|x|s\bigr)\, ds \end{aligned}$$

where J_m is the Bessel function of the first kind of order m. We now set $V_\tau^\sharp(x) := T_k \check{V}_\tau(x)$. Then $V_\tau^\sharp(x)$ satisfies the Helmholtz equation $\Delta V_\tau^\sharp + k^2 V_\tau^\sharp = 0$ in $\mathbb{R}^2$. One can show that $V_\tau^\sharp$ satisfies the following estimate in Γ

Lemma 3.1. [19] *We have*

$$V_\tau^\sharp(x) = \exp\left(\tau\eta(x)\right)\bigl(1 + R_0(x)\bigr) \text{ in } \Gamma, \tag{3.2}$$

where $R_0(x) = R_0(x;\tau)$ *satisfies*

$$|R_0(x)| \le \frac{1}{\tau}\, \frac{k^2|x|^2}{4\eta_{\mathrm{R}}(x)}, \qquad \left|\frac{\partial R_0}{\partial x_j}(x)\right| \le \frac{Nk^2|x|^{N+1}}{4\eta_{\mathrm{R}}(x)} + \frac{1}{\tau}\, \frac{k^2|x_j|}{2\eta_{\mathrm{R}}(x)} \quad \text{in } \Gamma.$$

Notice that here $V_\tau^\sharp(x)$ is only defined in $\Gamma \cap \Omega$. We now extend it to the whole domain Ω by using an appropriate cut-off. Let $l_s := \{x \in \Gamma : \eta_{\mathrm{R}}(x) = 1/s\}$ for $s > 0$. For $\varepsilon > 0$ small enough and $t^\sharp > 0$ large enough, we define the function $\phi_t \in C^\infty(\mathbb{R}^2)$ by

$$\phi_t(x) = \begin{cases} 1 & \text{for } x \in \overline{\displaystyle\bigcup_{0<s<t+\varepsilon/2} l_s},\ t \in [0, t^\sharp], \\ 0 & \text{for } x \in \mathbb{R}^2 \setminus \displaystyle\bigcup_{0<s<t+\varepsilon} l_s,\ t \in [0, t^\sharp] \end{cases}$$

and

$$|\partial_x^\alpha \phi_t(x)| \le C_\phi \quad \text{for } |\alpha| \le 2,\ x \in \Omega,\ t \in [0, t^\sharp]$$

for some positive constant C_ϕ depending only on Ω, N, $t^\sharp$ and ε. Next we define the function $V_{t,\tau}$ by

$$V_{t,\tau}(x) := \phi_t(x)\, \exp\left(-\frac{\tau}{t}\right) V_\tau^\sharp(x) \text{ for } x \in \overline{\Omega}.$$

Then we know by Lemma 3.1 that the dominant parts of $V_{t,\tau}$ and its derivatives are as follows:

$$V_{t,\tau}(x) = \begin{cases} 0 \text{ for } x \in \Omega \setminus \bigcup_{0<s<t+\varepsilon} l_s, \\ \exp\left(\tau\Big(-\frac{1}{t}+\eta(x)\Big)\right)\big(\phi_t(x)+S_0(x)h\big) \\ \qquad\qquad\qquad \text{for } x \in \Omega \cap \bigcup_{0<s<t+\varepsilon} l_s, \end{cases} \tag{3.3}$$

$$\nabla V_{t,\tau}(x) = \begin{cases} 0 \text{ for } x \in \Omega \setminus \bigcup_{0<s<t+\varepsilon} l_s, \\ \tau\exp\left(\tau\Big(-\frac{1}{t}+\eta(x)\Big)\right)\big(\phi_t(x)\nabla\eta(x)+\boldsymbol{S}(x)h\big) \\ \qquad\qquad\qquad \text{for } x \in \Omega \cap \bigcup_{0<s<t+\varepsilon} l_s \end{cases} \tag{3.4}$$

for $t \in (0, t^\sharp]$ and $\tau^{-1} \in (0,1]$, where $S_0(x) = S_0(x;t,\tau)$ and $\boldsymbol{S}(x) = \boldsymbol{S}(x;t,\tau)$ satisfy

$$|S_0(x)|,\ |\boldsymbol{S}(x)| \le C_{\mathrm{V}} \text{ for any } x \in \Omega \cap \bigcup_{0<s<t+\varepsilon} l_s,\ t \in (0,t^\sharp],\ \tau^{-1} \in (0,1]$$

with a positive constant C_{V} depending only on Ω, N, $t^\sharp$, ε and k. It should be remarked that the function $V_{t,\tau}$ does not satisfy the Helmholtz equation in Ω. Nonetheless, if we let $v_{0,t,\tau}$ be the solution to the Helmholtz equation in Ω with boundary value $f_{t,\tau} := V_{t,\tau}|_{\partial\Omega}$, then the error between $V_{t,\tau}$ and $v_{0,t,\tau}$ is exponentially small.

Lemma 3.2. *There exist constants $C_0, C_0' > 0$ and $a > 0$ such that*

$$\|v_{0,t,\tau} - V_{t,\tau}\|_{H^2(\Omega)} \le \tau C_0' e^{-\tau a_t} \le C_0 e^{-\tau a}$$

for any $\tau^{-1} \in (0,1]$, where the constants C_0 and C_0' depend only on Ω, k, N, $t^\sharp$ and ε; the constant a depends only on $t^\sharp$ and ε; and we set $a_t := 1/t - 1/(t+\varepsilon/2)$.

This lemma can be proved in the same way as Lemma 4.1 in [29].

Now we consider the energy gap

$$I(\tau,t) = \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{t,\tau}\, \overline{f}_{t,\tau}\, dS.$$

It can be shown that

$$I(\tau,t) \le k^2 \int_\Omega |w_{t,\tau}|^2\, dx + \int_D \gamma_D\, |\nabla v_{0,t,\tau}|^2\, dx, \tag{3.5}$$

$$I(\tau,t) \ge \int_D \frac{\gamma_D}{1+\gamma_D} |\nabla v_{0,t,\tau}|^2\, dx - k^2 \int_\Omega |w_{t,\tau}|^2\, dx, \tag{3.6}$$

where $v_{0,t,\tau}$ satisfies the Helmholtz equation in Ω with Dirichlet condition $v_{0,t,\tau}|_{\partial\Omega} = f_{t,\tau}$ and $w_{t,\tau} = v_{t,\tau} - v_{0,t,\tau}$ is the reflected solution, i.e.,

$$\begin{cases} \nabla\cdot(\widetilde{\gamma}\nabla w_{t,\tau}) + k^2 w_{t,\tau} = -\nabla\cdot((\widetilde{\gamma}-1)\nabla v_{0,t,\tau}) \text{ in } \Omega, \\ w_{t,\tau} = 0 \text{ on } \partial\Omega \end{cases} \tag{3.7}$$

(see [19, Lemma 4.1]). It is easy to see that

$$\int_\Omega |w_{t,\tau}|^2\,dx \le C\int_D |\nabla v_{0,t,\tau}|^2\,dx.$$

In other words, in view of (3.5), the upper bound of $I(t,\tau)$ solely depends on $\int_D|\nabla v_{0,t,\tau}|^2\,dx$.

To estimate the lower bound of $I(\tau,t)$, we proceed as above and introduce

$$I_{x_0,\alpha} := \int_{\partial D} |\partial_\nu v_{0,t,\tau}(x)|\,|x-x_0|^\alpha\,dS$$

for any $x_0\in\Omega$ and $0<\alpha<1$. The following estimate is crucial in determining the behavior of $I(\tau,t)$ when the level curve of η_R intersects D.

Lemma 3.3. [19, Lemma 3.7] *For any $x_0\in\Omega$, $0<\alpha<1$ and $2<q\le 4$, we have*

$$\int_\Omega |w_{t,\tau}|^2\,dx \le C_{q,\alpha}\Big(I_{x_0,\alpha}^2 + I_{x_0,\alpha}\|\nabla v_{0,t,\tau}\|_{L^q(D)} + \|v_{0,t,\tau}\|_{L^2(D)}^2\Big). \tag{3.8}$$

It should be noted that $w_{t,\tau}$ satisfies an elliptic equation with coefficients having jump interfaces. To get the desired estimate (3.8), we make use of Li-Vogelius' Hölder estimate for the this type of equations [17].

The enclosure method is now based on the following theorem regarding the behaviors of $I(\tau,t)$.

Theorem 3.4. [19, Theorem 4.1] *Assume $D\cap\Gamma\ne\emptyset$. Suppose that $\{x\in\Gamma : \eta_{\mathrm{R}}(x)=\Theta_D\}\cap\partial D$ consists only of one point x_0 and the relative curvature (see [19] for the definition) to $\eta_{\mathrm{R}}(x)=\Theta_D$ of ∂D at x_0 is not zero. Then there exist positive constants C_1, c_1 and τ_1 such that for any $0<t\le t^\sharp$ and $\tau\ge\tau_1$ the following holds:*

(I) *if $1/t>\Theta_D$ then*

$$|I(\tau,t)| \le \begin{cases} C_1\tau^2\exp\left(2\tau\Big(-\dfrac{1}{t}+\dfrac{1}{t+\varepsilon/2}\Big)\right) & \text{if } \Theta_D \le \dfrac{1}{t+\varepsilon/2}, \\ C_1\tau^2\exp\left(2\tau\Big(-\dfrac{1}{t}+\Theta_D\Big)\right) & \text{if } \dfrac{1}{t+\varepsilon/2} < \Theta_D < \dfrac{1}{t}. \end{cases}$$

(II) *if $1/t\le\Theta_D$ then*

$$I(\tau,t) \ge c_1\exp\left(2\tau\Big(-\frac{1}{t}+\Theta_D\Big)\right)\tau^{1/2}.$$

The proof of this theorem relies on estimates we obtained above. Moreover, even though we impose some restriction on the curvature of ∂D at x_0, one can show that the curvature assumption is always satisfied as long as N is large enough for C^2 boundary ∂D.

3.2. **An improvement by Sini and Yoshida.** In the enclosure method discussed above (for impenetrable or penetrable obstacles), two conditions are assumed, that is, the level curve of real part of the phase function in CGO solutions touches ∂D at one point and the nonvanishing of the relative curvature at the touching point. These two assumptions are removed by Sini and Yoshida in [25]. Roughly speaking, they use following estimates for the reflected solution w

$$\|w\|_{L^2(\Omega)} \le C_p \|v\|_{W^{1,p}(D)} \quad \text{with} \quad p<2 \tag{3.9}$$

for the penetrable obstacle, and

$$\|w\|_{L^2(\Omega\setminus\bar{D})} \le C_t \|v\|_{H^{-t+\frac{3}{2}}(D)} \quad \text{with} \quad t<1 \tag{3.10}$$

for the impenetrable obstacle. Here v satisfies the Helmholtz equation in Ω.

The derivation of (3.9) is based on Meyers' theorem [18] and the sharp Freidrichs inequality, while, the proof of (3.10) relies on layer potential techniques on Sobolev spaces and integral estimates of the p-powers of Green's function. We refer to [25] for details. Here we would like to see how (3.9) and (3.10) lead to the characteristic behaviors of the energy gap in the enclosure method. To illustrate the ideas, we follow [25] and only consider the CGO solutions with linear phases, i.e., $v(x;\tau,t) = e^{\tau(x\cdot\omega - t)+i\sqrt{\tau^2+k^2}x\cdot\omega^\perp}$. It is clear that v is a solution of the Helmholtz equation. Denote the energy gap

$$I(\tau,t) = \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) v\bar{v} dS.$$

The following behavior of I can be obtained.

Theorem 3.5. [25, Theorem 2.4] *Let $D \Subset \Omega$ with Lipschitz boundary ∂D. For both penetrable and impenetrable cases, we have*

(i)

$$\lim_{\tau\to\infty} I(\tau,t) = 0 \quad \text{if} \quad t > h_D(\omega),$$
$$\liminf_{\tau\to\infty} |I(\tau,h_D(\omega))| = \infty \ (n=2), \quad \liminf_{\tau\to\infty} |I(\tau,h_D(\omega))| > 0 \ (n=3),$$
$$\lim_{\tau\to\infty} |I(\tau,t)| = \infty \quad \text{if} \quad t < h_D(\omega).$$

(ii)

$$h_D(\omega) - t = \lim_{\tau\to\infty} \frac{\ln|I(\tau,t)|}{2\tau}.$$

To prove Theorem 3.5, it is enough to estimate the lower bound of $I(\tau,t)$ at $t = h_D(\omega)$ for $n=3$. Let $y \in \partial D \cap \{x\cdot\omega = h_D(\omega)\} := K$. Since K is compact, there exist $y_1,\cdots,y_N \in K$ such that

$$K \subset D_\delta \quad \text{for } \delta>0 \text{ sufficiently small},$$

where

$$D_\delta = \cup_{j=1}^N (D \cap B(y_j,\delta)).$$

It is obvious that $\int_{D\setminus D_\delta} |\nabla^m v|^p dx$ is exponentially small in τ for $m = 0, 1$. Therefore, to obtain the behaviors of $\int_D |\nabla^m v|^p dx$ in τ, it suffices to study the integrals over D_δ. Using the change of coordinates, it is tedious but not difficult to show that

$$\|v\|^2_{L^2(D)} \geq C\tau^{-2}, \quad \frac{\|\nabla v\|^2_{L^2(D)}}{\|v\|^2_{L^2(D)}} \geq C\tau^2 \tag{3.11}$$

and

$$\frac{\|v\|^2_{L^p(D)}}{\|v\|^2_{L^2(D)}} \leq C\tau^{1-2/p}, \quad \frac{\|\nabla v\|^2_{L^p(D)}}{\|v\|^2_{L^2(D)}} \leq C\tau^{3-2/p} \quad \text{with} \quad \max\{2-\varepsilon, 6/5\} < p \leq 2. \tag{3.12}$$

(see [25, Page 6-9]). Using (3.9) we get from (3.12) that

$$\frac{\|w\|^2_{L^p(D)}}{\|v\|^2_{L^2(D)}} \leq C\tau^{3-2/p}. \tag{3.13}$$

Recall that

$$I(\tau, t) \geq \int_D \frac{\gamma_D}{1+\gamma_D} |\nabla v|^2 \, dx - k^2 \int_\Omega |w|^2 \, dx.$$

Thus, combining (3.11) and (3.13) implies that

$$I(\tau, h_D(\omega)) \geq C\tau^2 \|v\|^2_{L^2(D)} \geq C' > 0$$

As for the impenetrable obstacle (sound hard), we recall that

$$-I(\tau, t) \geq \int_D |\nabla v|^2 dx - k^2 \int_{\Omega\setminus\bar{D}} |w|^2 \tag{3.14}$$

(see for example [8, Lemma 4.1]). Let $s = \frac{3}{2} - t$, then $\frac{1}{2} < s \leq \frac{3}{2}$ if $0 \leq t < 1$. From (3.10) and (3.14), we have that

$$-I(\tau, t) \geq \int_D |\nabla v|^2 dx - C\|v\|^2_{H^s(D)}.$$

Using the interpolation and Young's inequalities, one can choose appropriate parameters such that

$$-I(\tau, t) \geq C \int_D |\nabla v|^2 dx - C' \int_D |v|^2 dx$$

and thus

$$-I(\tau, h_D(\omega)) > 0$$

follows from (3.11).

3.3. **Elastic penetrable obstacles.** Recently, Kuan [16] extended the enclosure method to the reconstruction of a penetrable obstacle using elastic waves. Her result is in 2 dimensions, but it can be generalized to 3 dimensions without serious difficulties. Consider the elastic waves in $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$

$$\nabla \cdot (\sigma(u)) + k^2 u = 0 \quad \text{in} \quad \Omega, \tag{3.15}$$

where u is the displacement vector and

$$\sigma(u) = \lambda(\nabla \cdot u) I_2 + 2\mu\epsilon(u)$$

is the stress tensor. Here $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the infinitesimal strain tensor. Assume that

$$\lambda = \lambda_0 + \lambda_D \chi_D \quad \text{and} \quad \mu = \mu_0 + \mu_D \chi_D,$$

where D is an open subset of Ω with $\bar{D} \subset \Omega$ and λ_D, μ_D belong to $L^\infty(D)$. Assume that

$$\lambda_0 + \mu_0 > 0, \ \mu_0 > 0 \quad \text{in} \quad \Omega,$$
$$\lambda + \mu > 0, \ \mu > 0 \quad \text{in} \quad \Omega.$$

We would like to discuss the reconstruct of the shape of D from boundary measurements in the spirit of enclosure method.

Assume that $-k^2$ is not a Dirichlet eigenvalue of the Lamé operator $\nabla \cdot (\sigma(\cdot))$. Define the Dirichlet-to-Neumann (displacement-to-traction) map

$$\Lambda_D : u|_{\partial\Omega} \to \sigma(u)\nu|_{\partial\Omega}.$$

Let v satisfy the Lamé equation with Lamé coefficients λ_0, μ_0, i.e.,

$$\nabla \cdot (\sigma(v)) + k^2 v = 0 \quad \text{in} \quad \Omega \tag{3.16}$$

with

$$\sigma(v) = \lambda_0(\nabla \cdot v) I_2 + 2\mu_0\epsilon(v).$$

Likewise, we assume that $-k^2$ is not a Dirichlet eigenvalue of the free Lamé operator. We then define the corresponding Dirichlet-to-Neumann map

$$\Lambda_\emptyset : v|_{\partial\Omega} \to \sigma(v)\nu|_{\partial\Omega}.$$

Similar as above, in the enclosure method, we need to construct the CGO solutions for the Lamé equation (3.16). For simplicity, we assume that both λ_0 and μ_0 are constants. To construct the CGO solutions in this case, we take advantage of the Helmholtz decomposition and consider two Helmholtz equations

$$\begin{cases} \Delta\varphi + k_1^2\varphi = 0, \\ \Delta\psi + k_2^2\psi = 0, \end{cases} \tag{3.17}$$

where $k_1 = \left(\frac{k^2}{\lambda_0 + 2\mu_0}\right)^{1/2}$ and $k_2 = \left(\frac{k^2}{\mu_0}\right)^{1/2}$. Then $v = \nabla\varphi + \nabla^\perp\psi$ solves (3.16). Here $\nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)^T$. For (3.17), we can construct the CGO solutions having linear or polynomial phases, which will give us the CGO solutions v for (3.16).

We will not repeat the construction of CGO solutions here. We simply denote $v(\tau, t)$ the CGO solution. Similarly, we define the energy gap

$$I(\tau, t) = \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{\tau,t} \cdot \bar{f}_{\tau,t} dS,$$

where $f_{\tau,t} = v(\tau, t)|_{\partial\Omega}$. Let Γ be the domain where the real part of the phase function of v, denoted by $\rho(x)$, is positive. Let

$$h_D = \begin{cases} \sup\limits_{x \in D\cap\Gamma} \rho(x), & \text{if} \quad D \cap \Gamma \neq \emptyset, \\ 0, & \text{if} \quad D \cap \Gamma = \emptyset. \end{cases}$$

Assume appropriate jump conditions on λ_D and μ_D. Then the following behaviors of $I(\tau, t)$ are obtained in [16].

Theorem 3.6.

(i) $\lim\limits_{\tau\to\infty} I(\tau, t) = 0$ *if* $t > h_D$.

(ii) If $t = h_D$ *and* $\partial D \in C^{0,\alpha}, 1/3 < \alpha \leq 1$, *then* $\liminf\limits_{\tau\to\infty} |I(\tau, h_D(\omega))| = \infty$.

(iii) If $t < h_D$ *and* $\partial D \in C^0$, *then* $\lim\limits_{\tau\to\infty} |I(\tau, t)| = \infty$.

The proof of Theorem 3.6 is based on the following inequalities for the energy gap

$$\begin{aligned} I(\tau, t) \leq & \int_D (\lambda_D + \mu_D)|\nabla \cdot v|^2 dx \\ & + 2\int_D \mu_D \left|\epsilon(v) - \frac{1}{2}(\nabla \cdot v)I_2\right|^2 dx + k^2\|w\|^2_{L^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} I(\tau, t) \geq & \int_D \frac{(\lambda_0 + \mu_0)(\lambda_D + \mu_D)}{\lambda + \mu}|\nabla \cdot v|^2 dx \\ & + 2\int_D \frac{\mu_0\mu_D}{\mu} \left|\epsilon(v) - \frac{1}{2}(\nabla \cdot v)I_2\right|^2 dx - k^2\|w\|^2_{L^2(\Omega)}, \end{aligned}$$

where w is the reflected solution. The remaining arguments of the proof is similar to that in [25], which relies on the following L^p estimate

Lemma 3.7. [16, Lemma 4.2] *There exist constants* $C > 0$ *and* $1 \leq p_0 < 2$ *such that for* $p_0 < p \leq 2$,

$$\|w\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^p(D)}.$$

Finally, Lemma 3.7 can be proved by adopting Meyers' arguments [18].

4. Open problems

The enclosure method in the electromagnetic waves we discussed in Section 2.2 is for the case of impenetrable obstacle. Therefore, it is a legitimate project to study the penetrable case for the electromagnetic waves. However, the tools used in the acoustic waves, i.e., Li-Vogelius type estimates or Meyers type L^p estimates, are not available in the electromagnetic waves. The derivation of these estimates itself is an interesting problem. Another interesting problem is to extend the enclosure method to the plate or shell equations. The distinct feature of these equations is the appearance of the biharmonic operator Δ^2.

Finally, it is desirable to design stable and efficient algorithms for the enclosure method. There are two obvious difficulties. On one hand, the boundary data involves large parameter which gives rise to highly oscillatory functions. On the other hand, a reliable way of numerically determining whether $I(\tau, t)$ decays or blows up as $\tau \to \infty$ is yet to be found.

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