LANDIS' CONJECTURE FOR GENERAL SECOND ORDER ELLIPTIC EQUATIONS WITH SINGULAR LOWER ORDER TERMS IN THE PLANE

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ABSTRACT. In this article, we study a quantitative form of Landis' conjecture in the plane for solutions to second-order elliptic equations with variable coefficients and singular lower order terms. Precisely, let A be real-valued, bounded and elliptic, but not necessary symmetric or continuous. Assume that V and W_i are real-valued, satisfy some sign relations, and belong to L^p and L^{q_i}, respectively, for some $p \in (1,\infty]$ and $q_i \in (2,\infty]$ with $i = 1,2$. We consider real-valued solutions to equations of the form $-\operatorname{div}(A\nabla u + W_1 u) + W_2 \cdot \nabla u + Vu = 0$ in $\mathbb{R}^2$. If u is bounded and normalized in the sense that $|u(z)| \le \exp\left(c_0 |z|^\beta\right)$ and $u(0) = 1$, then for any R sufficiently large and any arbitrarily small $\varepsilon > 0$,

$$\inf_{|z_0|=R} ||u||_{L^\infty(B_1(z_0))} \ge \exp\left(-CR^{\beta(1+\varepsilon)} \log R\right),$$

where $\beta = \max\left\{1-\frac{1}{p}, 1-\frac{2}{q_1}, 1-\frac{2}{q_2}\right\}$. The integrability assumptions on V and W_i are nearly optimal in view of the scaling argument. We use the theory of elliptic boundary value problems to establish the existence of positive multipliers associated to the elliptic equation. Then the proofs rely on transforming the equations to Beltrami systems and applying a generalization of Hadamard's three-circle theorem.

1. INTRODUCTION

In this paper, we establish a quantitative version of Landis' conjecture for real-valued solutions to second-order, uniformly elliptic equations with singular lower order terms. Over an open, connected $\Omega \subset \mathbb{R}^2$, define the second-order divergence-form operator

$$L := -\operatorname{div}(A\nabla),$$

where we assume that $A = \left(a_{ij}\right)_{i,j=1}^2$ is real-valued, measurable, and is not necessarily symmetric. We also assume that A is uniformly elliptic and bounded, i.e., there exist $\lambda \in (0,1]$, $\Lambda > 0$ so that

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Davey is supported in part by the Simons Foundation Grant number 430198. Wang is supported in part by MOST 105-2115-M-002-014-MY3.

for every $z \in \Omega$,

$$a_{ij}(z)\,\xi^i\xi^j \geq \lambda\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2, \tag{1}$$

$$\left|a_{ij}(z)\right| \leq \Lambda. \tag{2}$$

We assume that the lower order terms are real-valued and belong to appropriate Lebesgue spaces:

$$V \in L^p(\Omega) \qquad \text{for some } p \in (1,\infty] \tag{3}$$

$$W_i \in L^{q_i}(\Omega) \qquad \text{for some } q_i \in (2,\infty] \qquad \text{for } i = 1,2. \tag{4}$$

In lieu of the non-negativity condition on V that has appeared in previous works (see for example [14], [6]), we impose the following sign conditions on the lower order terms,

$$V \geq 0 \quad \text{a.e.} \tag{5}$$

$$\int W_i \cdot \nabla\phi \geq 0 \quad \text{for every } \phi \in W^{1,q_i'}(\Omega) \text{ such that } \phi \geq 0 \qquad \text{for } i = 1,2, \tag{6}$$

where we use the prime notation to denote the Hölder conjugate exponent.

We study the unique continuation properties of real-valued solutions to the following second-order elliptic equation in the plane:

$$-\operatorname{div}(A\nabla u + W_1 u) + W_2 \cdot \nabla u + Vu = 0. \tag{7}$$

The main theorem of this article is the following.

Theorem 1. *Assume that conditions* (1) - (6) *are satisfied with* $\Omega = \mathbb{R}^2$*, where* $||V||_{L^p(\mathbb{R}^2)} \leq A_0$*,* $||W_1||_{L^{q_1}(\mathbb{R}^2)} \leq A_1$ *and* $||W_2||_{L^{q_2}(\mathbb{R}^2)} \leq A_2$*. Set* $\beta = \max\left\{1 - \frac{1}{p}, 1 - \frac{2}{q_1}, 1 - \frac{2}{q_2}\right\}$*. Let u be a real-valued solution to* (7) *in* $\mathbb{R}^2$ *for which*

$$|u(z)| \leq \exp\left(C_0\,|z|^\beta\right) \tag{8}$$

$$|u(0)| \geq 1. \tag{9}$$

Then for any $\varepsilon > 0$ *and any R sufficiently large, we have*

$$\inf_{|z_0|=R} ||u||_{L^\infty(B_1(z_0))} \geq \exp\left(-CR^{\beta(1+\varepsilon)}\log R\right), \tag{10}$$

where C depends on λ*,* Λ*,* p*,* q_1*,* q_2*,* A_0*,* A_1*,* A_2*,* C_0*, and* ε*.*

Remark. If $W_1 \equiv 0$, then we may replace conditions (5) – (6) with the following pair of conditions: There exists $\gamma > 0$ so that

$$\int_\Omega A\nabla u \cdot \nabla u + W_2\, u \cdot \nabla u + V\,|u|^2 \geq \gamma \int_\Omega |\nabla u|^2 \quad \text{for every } u \in W_0^{1,2}(\Omega) \tag{11}$$

$$\int_\Omega W_2 \cdot \nabla\phi + V\phi \geq 0 \quad \text{for every } \phi \in W_0^{1,q'}(\Omega) \cap L^{p'}(\Omega) \text{ such that } \phi \geq 0. \tag{12}$$

The first assumption ensures that we may construct a positive multiplier associated to the solution, while the second condition enables us to use the maximum principle to derive appropriate pointwise bounds.

Remark. In the extreme case where all lower order terms are essentially bounded (belong to L^∞) and at least one of the first order terms is trivial, if we assume that the coefficients of A are continuous, then we may take $\varepsilon = 0$. Moreover, we only need to impose the first positivity condition, (5). For the details of this proof, we refer to [6].

We point out now that the estimate in Theorem 1 is almost sharp in an exterior domain. Consider $u(r) = \exp(-r^\alpha)$ for some $\alpha \in (0,1)$ to be determined. A computation gives

$$\begin{aligned}
\nabla u(r) &= -\alpha r^{\alpha-1}\left(\frac{x}{r}, \frac{y}{r}\right) u(r) \\
\Delta u(r) &= \alpha^2 r^{2(\alpha-1)}\left(1 - r^{-\alpha}\right) u(r).
\end{aligned}$$

If we define

$$\begin{aligned}
V &= \frac{1}{2}\alpha^2 r^{2(\alpha-1)}\left(1 - r^{-\alpha}\right) \\
W_1 &= -\frac{1}{2}\alpha r^{\alpha-1}\left(\frac{x}{r}, \frac{y}{r}\right) \\
W_2 &= -\frac{1}{2}\alpha r^{\alpha-1}\left(1 - r^{-\alpha}\right)\left(\frac{x}{r}, \frac{y}{r}\right),
\end{aligned}$$

then

$$\operatorname{div}(\nabla u + W_1 u) = V u = W_2 \cdot \nabla u.$$

If $p, q_i = \infty$ for $i = 1,2$, set $\alpha = 1$ and note that $V, W_1, W_2 \in L^\infty\left(\mathbb{R}^2 \setminus B_1\right)$. If $p \in (1, \infty)$, for any $\delta \in (0, p-1)$, let $\alpha = 1 - \frac{1+\delta}{p}$ and we have

$$||V||^p_{L^p\left(\mathbb{R}^2 \setminus B_1\right)} \le C \int_1^\infty r^{-2(1+\delta)} r\, dr = \frac{C}{2\delta} < \infty.$$

Similarly, if $q_i \in (2, \infty)$, then for any $\delta \in (0, q_i - 2)$, let $\alpha = 1 - \frac{2+2\delta}{q_i}$ and we see that for $i = 1,2$,

$$||W_i||^{q_i}_{L^{q_i}\left(\mathbb{R}^2 \setminus B_1\right)} \le C \int_1^\infty r^{-(2+2\delta)} r\, dr = \frac{C}{2\delta} < \infty.$$

It follows that Theorem 1 is almost sharp in $\mathbb{R}^2 \setminus B_1$ with an arbitrarily small error. Moreover, since $V \ge 0$ and each W_i is differentiable with

$$-\operatorname{div} W_i = \frac{1}{2}\alpha^2 r^{\alpha-2} \ge 0,$$

then (5) and (6) are satisfied.

To prove Theorem 1, we first establish an order of vanishing estimate for a scaled version of equation (7). Then, using the scaling argument first developed in [4], we prove Theorem 1. We use the notation $B_r(z_0)$ to denote the ball of radius r centered at $z_0 \in \mathbb{R}^2$. Often, we abbreviate this notation and simply write B_r when the centre is understood from the context. For the order of vanishing estimate, we consider solutions to (7) in B_d, where d is a constant to be specified below (see (16)). The constant b will also be specified later on (see (57)), once we have introduced quasi-balls. Quasi-balls are sets associated with the levels sets of fundamental solutions, and are therefore appropriate generalizations of standard balls to the variable coefficient setting.

To prove Theorem 1, we establish the following order of vanishing estimate.

Theorem 2. *Assume that conditions* (1) - (6) *are satisfied with* $\Omega = B_d$, *and that* $||V||_{L^p(B_d)} \le M$, $||W_1||_{L^{q_1}(B_d)} \le K_1$, *and* $||W_2||_{L^{q_2}(B_d)} \le K_2$ *where either* $K_1 \ge 1$, $K_2 \ge 1$, *or* $M \ge 1$. *Let u be a real-valued solution to* (7) *in* B_d *that satisfies*

$$||u||_{L^\infty(B_d)} \le \exp\left[C_0\left(\sqrt{M} + K_1 + K_2\right)\right] \tag{13}$$

$$||u||_{L^\infty(B_b)} \ge 1. \tag{14}$$

Then for any $\varepsilon > 0$ *and any r sufficiently small,*

$$||u||_{L^\infty(B_r)} \ge r^{C\left(\sqrt{M}+K_1+K_2\right)^{1+\varepsilon}}, \tag{15}$$

where C depends on λ, Λ, p, q_1, q_2, C_0, *and* ε.

Remark. When $W_1 \equiv 0$, then we may replace conditions (5) and (6) with (11) and (12).

Remark. Again, in the extreme case where $p = q_i = \infty$, $W_j \equiv 0$ for $j \neq i$, and A is assumed to be continuous, the theorem above holds with $\varepsilon = 0$, see [6]. In this setting, (6) is not required.

Since we are working with real-valued solutions and equations in the plane, the best approach to proving these theorems is to the use the relationship between our solutions and the solutions to first order equations in the complex plane, Beltrami systems. Therefore, we will closely follow the proof ideas that were first developed in [14], with further generalizations in [6]. Since we are no longer working with bounded lower order terms, but rather with singular potentials, we also borrow some of the ideas that were presented in [15] where the authors considered drift equations with singular potentials.

Our results generalize those previously established in [14], [15], and [6] in three ways. First, our leading operator is no longer assumed to be Lipschitz continuous and symmetric as in [6]. (In [14] and [15], the leading operator is the Laplacian.) Here, we only assume that A is bounded and uniformly elliptic. Second, we allow all of the lower order terms to be unbounded. In [14] and [6], the two lower order terms, V and W, are assumed to be bounded; whereas in [15], one lower order term, W, can be unbounded, but V has to be zero. Third, we consider very general elliptic equations that can have two non-trivial first order terms. In [14], [15], and [6], it is always assumed that either $W_1 \equiv 0$ or $W_2 \equiv 0$. To deal with equations that have two non-trivial first-order terms, we follow Alessandrini in [1] and use two positive multipliers, instead of just one, to transform the PDE for u into a divergence-free equation.

As our lower order terms are not assumed to be bounded, our approach to the construction of the positive multipliers is completely new in this article. We use the existence of solutions to Dirichlet boundary value problems in combination with the maximum principle to argue that positive multipliers with appropriate pointwise bounds exist. We remark that our methods do not apply to the scale-invariant case of $V \in L^1\left(\mathbb{R}^2\right)$ or $W_i \in L^2\left(\mathbb{R}^2\right)$. This is not surprising since the counterexample of Kenig and Nadirashvili in [16] implies that weak unique continuation can fail for the operator $\Delta + V$ with $V \in L^1$. And for drift equations of the form $\Delta u + W \cdot \nabla u = 0$ in $\Omega \subset \mathbb{R}^2$, the counterexamples due to Mandache [19] and Koch and Tataru [18] show that weak unique continuation can fail for $W \in L^{2^-}$ and L^2_{weak}, respectively.

We point out that a similar problem was investigated by the first-named author and Zhu in [7] and [8]. In these papers, the authors studied the quantitative unique continuation properties of solutions to equations of the form (7) under the assumption that $L = -\Delta$, $W_1 \equiv 0$, and the other lower order terms belong to some admissible Lebesgue spaces. Since the proof techniques are based on certain $L^p - L^q$ Carleman estimates, the results apply to complex-valued solutions and equations in any dimension $n \geq 2$. Consequently, the estimates derived in [7] and [8] are not as sharp as those that we prove in the current paper. For a broader survey of related works, we refer the reader to [14] and the references therein.

The organization of this article is as follows. In Section 2, we discuss quasi-balls. Quasi-balls are a natural generalization of standard balls and they are associated to a uniformly elliptic divergence-form operator. Section 3 deals with the positive multipliers. In particular, we construct a very general positive multiplier, prove that it has appropriate pointwise bounds, satisfies generalized Caccioppoli-type inequalities, and then show that its logarithm also has good bounds in some L^t spaces. The Beltrami operators are introduced in Section 4. Much of this section resembles work that was previously done in [6], and we therefore omit some of the proofs. In Section 5, we use the tools that have been developed to prove Theorem 2. The proof of Theorem 1 is presented in Section 6.

In addition to the main content of this paper, we rely on some theory regarding elliptic boundary value problems, and this content has been relegated to the appendices. In Appendix A, we prove a pair of maximum principles. Appendix B presents a collection of results regarding the Green's functions for general elliptic operators in open, bounded, connected subsets of $\mathbb{R}^2$. This work is based on the constructions that appear in [11], [13], and [5]. We include this section for completeness since the specific representation that we sought was not available in the literature. The results of the appendices are used in Section 2 where we argue that the positive multipliers satisfy appropriate pointwise bounds.

Acknowledgement. Part of this research was carried out while the first author was visiting the National Center for Theoretical Sciences (NCTS) at National Taiwan University. The first author wishes to the thank the NCTS for their financial support and their kind hospitality during her visit to Taiwan.

2. QUASI-BALLS

Since we are working with variable-coefficient operators instead of the Laplacian, we will at times need to work with sets that are not classical balls. Therefore, we introduce the notion of quasi-balls.

Let $\mathscr{L}(\lambda, \Lambda)$ denote the set of all second-order elliptic operators acting on $\mathbb{R}^2$ that satisfy the ellipticity and boundedness conditions described by (1) and (2). Throughout this section, assume that $L \in \mathscr{L}(\lambda, \Lambda)$. We start by discussing the fundamental solutions of L. These results are based on the Appendix of [17].

Definition 1. *A function G is called a fundamental solution for L with pole at the origin if*

- $G \in H^{1,2}_{loc}\left(\mathbb{R}^2 \setminus \{0\}\right)$, $G \in H^{1,p}_{loc}\left(\mathbb{R}^2\right)$ *for all* $p < 2$, *and for every* $\varphi \in C^\infty_0\left(\mathbb{R}^2\right)$
$$\int a_{ij}(z) D_i G(z) D_j \varphi(z)\, dz = \varphi(0).$$
- $|G(z)| \leq C \log|z|$, *for some* $C > 0$, $|z| \geq C$.

Lemma 1 (Theorem A-2, [17])**.** *There exists a unique fundamental solution G for L, with pole at the origin and with the property that* $\lim\limits_{|z|\to\infty} G(z) - g(z) = 0$, *where g is a solution to* $Lg = 0$ *in*

$|z| > 1$ *with* $g = 0$ *on* $|z| = 1$. *Moreover, there are constants* $C_1, C_2, C_3, C_4, R_1 < 1 < R_2$, *that depend on* λ *and* Λ, *such that*

$$C_1 \log\left(\frac{1}{|z|}\right) \le -G(z) \le C_2 \log\left(\frac{1}{|z|}\right) \quad \textit{for } |z| < R_1$$
$$C_3 \log|z| \le G(z) \le C_4 \log|z| \quad \textit{for } |z| > R_2.$$

The level sets of G will be important to us.

Definition 2. *Define a function* $\ell : \mathbb{R}^2 \to (0,\infty)$ *as follows:* $\ell(z) = s$ *iff* $G(z) = \ln s$. *Then set*

$$Z_s = \left\{z \in \mathbb{R}^2 : G(z) = \ln s\right\} = \left\{z \in \mathbb{R}^2 : \ell(z) = s\right\}.$$

We refer to these level sets of G *as* **quasi-circles.** *That is,* Z_s *is the quasi-circle of radius* s. *We also define (closed)* **quasi-balls** *as*

$$Q_s = \left\{z \in \mathbb{R}^2 : \ell(z) \le s\right\}.$$

Open **quasi-balls** *are defined analogously. We may also use the notation* Q_s^L *and* Z_s^L *to remind ourselves of the underlying operator.*

The following lemma follows from the bounds given in Lemma 1. The details of the proof may be found in [6].

Lemma 2. *There are constants* $c_1, c_2, c_3, c_4, c_5, c_6, S_1 < 1 < S_2$, *that depend on* λ *and* Λ, *such that if* $z \in Z_s$, *then*

$$s^{c_1} \le |z| \le s^{c_2} \quad \textit{for } s \le S_1$$
$$c_5 s^{c_1} \le |z| \le c_6 s^{c_4} \quad \textit{for } S_1 < s < S_2$$
$$s^{c_3} \le |z| \le s^{c_4} \quad \textit{for } s \ge S_2.$$

Thus, the quasi-circle Z_s is contained in an annulus whose inner and outer radii depend on s, λ, and Λ. For future reference, it will be helpful to have a notation for the bounds on these inner and outer radii.

Definition 3. *Define*

$$\sigma(s;\lambda,\Lambda) = \sup\left\{r > 0 : B_r \subset \bigcap_{L \in \mathscr{L}(\lambda,\Lambda)} Q_s^L\right\}$$
$$\rho(s;\lambda,\Lambda) = \inf\left\{r > 0 : \bigcup_{L \in \mathscr{L}(\lambda,\Lambda)} Q_s^L \subset B_r\right\}$$

Remark. These functions are defined so that for any operator L in $\mathscr{L}(\lambda,\Lambda)$, $B_{\sigma(s;\lambda,\Lambda)} \subset Q_s^L \subset B_{\rho(s;\lambda,\Lambda)}$.

The quasi-balls and quasi-circles just defined above are centered at the origin since G is a fundamental solution with a pole at the origin. As a reminder, we may sometimes use the notation $Z_s(0)$ and $Q_s(0)$. If we follow the same process for any point $z_0 \in \mathbb{R}^2$, we may discuss the fundamental solutions with pole at z_0, and we may similarly define the quasi-circles and quasi-balls associated to these functions. We denote the quasi-circle and quasi-ball of radius s centred at z_0 by $Z_s(z_0)$ and $Q_s(z_0)$, respectively. Although $Q_s(z_0)$ is not necessarily a translation of $Q_s(0)$ for $z_0 \neq 0$, both sets are contained in annuli that are translations.

3. Positive multipliers

In [14] and [6], the first step in the proofs of the order of vanishing estimates is to establish that a positive multiplier associated to the operator (or its adjoint) exists and has suitable bounds. Unlike the settings in those papers, since our lower order terms are unbounded, we cannot simply construct positive super- and subsolutions in B_d, then argue that a positive solution exists. Therefore, our approach here is more involved. Instead, we use solutions to the Dirichlet boundary value problem for appropriately chosen boundary data and rely on the maximum principle to give us desirable bounds.

From now on, we set

$$d = \rho(7/5) + 2/5, \tag{16}$$

where $\rho(s) = \rho(s; \lambda, \Lambda)$ is as defined in the previous section. Throughout this section, assume that A satisfies (1) and (2), while V, W_1 and W_2 satisfy (3) and (4). Note that A^T also satisfies (1) and (2). Associated to an operator of the type $-\operatorname{div}(A\nabla + W_1) + W_2 \cdot \nabla + V$ is the bilinear form $B : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ given by

$$B[u,v] = \int_\Omega A\nabla u \cdot \nabla v + W_1 u \cdot \nabla v + W_2 \cdot \nabla u\, v + V\, u\, v. \tag{17}$$

In every case, we take $\Omega = B_d$.

Since we need the existence of solutions to various elliptic equations, the following lemma serves as a useful tool.

Lemma 3. *Let $g \in C^1\left(\overline{B_d}\right)$. Assume that the bilinear form given by* (17) *is bounded and coercive in $W_0^{1,2}(B_d)$. That is, there exist constants c and C so that for any $u, v \in W_0^{1,2}(B_d)$*

$$\begin{aligned} |B[u,v]| &\le C\,||u||_{W^{1,2}(B_d)}\,||v||_{W^{1,2}(B_d)} \\ B[v,v] &\ge c\,||v||^2_{W^{1,2}(B_d)}. \end{aligned}$$

Then there exists a weak solution $\phi \in W^{1,2}(B_d)$ to

$$\begin{cases} -\operatorname{div}(A\nabla\phi + W_1\phi) + W_2 \cdot \nabla\phi + V\phi = 0 & \text{in } B_d \\ \phi = g & \text{on } \partial B_d \end{cases}. \tag{18}$$

Proof. To establish that a solution to (18) exists, we prove that there exists a $\psi \in W_0^{1,2}(B_d)$ for which

$$-\operatorname{div}(A\nabla\psi + W_1\psi) + W_2 \cdot \nabla\psi + V\psi = -\operatorname{div} G + f \ \text{ in } B_d, \tag{19}$$

where $G \in L^q(B_d)$ and $f \in L^p(B_d)$ for some $q \in (2, \infty]$ and $p \in (1, \infty]$. With $\psi = \phi - g$, we have $G = -A\nabla g - W_1 g$ and $f = -W_2 \cdot \nabla g - Vg$, and this gives the claimed result since $g \in C^1\left(\overline{B_d}\right)$ and B_d is bounded.

To show that (19) is solvable, we need to show that for any $v \in W_0^{1,2}(B_d)$, there exists a $\psi \in W_0^{1,2}(B_d)$ for which

$$B[\psi, v] = \int_{B_d} G \cdot \nabla v + f v. \tag{20}$$

For any $v \in W_0^{1,2}(B_d)$, consider the linear functional

$$v \mapsto \int_{B_d} G \cdot \nabla v + f v. \tag{21}$$

If $p \geq 2$, then

$$|\int_{B_d} G \cdot \nabla v + f v| \leq ||G||_{L^q(B_d)} ||\nabla v||_{L^2(B_d)} |B_d|^{\frac{1}{2}-\frac{1}{q}} + ||f||_{L^p(B_d)} ||v||_{L^2(B_d)} |B_d|^{\frac{1}{2}-\frac{1}{p}}.$$

On the other hand, if $p < 2$, then $p' > 2$ and

$$\begin{aligned} |\int_{B_d} G \cdot \nabla v + f v| &\leq ||G||_{L^q(B_d)} ||\nabla v||_{L^2(B_d)} |B_d|^{\frac{1}{2}-\frac{1}{q}} + ||f||_{L^p(B_d)} ||v||_{L^{p'}(B_d)} \\ &\leq \left(||G||_{L^q(B_d)} |B_d|^{\frac{1}{2}-\frac{1}{q}} + C_p ||f||_{L^p(B_d)} \right) ||\nabla v||_{L^2(B_d)}, \end{aligned}$$

where the last line follows from an application of the Sobolev inequality with $2^* = p' \in (2, \infty)$. Hereafter, we use the notation 2^* to denote the Sobolev exponent of 2, and it will be chosen in $(2, \infty)$. In either case,

$$|\int_{B_d} G \cdot \nabla v + f v| \leq C ||v||_{W^{1,2}(B_d)},$$

so the functional defined by (21) is bounded on $W_0^{1,2}(B_d)$.

By assumption, $B[\cdot, \cdot]$ is a bounded, coercive form on $W_0^{1,2}(B_d)$. Therefore, we may apply the Lax-Milgram theorem to conclude that there exists a unique $\psi \in W_0^{1,2}(B_d)$ that satisfies (20). Consequently, (19) has a unique solution, and therefore, (18) is solvable. □

Using the lemma above, we now prove that a general positive multipliers exists. With an appropriate choice of boundary data, the maximum principle implies that this positive multiplier has the required pointwise bounds from above and below.

Lemma 4. *Assume that $||V||_{L^p(B_d)} \leq M$, $||W_1||_{L^{q_1}(B_d)} \leq K_1$, $||W_2||_{L^{q_2}(B_d)} \leq K_2$, and* (5), (6) *hold. Then there exists a weak solution $\phi \in W^{1,2}(B_d)$ to*

$$-\operatorname{div}\left(A^T \nabla \phi + W_2 \phi\right) + W_1 \cdot \nabla \phi + V \phi = 0 \quad \text{in } B_d \tag{22}$$

with the property that

$$e^{-c\left(\sqrt{M}+K_1+K_2\right)} \leq \phi(z) \leq e^{c\left(\sqrt{M}+K_1+K_2\right)} \quad \text{for a.e. } z \in B_d, \tag{23}$$

where c is some positive constant.

Proof of Lemma 4. Let $g \in C^1\left(\overline{B_d}\right)$ be a positive function for which

$$\kappa e^{-c\left(\sqrt{M}+K_1+K_2\right)} \leq g(z) \leq e^{c\left(\sqrt{M}+K_1+K_2\right)} \quad \text{for all } z \in \partial B_d,$$

where the constant κ will be specified below. Let $\phi \in W^{1,2}(B_d)$ be the weak solution to the following Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div}\left(A^T \nabla \phi + W_2 \phi\right) + W_1 \cdot \nabla \phi + V\phi = 0 & \text{in } B_d \\ \phi = g & \text{on } \partial B_d. \end{cases} \tag{24}$$

To establish that a solution to (24) exists, we need to check that the associated bilinear form is bounded above and below, then we may apply Lemma 3 (with A being replaced by A^T and the roles of W_1 and W_2 interchanged). For any $u, v \in W_0^{1,2}(B_d)$, (2) and Hölder's inequality imply that

$$\begin{aligned} |B[u,v]| &= \int_{B_d} A^T \nabla u \cdot \nabla v + W_2 u \cdot \nabla v + W_1 \cdot \nabla u\, v + V u v \\ &\le \Lambda ||\nabla u||_{L^2(B_d)} ||\nabla v||_{L^2(B_d)} + ||W_1||_{L^{q_1}(B_d)} ||\nabla u||_{L^2(B_d)} ||v||_{L^{\frac{2q_1}{q_1-2}}(B_d)} \\ &+ ||W_2||_{L^{q_2}(B_d)} ||\nabla v||_{L^2(B_d)} ||u||_{L^{\frac{2q_2}{q_2-2}}(B_d)} + ||V||_{L^p(B_d)} ||u||_{L^{\frac{2p}{p-1}}(B_d)} ||v||_{L^{\frac{2p}{p-1}}(B_d)} \\ &\le \left(\Lambda + C_{q_1} ||W_1||_{L^{q_1}(B_d)} + C_{q_2} ||W_2||_{L^{q_2}(B_d)} + C_p ||V||_{L^p(B_d)}\right) ||\nabla u||_{L^2(B_d)} ||\nabla v||_{L^2(B_d)}, \end{aligned}$$

where we have used the Sobolev inequality three times with $2^* = \frac{2q_i}{q_i-2} \in (2,\infty)$ for $i = 1,2$ and $2^* = \frac{2p}{p-1} \in (2,\infty)$ to reach the last line. Therefore,

$$|B[u,v]| \le C ||u||_{W^{1,2}(B_d)} ||v||_{W^{1,2}(B_d)}.$$

From (1), (5), (6), it follows that

$$B[v,v] = \int_{B_d} A^T \nabla v \cdot \nabla v + (W_1 + W_2) \cdot \nabla v\, v + V |v|^2 \ge \lambda \int_{B_d} |\nabla v|^2.$$

The Poincaré inequality immediately implies that for any $v \in W_0^{1,2}(B_d)$,

$$B[v,v] \ge c ||v||^2_{W^{1,2}(B_d)}.$$

In conclusion, $B[\cdot,\cdot]$ is a bounded, coercive form on $W_0^{1,2}(B_d)$. It follows from Lemma 3 that (24) is solvable.

It remains to show that ϕ satisfies the stated pointwise bounds a.e. Since (5) and (6) for $i = 2$ imply (12), then by Theorem 4 in Appendix A and that $g = g^+$,

$$\sup_{z \in B_d} \phi(z) \le \sup_{z \in \partial B_d} g^+(z) = \sup_{z \in \partial B_d} g(z) \le e^{c\left(\sqrt{M} + K_1 + K_2\right)},$$

proving the upper bound. To show that ϕ satisfies the stated lower bound a.e., define an auxiliary function $v \in W^{1,2}(B_d)$ that weakly solves the related boundary value problem,

$$\begin{cases} -\operatorname{div}\left(A^T \nabla v\right) + W_1 \cdot \nabla v = 0 & \text{in } B_d \\ v = g & \text{on } \partial B_d \end{cases}. \tag{25}$$

Since the associated bilinear form is bounded (by an argument similar to the one above) and coercive (by ellipticity (1) in combination with assumption (6) for $i = 1$), such a v exists. With $w = \phi - v \in W_0^{1,2}(B_d)$, we see that

$$\begin{cases} -\operatorname{div}\left(A^T \nabla w\right) + W_1 \cdot \nabla w &= \operatorname{div}(W_2 \phi) - V\phi & \text{in } B_d \\ w &= 0 & \text{on } \partial B_d \end{cases}. \tag{26}$$

Let $\Gamma(z,\zeta)$ denote the Green's function for the operator $-\operatorname{div}(A\nabla+W_1)$ in B_d, and let $\Gamma^*(z,\zeta)$ denote the Green's function for the operator $-\operatorname{div}\left(A^T\nabla\right)+W_1\cdot\nabla$ in B_d. Note that $\Gamma^*(\zeta,z)=\Gamma(z,\zeta)$. The assumption (6) for W_1 ensures that such a Green's function exists (see Appendix B). According to Theorem 5 in Appendix B (see also Definition 4), (26) implies that

$$w(z)=-\int_{B_d}\left[D_\zeta\Gamma(z,\zeta)\cdot W_2(\zeta)+\Gamma(z,\zeta)V(\zeta)\right]\phi(\zeta)\,d\zeta.$$

Therefore

$$v(z)=\phi(z)+\int_{B_d}\left[D_\zeta\Gamma(z,\zeta)\cdot W_2(\zeta)+\Gamma(z,\zeta)V(\zeta)\right]\phi(\zeta)\,d\zeta$$

so that

$$\sup_{z\in B_d}v(z)\le\left[1+\int_{B_d}\left|D_\zeta\Gamma(z,\zeta)\right||W_2(\zeta)|\,d\zeta+\int_{B_d}|\Gamma(z,\zeta)||V(\zeta)|\,d\zeta\right]||\phi||_{L^\infty(B_d)}.$$

By Hölder's inequality, we have

$$\int_{B_d}|\Gamma(z,\zeta)||V(\zeta)|\,d\zeta\le\left(\int_{B_d}|\Gamma(z,\zeta)|^{p'}d\zeta\right)^{\frac{1}{p'}}\left(\int_{B_d}|V(\zeta)|^p\,d\zeta\right)^{\frac{1}{p}}\le M\,||\Gamma(z,\cdot)||_{L^{p'}(B_{2d}(z)\cap B_d)}$$

and

$$\int_{B_d}\left|\nabla_\zeta\Gamma(z,\zeta)\right||W_2(\zeta)|\,d\zeta\le K_2\,||D\Gamma(z,\cdot)||_{L^{q_2'}(B_{2d}(z)\cap B_d)}.$$

Since $p>1$, then $p'\in[1,\infty)$, so it follows from (B.17) in Theorem 5 that $||\Gamma(z,\cdot)||_{L^{p'}(B_{2d}(z)\cap B_d)}=C_p$. Similarly, since $q_2>2$, then $q_2'\in[1,2)$ and (B.18) implies that $||D\Gamma(z,\cdot)||_{L^{q_2'}(B_{2d}(z)\cap B_d)}=C_{q_2}$. We set $\kappa=1+C_{q_2}K_2+C_pM$. Combining the observations above with Corollary 4, we have

$$\kappa e^{-c\left(\sqrt{M}+K_1+K_2\right)}\le\inf_{z\in\partial B_d}g(z)\le v(z)\le\kappa\,||\phi||_{L^\infty(B_d)}$$

and we conclude that $e^{-c\left(\sqrt{M}+K_1+K_2\right)}\le||\phi||_{L^\infty(B_d)}$. Since ϕ is assumed to be real-valued and must be continuous, then either $\phi(z)\ge e^{-c\left(\sqrt{M}+K_1+K_2\right)}$ for a.e. $z\in B_d$, or $\phi(z)\le-e^{-c\left(\sqrt{M}+K_1+K_2\right)}$ for a.e. $z\in B_d$. However, we know that $\phi=g$ along ∂B_d, where g is a positive function. Therefore, $\phi(z)\ge e^{-c\left(\sqrt{M}+K_1+K_2\right)}$ for a.e. $z\in B_d$, proving the lemma. □

In addition to the pointwise bounds for ϕ that were established in the previous lemmas, we also prove gradient estimates for all solutions to (7). A similar argument shows that analogous bounds hold for ϕ, and hence for all solutions to equations of the form (22). The following estimates follow from a standard integration by parts argument (a Caccioppoli estimate) in combination with Theorem 2 from [20].

Lemma 5. *For any $r>0$ and $\alpha>1$ for which $\alpha r<2$, let v be a weak solution to (7) in $B_{\alpha r}$. Assume that $||V||_{L^p(B_{\alpha r})}\le M$, $||W_1||_{L^{q_1}(B_{\alpha r})}\le K_1$, and $||W_2||_{L^{q_2}(B_{\alpha r})}\le K_2$. Then for any $t\in[2,\tau_0]$,*

$$\left(\int_{B_r}|\nabla v|^t\right)^{\frac{1}{t}}\le Cr^{\frac{2}{t}-1}\left(1+r^{2-\frac{2}{p}}M+r^{2-\frac{4}{q_1}}K_1^2+r^{2-\frac{4}{q_2}}K_2^2\right)||v||_{L^\infty(B_{\alpha r})},\tag{27}$$

where C depends on λ, Λ, p, q_1, q_2, α, and t, where

$$\tau_0 = \begin{cases} \min\left\{q_1, q_2, \dfrac{2p}{2-p}\right\} & \text{if} \quad 1 < p < 2, \\ \min\{q_1, q_2\} \quad \text{if} \quad p \geq 2. \end{cases} \tag{28}$$

Proof. We start with a Caccioppoli estimate, i.e. with $t = 2$. Let $\eta \in C_c^\infty(B_{\alpha r})$ be such that $\eta \equiv 1$ in B_r and $|\nabla \eta| \leq \frac{C}{(\alpha-1)r}$. Take $v\eta^2$ to be the test function. Then

$$\int (A\nabla v + W_1 v) \cdot \nabla\left(v\eta^2\right) + (W_2 \cdot \nabla v + Vv)\, v\eta^2 = 0$$

so that with the use of (1), Hölder and Cauchy inequalities, we have

$$\begin{aligned}
&\lambda \int |\nabla v|^2 \eta^2 \leq \int A\nabla v \cdot \nabla v\, \eta^2 \\
&= -2\int A\nabla v \cdot \nabla \eta\, v\eta - \int W_1 \cdot \nabla v\, v\eta^2 - 2\int W_1 \cdot \nabla \eta\, |v|^2 \eta - \int (W_2 \cdot \nabla v + Vv)\, v\eta^2 \\
&\leq 2\Lambda \int |\nabla v|\, |\nabla \eta|\, |v|\, \eta + 2\int |W_1|\, |\nabla \eta|\, |v|^2 \eta + \int (|W_1| + |W_2|)\, |v|\, |\nabla v|\, \eta^2 + \int |V|\, v^2 \eta^2 \\
&\leq \frac{\lambda}{2}\int |\nabla v|^2 \eta^2 + \left[\left(\frac{4\Lambda^2}{\lambda} + \frac{\lambda}{2}\right)\int |\nabla \eta|^2 + \int |V|\, \eta^2 + \frac{2}{\lambda}\int \left(2|W_1|^2 + |W_2|^2\right)\eta^2\right] ||v||^2_{L^\infty(B_{\alpha r})} \\
&\leq \frac{\lambda}{2}\int |\nabla v|^2 \eta^2 + C\left[\left(\frac{4\Lambda^2}{\lambda} + \frac{\lambda}{2}\right)\frac{\alpha+1}{\alpha-1} + M(\alpha r)^{2-\frac{2}{p}} + \frac{4}{\lambda}K_1^2 (\alpha r)^{2-\frac{4}{q_1}} + \frac{2}{\lambda}K_2^2(\alpha r)^{2-\frac{4}{q_2}}\right] ||v||^2_{L^\infty(B_{\alpha r})}.
\end{aligned}$$

After simplifying, we reach (27) with $t = 2$.

Let $\beta = \frac{1}{2}(\alpha + 1)$. Since $\operatorname{div}(A\nabla v) = -\operatorname{div}(W_1 v) + W_2 \cdot \nabla v + Vv$, then an application of Theorem 2 from [20] shows that for τ_0 given in (28)

$$\begin{aligned}
\left(\int_{B_r} |\nabla v|^{\tau_0}\right)^{\frac{1}{\tau_0}} &\leq C\left(r^{\frac{2}{\tau_0}-1}||\nabla v||_{L^2(B_{\beta r})} + r^{\frac{2}{\tau_0}-1}||v||_{L^\infty(B_{\beta r})}\right) \\
&\quad + C\left(r^{\frac{2}{\tau_0}-\frac{2}{p}+1}||Vv||_{L^p(B_{\beta r})} + ||W_1 v||_{L^{\tau_0}(B_{\beta r})} + r^{\frac{2}{\tau_0}-\frac{q_2+2}{q_2}+1}||W_2 \cdot \nabla v||_{L^{\frac{2q_2}{q_2+2}}(B_{\beta r})}\right) \\
&\leq Cr^{\frac{2}{\tau_0}-1}\left[\left(1 + r^{1-\frac{2}{q_2}}K_2\right)||\nabla v||_{L^2(B_{\beta r})} + \left(1 + r^{2-\frac{2}{p}}M + r^{1-\frac{2}{q_1}}K_1\right)||v||_{L^\infty(B_{\alpha r})}\right] \\
&\leq Cr^{\frac{2}{\tau_0}-1}\left(1 + r^{2-\frac{2}{p}}M + r^{2-\frac{4}{q_1}}K_1^2 + r^{2-\frac{4}{q_2}}K_2^2\right)||v||_{L^\infty(B_{\alpha r})},
\end{aligned}$$

where we have used Hölder's inequality, the bounds on V, W_1 and W_2, and the result above for $t = 2$. Hölder's inequality leads to the conclusion for general $t \in (2, \tau_0)$. $\square$

For the positive function ϕ given in Lemma 4, define $\Phi = \log \phi$. From Lemma 4, it is clear that $|\Phi(z)| \leq c\left(\sqrt{M} + K_1 + K_2\right)$ for a.e. $z \in B_d$. Since $\operatorname{div}\left(A^T \nabla \phi\right) = W_1 \cdot \nabla \phi + V\phi - \operatorname{div}(W_2 \phi)$ weakly in B_d, then

$$\operatorname{div}\left(A^T \nabla \Phi\right) + (W_2 - W_1) \cdot \nabla \Phi + A^T \nabla \Phi \cdot \nabla \Phi = V - \operatorname{div} W_2 \quad \text{weakly in } B_d. \tag{29}$$

The following estimates for $\nabla\Phi$ will be crucial to our proofs. We begin with an L^2-estimate for $\nabla\Phi$.

Lemma 6. *Let $\Phi = \log\phi$, where ϕ is the positive multiplier given in Lemma 4. Then*

$$||\nabla\Phi||_{L^2\left(B_{\rho(7/5)+1/5}\right)} \le C\left(\sqrt{M}+K_1+K_2\right),$$

where $C(\lambda,\Lambda,p,q_1,q_2)$.

Proof. Recall that $d = \rho(7/5)+2/5$. Let $\theta \in C_0^\infty(B_d)$ be a cutoff function for which $\theta \equiv 1$ in $B_{\rho(7/5)+1/5}$. Multiply (29) by θ^2, then integrate by parts to get

$$\begin{aligned}\lambda\int|\nabla\Phi|^2\theta^2 &\le \int A^T\nabla\Phi\cdot\nabla\Phi\theta^2\\ &= \int V\theta^2+2\int W_2\cdot\theta\nabla\theta+2\int A^T\nabla\Phi\theta\nabla\theta-\int(W_2-W_1)\cdot\nabla\Phi\theta^2\\ &\le \int V\theta^2+2\int W_2\cdot\theta\nabla\theta+\frac{\lambda}{4}\int|\nabla\Phi|^2\theta^2+\frac{4\Lambda^2}{\lambda}\int|\nabla\theta|^2\\ &+\frac{\lambda}{4}\int|\nabla\Phi|^2\theta^2+\frac{1}{\lambda}\int|W_2-W_1|^2\theta^2.\end{aligned}$$

Rearranging and repeatedly applying the Hölder inequality, we see that

$$\begin{aligned}\frac{\lambda}{2}\int|\nabla\Phi|^2\theta &\le M\left|\left|\theta^2\right|\right|_{L^{p'}(B_d)}+2K_2||\nabla\theta||_{L^{q_2'}(B_d)}+\frac{4\Lambda^2}{\lambda}||\nabla\theta||^2_{L^2(B_d)}\\ &+\frac{2}{\lambda}K_2^2\left|\left|\theta^2\right|\right|_{L^{(q_2/2)'}(B_d)}+\frac{2}{\lambda}K_1^2\left|\left|\theta^2\right|\right|_{L^{(q_1/2)'}(B_d)}\end{aligned}$$

Since either $K_1 \ge 1$, $K_2 \ge 1$, or $M \ge 1$, then

$$\int_{B_{\rho(7/5)+1/5}}|\nabla\Phi|^2 \le C\left(M+K_1^2+K_2^2\right),$$

where C depends on λ, Λ, p, q_1 and q_2, as required. □

Now we prove that $\nabla\Phi$ belongs to L^{t_0} for some $t_0 > 2$.

Lemma 7. *Let $\Phi = \log\phi$, where ϕ is the positive multiplier given in Lemma 4. Then there exists $t_0 > 2$ such that $||\nabla\Phi||_{L^{t_0}\left(B_{\rho(7/5)}\right)} \le C\left(\sqrt{M}+K_1+K_2\right)^{\frac{2}{\mu}\left(3-\frac{2}{t_0}\right)}$, where $\mu = \min\left\{2-\frac{2}{p}, 2-\frac{4}{q_1}, 2-\frac{4}{q_2}\right\}$ and C depends on λ, Λ, p, q_1, q_2 and t_0.*

Proof. We rescale equation (29). Set $\varphi = \frac{\Phi}{C\left(\sqrt{M}+K_1+K_2\right)}$ for some $C > 0$. Then (29) is equivalent to

$$\varepsilon\,\mathrm{div}\left(A^T\nabla\varphi+\widetilde{W}_2\right)+A^T\nabla\varphi\cdot\nabla\varphi = \left(\widetilde{W}_1-\widetilde{W}_2\right)\cdot\nabla\varphi+\widetilde{V}, \tag{30}$$

where $\varepsilon = \frac{1}{C\left(\sqrt{M}+K_1+K_2\right)}$, $\widetilde{W}_i = \frac{W_i}{C\left(\sqrt{M}+K_1+K_2\right)}$ for $i = 1,2$, and $\widetilde{V} = \frac{V}{C^2\left(\sqrt{M}+K_1+K_2\right)^2}$. We'll choose C sufficiently large so that

$$\left|\left|\widetilde{V}\right|\right|_{L^p(B_d)} \le 1, \quad \left|\left|\widetilde{W}_i\right|\right|_{L^{q_i}(B_d)} \le 1, \quad ||\varphi||_{L^\infty(B_d)} \le 1, \quad \int_{B_{\rho(7/5)+1/5}}|\nabla\varphi|^2 \le 1, \tag{31}$$

where the last bound is possible because of Lemma 6.

Claim 1. *Let $c > 0$ be such that for any $z \in B_{\rho(7/5)}$, $B_{2c/5}(z) \subset B_{\rho(7/5)+1/5}$. For any $z \in B_{\rho(7/5)}$ and $\varepsilon < r < c/5$, we have*

$$\int_{B_r(z)} |\nabla \varphi|^2 \leq C r^{\mu},$$

where $\mu = \min\left\{2 - \frac{2}{p}, 2 - \frac{4}{q_1}, 2 - \frac{4}{q_2}\right\}$.

Proof of Claim 1. It suffices to take $z = 0$. Let $\eta \in C_0^{\infty}(B_{2r})$ be a cutoff function such that $\eta \equiv 1$ in B_r. By the divergence theorem,

$$\begin{aligned} 0 &= \varepsilon \int \operatorname{div}\left[\left(A^T \nabla \varphi + \widetilde{W}_2\right)\eta^2\right] \\ &= \varepsilon \int \operatorname{div}\left(A^T \nabla \varphi + \widetilde{W}_2\right)\eta^2 + 2\varepsilon \int \eta \nabla \eta \cdot \left(A^T \nabla \varphi\right) + 2\varepsilon \int \eta \nabla \eta \cdot \widetilde{W}_2 \end{aligned} \tag{32}$$

We now estimate each of the three terms. By (30) and (31),

$$\begin{aligned} &\varepsilon \int \operatorname{div}\left(A^T \nabla \varphi + \widetilde{W}_2\right)\eta^2 \\ &= -\int A^T \nabla \varphi \cdot \nabla \varphi \eta^2 + \int \widetilde{V} \eta^2 + \int \left(\widetilde{W}_1 - \widetilde{W}_2\right) \cdot \nabla \varphi \eta^2 \\ &\leq -\lambda \int |\nabla \varphi|^2 \eta^2 + \left|\left|\widetilde{V}\right|\right|_{L^p(B_d)} \left(\int_{B_{2r}} 1\right)^{1-\frac{1}{p}} \\ &+ \left(\int |\nabla \varphi|^2 \eta^2\right)^{\frac{1}{2}} \left[\left|\left|\widetilde{W}_1\right|\right|_{L^{q_1}(B_d)} \left(\int_{B_{2r}} 1\right)^{1-\frac{1}{q_1}-\frac{1}{2}} + \left|\left|\widetilde{W}_2\right|\right|_{L^{q_2}(B_d)} \left(\int_{B_{2r}} 1\right)^{1-\frac{1}{q_2}-\frac{1}{2}}\right] \\ &\leq -\frac{\lambda}{2} \int |\nabla \varphi|^2 \eta^2 + C r^{2-\frac{2}{p}} + \frac{C}{2\lambda}\left(r^{2-\frac{4}{q_1}} + r^{2-\frac{4}{q_2}}\right). \end{aligned} \tag{33}$$

By Cauchy-Schwarz and Young's inequality,

$$\begin{aligned} \left|2\varepsilon \int \eta \nabla \eta \cdot \left(A^T \nabla \varphi\right)\right| &\leq 2\varepsilon \Lambda \int \eta |\nabla \eta| |\nabla \varphi| \leq 2\varepsilon \Lambda \left(\int |\nabla \varphi|^2\right)^{1/2} \left(\int \eta^2 |\nabla \eta|^2\right)^{1/2} \\ &\leq \frac{C\Lambda^2}{\lambda} \varepsilon^2 + \frac{\lambda}{200} \int_{B_{2r}} |\nabla \varphi|^2. \end{aligned} \tag{34}$$

Similarly, by Hölder and Young's inequality,

$$\left|2\varepsilon \int \eta \nabla \eta \cdot \widetilde{W}_2\right| \leq 2\varepsilon \left|\left|\widetilde{W}_2\right|\right|_{L^{q_2}(B_d)} \left(\int_{B_{2r} \setminus B_r} |\nabla \eta|^{\frac{q_2}{q_2-1}}\right)^{1-\frac{1}{q_2}} \leq 2C\varepsilon r^{1-\frac{2}{q_2}} \leq C\varepsilon^2 + C r^{2-\frac{4}{q_2}}. \tag{35}$$

Combining (32)-(35) gives

$$\int_{B_r} |\nabla \varphi|^2 \leq C\varepsilon^2 + C\left(r^{2-\frac{2}{p}} + r^{2-\frac{4}{q_1}} + r^{2-\frac{4}{q_2}}\right) + \frac{1}{100}\int_{B_{2r}} |\nabla \varphi|^2 \leq C r^{\mu} + \frac{1}{100}\int_{B_{2r}} |\nabla \varphi|^2, \tag{36}$$

since $\mu = \min\left\{2 - \frac{2}{p}, 2 - \frac{4}{q_1}, 2 - \frac{4}{q_2}\right\} \leq 2$ and $\varepsilon < r < \frac{c}{5} < 1$.

If $r^{\mu} \geq \frac{1}{100}$, then by the last estimate of (31), the inequality above implies that

$$\int_{B_r} |\nabla \varphi|^2 \leq C r^{\mu}.$$

Otherwise, if $r^{\mu} < \frac{1}{100}$, choose $k \in \mathbb{N}$ so that

$$\frac{c}{5} \leq 2^k r \leq \frac{2c}{5}.$$

Since $r^{\mu} \geq \left(\frac{c}{2^k 5}\right)^{\mu} \geq \left(\frac{c}{2^k 5}\right)^2 = \left(\frac{c^2}{4^k 25}\right) \geq C\left(\frac{1}{100}\right)^k$, then it follows from repeatedly applying (36) that

$$\int_{B_r} |\nabla \varphi|^2 \leq C r^{\mu} + C\left(\frac{1}{100}\right)^k \int_{B_{2^k r}} |\nabla \varphi|^2 \leq C r^{\mu},$$

proving the claim. $\square$

We now use Claim 1 to give an L^{t_0} bound for $\nabla \varphi$ in $B_{\rho(7/5)}$. Define

$$\varphi_{\varepsilon}(z) = \frac{1}{\varepsilon} \varphi\left(\varepsilon^{\frac{2}{\mu}} z\right), \qquad A_{\varepsilon}(z) = A\left(\varepsilon^{\frac{2}{\mu}} z\right), \qquad L_{\varepsilon}^{*} = \operatorname{div} A_{\varepsilon}^{T} \nabla.$$

Then

$$\begin{aligned} \nabla \varphi_{\varepsilon}(z) &= \varepsilon^{\frac{2}{\mu}-1} \nabla \varphi\left(\varepsilon^{\frac{2}{\mu}} z\right) \\ L_{\varepsilon}^{*} \varphi_{\varepsilon}(z) &= \varepsilon^{\frac{4}{\mu}-1} \operatorname{div}\left(A^{T}\left(\varepsilon^{\frac{2}{\mu}} z\right) \nabla \varphi\left(\varepsilon^{\frac{2}{\mu}} z\right)\right). \end{aligned}$$

It follows from (30) that

$$\begin{aligned} L_{\varepsilon}^{*} \varphi_{\varepsilon}(z) + A_{\varepsilon}^{T} \nabla \varphi_{\varepsilon} \cdot \nabla \varphi_{\varepsilon} &= \varepsilon^{\frac{4}{\mu}-2}\left[\varepsilon \operatorname{div}\left(A^{T}\left(\varepsilon^{\frac{2}{\mu}} z\right) \nabla \varphi\left(\varepsilon^{\frac{2}{\mu}} z\right)\right) + A\left(\varepsilon^{\frac{2}{\mu}} z\right) \nabla \varphi\left(\varepsilon^{\frac{2}{\mu}} z\right) \cdot \nabla \varphi\left(\varepsilon^{\frac{2}{\mu}} z\right)\right] \\ &= \widetilde{V}_{\varepsilon}(z) + \left(\widetilde{W}_{1,\varepsilon}(z) - \widetilde{W}_{2,\varepsilon}(z)\right) \cdot \nabla \varphi_{\varepsilon}(z) - \operatorname{div} \widetilde{W}_{2,\varepsilon}(z), \end{aligned}$$

where

$$\begin{aligned} \widetilde{V}_{\varepsilon}(z) &:= \varepsilon^{\frac{4}{\mu}-2} \widetilde{V}\left(\varepsilon^{\frac{2}{\mu}} z\right) \\ \widetilde{W}_{i,\varepsilon}(z) &:= \varepsilon^{\frac{2}{\mu}-1} \widetilde{W}_i\left(\varepsilon^{\frac{2}{\mu}} z\right) \quad \text{ for } i = 1,2. \end{aligned}$$

With $\delta = \varepsilon^{\frac{2}{\mu}} \leq \frac{\rho(7/5)+1/5}{2}$, note that

$$\begin{aligned} \left|\left|\widetilde{V}_{\varepsilon}\right|\right|_{L^p(B_1)} &= \left(\int_{B_1} \left|\widetilde{V}_{\varepsilon}(z)\right|^p dz\right)^{\frac{1}{p}} = \left(\int_{B_1} \left|\varepsilon^{2\left(\frac{2}{\mu}-1\right)} \widetilde{V}\left(\varepsilon^{\frac{2}{\mu}} z\right)\right|^p dz\right)^{\frac{1}{p}} \\ &= \varepsilon^{\frac{2}{\mu}\left(2-\frac{2}{p}-\mu\right)} \left(\int_{B_1} \left|\widetilde{V}\left(\varepsilon^{\frac{2}{\mu}} z\right)\right|^p d\left(\varepsilon^{\frac{2}{\mu}} z\right)\right)^{\frac{1}{p}} \leq \left|\left|\widetilde{V}\right|\right|_{L^p(B_{\delta})} \leq 1 \end{aligned}$$

and for $i=1,2$,

$$\left|\left|\widetilde{W}_{i,\varepsilon}\right|\right|_{L^{q_i}(B_1)} = \left(\int_{B_1}\left|\widetilde{W}_{i,\varepsilon}(z)\right|^{q_i}dz\right)^{\frac{1}{q_i}} = \left(\int_{B_1}\left|\varepsilon^{\frac{2}{\mu}-1}\widetilde{W}_i\left(\varepsilon^{\frac{2}{\mu}}z\right)\right|^{q_i}dz\right)^{\frac{1}{q_i}}$$
$$= \varepsilon^{\frac{1}{\mu}\left(2-\frac{4}{q_i}-\mu\right)}\left(\int_{B_1}\left|\widetilde{W}_i\left(\varepsilon^{\frac{2}{\mu}}z\right)\right|^{q_i}d\left(\varepsilon^{\frac{2}{\mu}}z\right)\right)^{\frac{1}{q_i}} \le \left|\left|\widetilde{W}_i\right|\right|_{L^{q_i}(B_\delta)} \le 1.$$

Moreover,

$$\int_{B_2}|\nabla\varphi_\varepsilon|^2 \le \varepsilon^{2\left(\frac{2}{\mu}-1\right)}\int_{B_2}\left|\nabla\varphi\left(\varepsilon^{\frac{2}{\mu}}z\right)\right|^2 dz = \frac{1}{\varepsilon^2}\int_{B_{2\delta}}|\nabla\varphi|^2 \le \frac{1}{\varepsilon^2}C\left(2\varepsilon^{\frac{2}{\mu}}\right)^{\mu} = C,$$

where we have used Claim 1. It follows from Theorem 2.3 and Proposition 2.1 in Chapter V of [9] that there exists $t_0>2$ such that

$$||\nabla\varphi_\varepsilon||_{L^{t_0}(B_1)} \le C. \tag{37}$$

Recalling the definition of φ_ε, we see that

$$C \ge ||\nabla\varphi_\varepsilon||_{L^{t_0}(B_1)} = \varepsilon^{\frac{2}{\mu}-1-\frac{4}{\mu t_0}}||\nabla\varphi||_{L^{t_0}(B_\delta)} = \frac{\varepsilon^{\frac{2}{\mu}-1-\frac{4}{\mu t_0}}}{C\left(\sqrt{M}+K_1+K_2\right)}||\nabla\Phi||_{L^{t_0}(B_\delta)}.$$

Since $\varepsilon = \frac{1}{C\left(\sqrt{M}+K_1+K_2\right)}$, then we conclude that

$$||\nabla\Phi||_{L^{t_0}(B_\delta)} \le C\left(\sqrt{M}+K_1+K_2\right)^{\frac{2}{\mu}\left(1-\frac{2}{t_0}\right)}.$$

Since this derivation works for any $z\in B_{\rho(7/5)}$and we may cover $B_{\rho(7/5)}$ with N balls of radius δ, where $N\sim\delta^{-2}=\varepsilon^{-4/\mu}\sim\left(\sqrt{M}+K_1+K_2\right)^{4/\mu}$, then the result follows. □

Using an interpolation argument in combination with the estimates just proved, we establish L^t bounds for $\nabla\Phi$.

Lemma 8. *Let $\Phi=\log\phi$, where ϕ is the positive multiplier given in Lemma 4. Let $t_0>0$ be the exponent provided in Lemma 7. For any $t\in[2,t_0]$,*

$$||\nabla\Phi||_{L^t\left(B_{\rho(7/5)}\right)} \le C\left(\sqrt{M}+K_1+K_2\right)^{1+\left[\frac{2}{\mu}\left(3-\frac{2}{t_0}\right)-1\right]\frac{t_0}{t}\left(\frac{t-2}{t_0-2}\right)},$$

where the constant C depends on λ, Λ, p, q_1, q_2, t_0, and t.

Proof. Take $t\in(2,t_0)$ since the endpoint estimates are given in Lemmas 6 and 7. Choose $\gamma\in(0,1)$ so that $t=2\gamma+t_0(1-\gamma)$, i.e. $\gamma=\frac{t_0-t}{t_0-2}$. Then by the Hölder inequality along with Lemmas 6 and

7,

$$
\begin{aligned}
\|\nabla\Phi\|_{L^t\left(B_{\rho(7/5)}\right)} &= \left(\int_{B_{\rho(7/5)}} |\nabla\Phi|^{2\gamma}|\nabla\Phi|^{t_0(1-\gamma)}\right)^{\frac{1}{t}} \leq \left[\left(\int_{B_{\rho(7/5)}} |\nabla\Phi|^2\right)^{\gamma}\left(\int_{B_{\rho(7/5)}} |\nabla\Phi|^{t_0}\right)^{1-\gamma}\right]^{\frac{1}{t}} \\
&= \|\nabla\Phi\|_{L^2\left(B_{\rho(7/5)}\right)}^{\frac{2\gamma}{t}} \|\nabla\Phi\|_{L^{t_0}\left(B_{\rho(7/5)}\right)}^{\frac{t_0(1-\gamma)}{t}} \\
&\leq \left[C\left(\sqrt{M}+K_1+K_2\right)\right]^{\frac{2\gamma}{t}}\left[C\left(\sqrt{M}+K_1+K_2\right)^{\frac{2}{\mu}\left(3-\frac{2}{t_0}\right)}\right]^{\frac{t_0(1-\gamma)}{t}} \\
&= C\left(\sqrt{M}+K_1+K_2\right)^{\frac{2\gamma}{t}+\frac{2}{\mu}\left(3-\frac{2}{t_0}\right)\frac{t_0(1-\gamma)}{t}}.
\end{aligned}
$$

Since $\frac{2}{\mu}\left(3-\frac{2}{t_0}\right) > 1$, then simplifying the exponent gives

$$
\frac{2\gamma}{t}+\left[1+\frac{2}{\mu}\left(3-\frac{2}{t_0}\right)-1\right]\frac{t_0(1-\gamma)}{t} = 1+\left[\frac{2}{\mu}\left(3-\frac{2}{t_0}\right)-1\right]\frac{t_0}{t}\left(\frac{t-2}{t_0-2}\right),
$$

as required. □

Corollary 1. *Let $\Phi = \log\phi$, where ϕ is the positive multiplier given in Lemma 4. Then for any $\varepsilon > 0$, there exists $t > 2$, depending on ε, p, q_1, q_2, and t_0, such that*

$$
\|\nabla\Phi\|_{L^t\left(B_{\rho(7/5)}\right)} \leq C\left(\sqrt{M}+K_1+K_2\right)^{1+\varepsilon}, \tag{38}
$$

where C depends on λ, Λ, p, q_1, q_2, t_0, and t.

4. The Beltrami operators

We define a Beltrami operator that allows us to the reduce the second-order equation to a first order system. For a complex-valued function $f = u+iv$, define

$$
Df = \bar{\partial}f+\eta(z)\,\partial f+\nu(z)\,\overline{\partial f}, \tag{39}
$$

where

$$
\begin{aligned}
\bar{\partial} &= \tfrac{1}{2}\left(\partial_x+i\partial_y\right) \\
\partial &= \tfrac{1}{2}\left(\partial_x-i\partial_y\right) \\
\eta(z) &= \frac{a_{11}-a_{22}}{\det(A+I)}+i\frac{a_{12}+a_{21}}{\det(A+I)} \qquad (40)\\
\nu(z) &= \frac{\det A-1}{\det(A+I)}+i\frac{a_{21}-a_{12}}{\det(A+I)}. \qquad (41)
\end{aligned}
$$

Lemma 9. *For η,ν defined above, there exists $K < 1$ so that*

$$
|\eta(z)|+|\nu(z)| \leq K.
$$

The following proof is purely computational and relies on the assumption (1).

Proof. We have

$$|\eta| = \frac{\sqrt{(a_{11}-a_{22})^2+(a_{12}+a_{21})^2}}{\det(A+I)} = \frac{\sqrt{\operatorname{tr}A^2-4a_{11}a_{22}+(a_{12}+a_{21})^2}}{\det A+\operatorname{tr}A+1}$$

$$|\nu| = \frac{\sqrt{(\det A-1)^2+(a_{21}-a_{12})^2}}{\det(A+I)} = \frac{\sqrt{(\det A+1)^2-4a_{11}a_{22}+(a_{21}+a_{12})^2}}{\det A+\operatorname{tr}A+1}.$$

Note that it follows from (1) that

$$a_{11}a_{22}-\frac{1}{4}(a_{12}+a_{21})^2 \geq \lambda^2.$$

Therefore, we see that

$$|\eta(z)|+|\nu(z)| \leq \frac{\sqrt{\operatorname{tr}A^2-4\lambda^2}+\sqrt{(\det A+1)^2-4\lambda^2}}{\operatorname{tr}A+\det A+1} =: K.$$

□

Let $f=u+iv$. A computation shows that

$$Df = \frac{(a_{11}+\det A)+ia_{21}}{\det(A+I)}u_x + \frac{a_{12}+i(a_{22}+\det A)}{\det(A+I)}u_y + \frac{(a_{11}+1)+ia_{12}}{\det(A+I)}iv_x + \frac{a_{21}+i(a_{22}+1)}{\det(A+I)}iv_y.$$

This presentation will be useful in subsequent sections.

In addition to the operator D, we will also make use of an operator that is related to D through some function w. For a given function w, set

$$\eta_w(z) = \begin{cases} \eta(z)+\nu(z)\frac{\overline{\partial w}}{\partial w} & \text{for } \partial w \neq 0 \\ \eta(z)+\nu(z) & \text{otherwise} \end{cases},$$

where η and ν are as defined in (40) and (41), respectively. By Lemma 9, it follows that $|\eta_w| \leq K$. Define

$$D_w f = \overline{\partial} f + \eta_w(z)\,\partial f. \tag{42}$$

If $\eta_w(z) = \alpha_w(z)+i\beta_w(z)$, then

$$\begin{aligned} D_w &= \frac{1}{2}\left[\partial_x + i\partial_y + (\alpha_w+i\beta_w)(\partial_x - i\partial_y)\right] \\ &= \frac{1+\alpha_w+i\beta_w}{2}\partial_x + \frac{\beta_w+i(1-\alpha_w)}{2}\partial_y \end{aligned} \tag{43}$$

Bertrami operators of this form will be used in the proofs of the main theorems.

At times, the dependence on w will not be important to our arguments, so we define

$$\hat{D} = \frac{1+\alpha+i\beta}{2}\partial_x + \frac{\beta+i(1-\alpha)}{2}\partial_y, \tag{44}$$

where α,β are assumed to be functions of z such that $\alpha^2+\beta^2 \leq K < 1$. Associated to $\hat{D}$ is the symmetric second-order elliptic operator $\hat{L} = \operatorname{div}\left(\hat{A}\nabla\right)$ with

$$\hat{A} = \begin{bmatrix} \frac{(1+\alpha)^2+\beta^2}{1-\alpha^2-\beta^2} & \frac{2\beta}{1-\alpha^2-\beta^2} \\ \frac{2\beta}{1-\alpha^2-\beta^2} & \frac{(1-\alpha)^2+\beta^2}{1-\alpha^2-\beta^2} \end{bmatrix} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{12} & \hat{a}_{22} \end{bmatrix}. \tag{45}$$

A computation shows that

$$\left[\frac{(1+\alpha)^2+\beta^2}{1-\alpha^2-\beta^2}\right]\left[\frac{(1-\alpha)^2+\beta^2}{1-\alpha^2-\beta^2}\right]-\frac{1}{4}\left[\frac{2\beta}{1-\alpha^2-\beta^2}+\frac{2\beta}{1-\alpha^2-\beta^2}\right]^2$$
$$=\frac{1}{(1-\alpha^2-\beta^2)^2}\left[\left(1-\alpha^2\right)^2+2\beta^2\left(1+\alpha^2\right)+\beta^4-4\beta^2\right]$$
$$=\frac{1-2\alpha^2+\alpha^4-2\left(1-\alpha^2\right)\beta^2+\beta^4}{(1-\alpha^2-\beta^2)^2}=1.$$

Therefore, $\hat{A}$ satisfies the same ellipticity and boundedness given in (1) and (2) with possibly different constants λ, Λ.

Remark. Note that if D is given as in (39) and $Df=0$, then $D_w f=0$ with $w=f$, where D_w is defined in (42).

4.1. **A Hadamard three-quasi-circle theorem.** Within this subsection, we present the Hadamard three-quasi-circle theorem. We originally proved this result in [6, Theorem 4.5], but include the proof here for completeness. The related lemmas are all presented, but we refer the reader to [6] for their computational proofs.

The following lemmas show that $\hat{D}$ relates to $\hat{L}$ in some of the same ways that $\overline{\partial}$ relates to Δ. These properties will allow us to prove the Hadamard three-quasi-circle theorem.

Lemma 10. [6, Lemma 4.2] *If $\hat{D}f=0$, where $f(x,y)=u(x,y)+iv(x,y)$ for real-valued u and v, then*

$$\hat{L}u=0=\hat{L}v.$$

We find another parallel with the Laplace equation. As in the case of $\hat{L}=\Delta$, the logarithm of the norm of f is a subsolution to the second-order equation whenever $\hat{D}f=0$. To see this, it suffices to prove that

Lemma 11. [6, Lemma 4.3] *If $\hat{D}f=0$ and $f\neq 0$, then $\hat{L}[\log|f(z)|]=0$.*

Using the fundamental solution $\hat{G}$ for the operator $\hat{L}$, we can now prove the following Hadamard three-quasi-ball inequality. We would like to mention that similar theorems were proved by Alessandrini and Escauriaza in [2], see Propositions 1, 2, using quasi-regular mappings.

Theorem 3. *Let f be a function for which $\hat{D}f=0$ in Q_{s_0}. Set*

$$M(s)=\max\{|f(z)|:z\in Z_s\}.$$

Then for any $0<s_1<s_2<s_3<s_0$,

$$\log\left(\frac{s_3}{s_1}\right)\log M(s_2)\leq\log\left(\frac{s_3}{s_2}\right)\log M(s_1)+\log\left(\frac{s_2}{s_1}\right)\log M(s_3). \tag{46}$$

Proof. Let $\mathscr{A}_{s_1,s_3}=\{z:s_1\leq\ell(z)\leq s_3\}=\overline{Q_{s_3}\setminus Q_{s_1}}$, where ℓ is associated to $\hat{G}$, the fundamental solution of $\hat{L}$. By Lemma 2, this set is contained in an annulus with inner and outer radius depending on s_1, s_3, λ, and Λ. In particular, it is bounded and does not contain the origin.

Therefore, $\hat{G}(z)$ is bounded on $\mathscr{A}_{s_1,s_3}$. Let z_0 be in the interior of $\mathscr{A}_{s_1,s_3}$. If $f(z_0)=0$, then $a\hat{G}(z_0)+\log|f(z_0)|=-\infty$ for any $a\in\mathbb{R}$. On the other hand, if $f(z_0)\neq 0$, then Lemma 11 implies that $\hat{L}\left[a\hat{G}(z)+\log|f(z)|\right]=0$ for z near z_0. By the maximum principle, z_0 cannot be an extremal point. Therefore, $a\hat{G}(z)+\log|f(z)|$ takes it maximum value on the boundary of $\mathscr{A}_{s_1,s_3}$. We will choose the constant $a\in\mathbb{R}$ so that

$$\max\left\{a\hat{G}(z)+\log|f(z)| : z\in Z_{s_1}\right\}=\max\left\{a\hat{G}(z)+\log|f(z)| : z\in Z_{s_3}\right\},$$

or rather

$$\log\left(s_1^a M(s_1)\right)=\log\left(s_3^a M(s_3)\right).$$

It follows that for any $z\in\mathscr{A}_{s_1,s_3}$,

$$a\hat{G}(z)+\log|f(z)|\le\log\left(s_i^a M(s_i)\right)\quad\text{for } i=1,3.$$

Furthermore, for any $s_2\in(s_1,s_3)$,

$$\max\left\{a\hat{G}(z)+\log|f(z)| : z\in Z_{s_2}\right\}\le\log\left(s_i^a M(s_i)\right)\quad\text{for } i=1,3,$$

or

$$\log\left(s_2^a M(s_2)\right)\le\log\left(s_i^a M(s_i)\right)\quad\text{for } i=1,3.$$

Consequently,

$$s_2^a M(s_2)\le s_i^a M(s_i)\quad\text{for } i=1,3,$$

so that for any $\tau\in(0,1)$, since $s_1^a M(s_1)=s_3^a M(s_3)$, then

$$\begin{aligned}
&s_2^a M(s_2)\le\left[s_1^a M(s_1)\right]^{\tau}\left[s_3^a M(s_3)\right]^{1-\tau}\\
&\left[M(s_2)\right]^{\log\left(\frac{s_3}{s_1}\right)}\le\left[\left(\frac{s_1}{s_2}\right)^a M(s_1)\right]^{\tau\log\left(\frac{s_3}{s_1}\right)}\left[\left(\frac{s_3}{s_2}\right)^a M(s_3)\right]^{(1-\tau)\log\left(\frac{s_3}{s_1}\right)}.
\end{aligned}$$

We choose τ so that $\tau\log\left(\frac{s_3}{s_1}\right)=\log\left(\frac{s_3}{s_2}\right)$. Then $(1-\tau)\log\left(\frac{s_3}{s_1}\right)=\log\left(\frac{s_2}{s_1}\right)$ and

$$\left[\left(\frac{s_1}{s_2}\right)^a\right]^{\tau\log\left(\frac{s_3}{s_1}\right)}\left[\left(\frac{s_3}{s_2}\right)^a\right]^{(1-\tau)\log\left(\frac{s_3}{s_1}\right)}=\exp\left[a\log\left(\frac{s_3}{s_2}\right)\log\left(\frac{s_1}{s_2}\right)+a\log\left(\frac{s_2}{s_1}\right)\log\left(\frac{s_3}{s_2}\right)\right]=1.$$

Therefore,

$$M(s_2)^{\log\left(\frac{s_3}{s_1}\right)}\le M(s_1)^{\log\left(\frac{s_3}{s_2}\right)}M(s_3)^{\log\left(\frac{s_2}{s_1}\right)}.$$

Taking logarithms completes the proof. $\square$

Corollary 2. *Let f satisfy $\hat{D}f=0$ in Q_{s_0}. Then for $0<s_1<s_2<s_3<s_0$*

$$||f||_{L^\infty(Q_{s_2})}\le\left(||f||_{L^\infty(Q_{s_1})}\right)^{\theta}\left(||f||_{L^\infty(Q_{s_3})}\right)^{1-\theta},$$

where

$$\theta=\frac{\log(s_3/s_2)}{\log(s_3/s_1)}.$$

Remark. From Remark 4, we know that if $Df=0$, then $D_f f=0$. Hence Corollary 2 applies to such f.

4.2. **The similarity principle.** This subsection is similar to Section 4.4 of [6]. As usual, we include it here for the sake of completeness. The approach here is based on the work of Bojarksi, as presented in [3]. Define the operators

$$T\omega(z) = -\frac{1}{\pi}\int_{\Omega}\frac{\omega(\zeta)}{\zeta - z}d\zeta$$

$$S\omega(z) = -\frac{1}{\pi}\int_{\Omega}\frac{\omega(\zeta)}{(\zeta - z)^2}d\zeta.$$

We use the of the following results, collected from [3].

Lemma 12. *Suppose that $g \in L^t$ for some $t > 2$. Then Tg exists everywhere as an absolutely convergent integral and Sg exists almost everywhere as a Cauchy principal limit. The following relations hold:*

$$\begin{aligned}
&\bar{\partial}(Tg) = g\\
&\partial(Tg) = Sg\\
&|Tg(z)| \le c_t\,||g||_{L^t}\\
&||Sg||_{L^t} \le C_t\,||g||_{L^t}\\
&\lim_{t\to 2^+} C_t = 1.
\end{aligned}$$

Lemma 13 (see Theorems 4.1, 4.3 [3])**.** *Let w be a generalized solution (possibly admitting isolated singularities) to*

$$\bar{\partial}w + q_1(z)\,\partial w + q_2(z)\,\overline{\partial w} = A(z)\,w + B(z)\,\bar{w}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. Assume that $|q_1(z)| + |q_2(z)| \le \alpha_0 < 1$ in Ω, and A, B belong to $L^t(\Omega)$ for some $t > 2$. Then $w(z)$ is given by

$$w(z) = f(z)\,e^{\phi(z)},$$

where f is a solution to

$$\bar{\partial}f + q_0(z)\,\partial f = 0$$

and

$$\phi(z) = T\omega(z).$$

Here, q_0 is defined by

$$q_0(z) = \begin{cases} q_1(z) + q_2(z)\,\frac{\overline{\partial w}}{\partial w} & \text{for } \partial w \neq 0\\ q_1(z) + q_2(z) & \text{otherwise}\,, \end{cases} \tag{47}$$

and ω solves

$$\omega + q_0 S\omega = h \tag{48}$$

with

$$h(z) = \begin{cases} A(z) + B(z)\,\frac{\bar{w}}{w} & \text{for } w(z) \neq 0 \text{ and } w(z) \neq \infty\\ A(z) + B(z) & \text{otherwise}\,. \end{cases}$$

The proof ideas are available in [3] and detailed arguments can be found in [6].

Corollary 3. *Let w be a generalized solution (possibly admitting isolated singularities) to*

$$\bar{\partial} w + q_1(z)\,\partial w + q_2(z)\,\overline{\partial w} = A(z)\,w + B(z)\,\bar{w}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. Assume that $|q_1(z)| + |q_2(z)| \le \alpha_0 < 1$ in Ω, and A, B belong to $L^t(\Omega)$ for some $t > 2$. Then $w(z)$ is given by

$$w(z) = f(z)\,g(z),$$

where f is a solution to

$$\bar{\partial} f + q_0(z)\,\partial f = 0$$

and

$$\exp\left[-C\left(\|A\|_{L^t(\Omega)} + \|B\|_{L^t(\Omega)}\right)\right] \le |g(z)| \le \exp\left[C\left(\|A\|_{L^t(\Omega)} + \|B\|_{L^t(\Omega)}\right)\right].$$

Proof. From the previous lemma, we have that $g(z) = \exp(T\omega(z))$, where ω is the unique solution to (48). Since $C_t\alpha_0 < 1$, then a fixed point argument implies that $\|\omega\|_{L^t} \le C\|h\|_{L^t}$. It follows from the third fact in Lemma 12 that

$$|T\omega(z)| \le C\|h\|_{L^t} \le C\left[\|A\|_{L^t(\Omega)} + \|B\|_{L^t(\Omega)}\right],$$

where C depends on Ω. The conclusion follows. $\square$

5. The order of vanishing estimates

Here we present the proof of the order of vanishing estimate, that of Theorem 2. One of the novelties with this proof is that instead of using one positive multiplier to transform the equation for u into a divergence-free equation, we rely upon two positive multipliers. This idea was inspired by the methods in [1].

Let u be a solution to (7) in $B_d \subset \mathbb{R}^2$. That is,

$$-\operatorname{div}(A\nabla u + W_1 u) + W_2 \cdot \nabla u + Vu = 0 \quad \text{in } B_d.$$

Assumptions (1), (5) and (6) imply that Lemma 4 is applicable, and therefore there exists a function ϕ satisfying (23) that weakly solves

$$-\operatorname{div}\left(A^T\nabla\phi + W_2\phi\right) + W_1 \cdot \nabla\phi + V\phi = 0 \quad \text{in } B_d.$$

Similarly, Lemma 4 shows that there exists a function ψ that weakly solves

$$-\operatorname{div}(A\nabla\psi) + W_2 \cdot \nabla\psi + V\psi = 0 \quad \text{in } B_d.$$

Bounds analogous to (23) hold for ψ.

Remark. If either $W_1 \equiv 0$ or $W_2 \equiv 0$, then the second positive multiplier, ψ, is not required, and we set $\psi = \phi$ in all arguments below. If $W_1 \equiv 0$, then only conditions (11) and (12) are required for the application of Lemma 4.

With $v = \dfrac{u}{\psi}$, $\Phi = \log\phi$, $\Psi = \log\psi$, and $b = A\nabla\Psi - A^T\nabla\Phi + W_1 - W_2$, we have

$$\begin{aligned}
&\operatorname{div}\left[\phi\,\psi\left(A\nabla v + bv\right)\right] \\
&= \operatorname{div}\left[\phi\left(A\nabla u - uA^T\nabla\Phi + uW_1 - uW_2\right)\right] \\
&= \operatorname{div}\left(\phi A\nabla u - uA^T\nabla\phi + u\phi W_1 - u\phi W_2\right) \\
&= \nabla\phi\cdot A\nabla u + \phi\operatorname{div}\left(A\nabla u\right) - \nabla u\cdot A^T\nabla\phi - u\operatorname{div}\left(A^T\nabla\phi\right) + \operatorname{div}\left(u\phi W_1 - u\phi W_2\right) \\
&= \phi\left[Vu + W_2\cdot\nabla u - \operatorname{div}\left(W_1 u\right)\right] - u\left[V\phi + W_1\cdot\nabla\phi - \operatorname{div}\left(W_2\phi\right)\right] + \operatorname{div}\left(u\phi W_1 - u\phi W_2\right) \\
&= 0.
\end{aligned}$$

Therefore, the PDE for u can be transformed into a divergence-free equation. Let $\tilde{v}$ be the stream function associated to the vector $\phi\,\psi\left(A\nabla v + bv\right)$ with $\tilde{v}(0,0) = 0$. That is, for every $(x,y) \in B_d$,

$$\tilde{v}(x,y) = \int_0^1 \left[-\phi\,\psi\left(a_{21}v_x + a_{22}v_y + b_2 v\right)(tx,ty)\,x + \phi\,\psi\left(a_{11}v_x + a_{12}v_y + b_1 v\right)(tx,ty)\,y\right]dt. \quad (49)$$

To verify the validity of (49), we let $P = (P_1, P_2) = \phi\,\psi\left(A\nabla v + bv\right)$, then

$$\tilde{v}(x,y) = -\int_0^1 P_2(tx,ty)\,x\,dt + \int_0^1 P_1(tx,ty)\,y\,dt.$$

So we have

$$\begin{aligned}
\tilde{v}_x(x,y) &= -\int_0^1 \partial_1 P_2(tx,ty)\,tx\,dt - \int_0^1 P_2(tx,ty)\,dt + \int_0^1 \partial_1 P_1(tx,ty)\,ty\,dt \\
&= -\int_0^1 \partial_1 P_2(tx,ty)\,tx\,dt - \int_0^1 P_2(tx,ty)\,dt - \int_0^1 \partial_2 P_2(tx,ty)\,ty\,dt \\
&= -\int_0^1 \left[\partial_t P_2(tx,ty)\,t + P_2(tx,ty)\right]dt = -\int_0^1 \partial_t\left[P_2(tx,ty)\,t\right]dt = -P_2(x,y)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{v}_y(x,y) &= -\int_0^1 \partial_2 P_2(tx,ty)\,tx\,dt + \int_0^1 \partial_2 P_1(tx,ty)\,ty\,dt + \int_0^1 P_1(tx,ty)\,dt \\
&= \int_0^1 \partial_1 P_1(tx,ty)\,tx\,dt + \int_0^1 \partial_2 P_1(tx,ty)\,ty\,dt + \int_0^1 P_1(tx,ty)\,dt \\
&= \int_0^1 \left[\partial_t P_1(tx,ty)\,t + P_1(tx,ty)\right]dt = \int_0^1 \partial_t\left[P_1(tx,ty)\,t\right]dt = P_1(x,y).
\end{aligned}$$

That is,

$$\begin{cases} \tilde{v}_y &= \phi\,\psi\left(a_{11}v_x + a_{12}v_y + b_1 v\right) \\ -\tilde{v}_x &= \phi\,\psi\left(a_{21}v_x + a_{22}v_y + b_2 v\right). \end{cases} \quad (50)$$

Lemma 14. *For any r and $\kappa > 1$ such that $\kappa r \le d$, there is a constant c, depending on λ, Λ, and a constant C, depending on λ, Λ, κ, for which*

$$||\tilde{v}||_{L^1(B_r)} \le Cre^{c\left(\sqrt{M}+K_1+K_2\right)}\,||u||_{L^2(B_{\kappa r})}.$$

Proof. As above, we use the notation

$$\tilde{v}(x,y) = -\int_0^1 P_2(tx,ty)\,x\,dt + \int_0^1 P_1(tx,ty)\,y\,dt.$$

It follows that

$$\begin{aligned}
\|\tilde{v}\|_{L^1(B_r)} &= \int_{B_r} \left| -\int_0^1 P_2(tx,ty)\,x\,dt + \int_0^1 P_1(tx,ty)\,y\,dt \right| dz \\
&\le \int_{B_r}\int_0^1 |P_2(tx,ty)\,x|\,dtdz + \int_{B_r}\int_0^1 |P_1(tx,ty)\,y|\,dtdz \\
&\le r\int_{B_r}\int_0^1 |P_2(tx,ty)|\,dtdz + r\int_{B_r}\int_0^1 |P_1(tx,ty)|\,dtdz.
\end{aligned}$$

A computation shows that

$$\begin{aligned}
P_1 &= (a_{11}u_x + a_{12}u_y)\,\phi - u\,(a_{11}\phi_x + a_{21}\phi_y) + u\,(W_{1,1} - W_{2,1})\,\phi \\
P_2 &= (a_{21}u_x + a_{22}u_y)\,\phi - u\,(a_{12}\phi_x + a_{22}\phi_y) + u\,(W_{1,2} - W_{2,2})\,\phi,
\end{aligned}$$

where we use the notation $W_i = (W_{i,1}, W_{i,2})$ for $i = 1,2$. By Lemma 5 applied to u and ϕ, along with the assumption that each $W_i \in L^{q_i}$, it follows that each $P_i \in L^{q_0}$ for some $q_0 \in (2, \tau_0]$. Interchanging the order of integration, applying Hölder's inequality, then simplifying, we see that

$$\begin{aligned}
\int_{B_r}\int_0^1 |P_1(tx,ty)|\,dtdz &= \int_0^1 \frac{1}{t^2}\int_{B_{tr}} |P_1(x,y)|\,dzdt \le \int_0^1 \frac{1}{t^2}\left(\int_{B_{tr}} |P_1(x,y)|^{q_0}\,dz\right)^{\frac{1}{q_0}} |B_{tr}|^{1-\frac{1}{q_0}}\,dt \\
&= Cr^{2-\frac{2}{q_0}}\int_0^1 \|P_1\|_{L^{q_0}(B_{tr})}\,t^{-\frac{2}{q_0}}\,dt \le Cr^{2-\frac{2}{q_0}}\,\|P_1\|_{L^{q_0}(B_r)}\int_0^1 t^{-\frac{2}{q_0}}\,dt.
\end{aligned}$$

Since $q_0 > 2$, then $\int_0^1 t^{-\frac{2}{q_0}}\,dt$ converges and $\int_{B_r}\int_0^1 |P_1(tx,ty)|\,dtdz \le Cr^{2-\frac{2}{q_0}}\,\|P_1\|_{L^{q_0}(B_r)}$. A similar estimate holds for $\int_{B_r}\int_0^1 |P_2(tx,ty)|^2\,dtdz$, so we conclude that

$$\|\tilde{v}\|_{L^1(B_r)} \le Cr^{3-\frac{2}{q_0}}\left(\|P_1\|_{L^{q_0}(B_r)} + \|P_2\|_{L^{q_0}(B_r)}\right).$$

Let $\beta = \frac{1}{2}(\kappa+1)$. Therefore, for $i = 1,2$, with an application of (2) and Lemma 5, we see that

$$\begin{aligned}
\|P_i\|_{L^{q_0}(B_r)} &\le \Lambda\left(\|\nabla u\|_{L^{q_0}(B_r)}\|\phi\|_{L^\infty(B_r)} + \|u\|_{L^\infty(B_r)}\|\nabla\phi\|_{L^{q_0}(B_r)}\right) \\
&+ \left(\left\|W_{1,i}\right\|_{L^{q_1}(B_r)}|B_r|^{\frac{1}{q_0}-\frac{1}{q_1}} + \left\|W_{2,i}\right\|_{L^{q_2}(B_r)}|B_r|^{\frac{1}{q_0}-\frac{1}{q_2}}\right)\|u\|_{L^\infty(B_r)}\|\phi\|_{L^\infty(B_r)} \\
&\le Cr^{\frac{2}{q_0}-1}\left(1 + r^{2-\frac{2}{p}}M + r^{2-\frac{4}{q_1}}K_1^2 + r^{2-\frac{4}{q_2}}K_2^2\right)\|u\|_{L^\infty(B_{\beta r})}\|\phi\|_{L^\infty(B_{\beta r})} \\
&+ \left(K_1 r^{\frac{2}{q_0}-\frac{2}{q_1}} + K_2 r^{\frac{2}{q_0}-\frac{2}{q_2}}\right)\|u\|_{L^\infty(B_r)}\|\phi\|_{L^\infty(B_r)} \\
&\le Cr^{\frac{2}{q_0}-1}\left(1 + r^{2-\frac{2}{p}}M + r^{2-\frac{4}{q_1}}K_1^2 + r^{2-\frac{4}{q_2}}K_2^2\right)\|u\|_{L^\infty(B_{\beta r})}\,e^{c\left(\sqrt{M}+K_1+K_2\right)},
\end{aligned}$$

where we have used the pointwise bounds on ϕ from Lemma 4. It follows that

$$\begin{aligned}||\tilde{v}||_{L^1(B_r)} &\le Cr^2\left(1+r^{2-\frac{2}{p}}M+r^{2-\frac{4}{q_1}}K_1^2+r^{2-\frac{4}{q_2}}K_2^2\right)||u||_{L^\infty\left(B_{\beta r}\right)}e^{c\left(\sqrt{M}+K_1+K_2\right)}\\ &\le Cr\,||u||_{L^2(B_{\kappa r})}e^{c\left(\sqrt{M}+K_1+K_2\right)},\end{aligned}$$

where the last inequality follows from an application of interior estimates for elliptic equations, which can be derived by De Giorgi's method (for example, see the proof of [5, Lemma 5.1] for related arguments). □

With $w=\phi\psi v+i\tilde{v}$ and D as defined in (39), we see that in $B_d\Supset B_{\rho(7/5)}$,

$$\begin{aligned}Dw &= D\phi\psi v+\phi D\psi v+\phi\psi Dv+D(i\tilde{v})\\ &= \left[D(\log\phi)+D(\log\psi)\right]\phi\psi v\\ &+\phi\psi\left[\frac{(a_{11}+\det A)+ia_{21}}{\det(A+I)}v_x+\frac{a_{12}+i(a_{22}+\det A)}{\det(A+I)}v_y\right]\\ &+\frac{(a_{11}+1)+ia_{12}}{\det(A+I)}i\tilde{v}_x+\frac{a_{21}+i(a_{22}+1)}{\det(A+I)}i\tilde{v}_y\\ &= \left[D(\log\phi)+D(\log\psi)+ib_1\frac{a_{21}+i(a_{22}+1)}{\det(A+I)}-ib_2\frac{(a_{11}+1)+ia_{12}}{\det(A+I)}\right]\phi\psi v\\ &+\phi\psi\left[\frac{(a_{11}+\det A)+ia_{21}}{\det(A+I)}+\frac{-ia_{21}(a_{11}+1)+a_{12}a_{21}}{\det(A+I)}+\frac{ia_{21}a_{11}-a_{11}(a_{22}+1)}{\det(A+I)}\right]v_x\\ &+\phi\psi\left[\frac{a_{12}+i(a_{22}+\det A)}{\det(A+I)}+\frac{-ia_{22}(a_{11}+1)+a_{12}a_{22}}{\det(A+I)}+\frac{ia_{12}a_{21}-a_{12}(a_{22}+1)}{\det(A+I)}\right]v_y\\ &= (\alpha+\beta_1-\beta_2)(w+\bar{w}),\end{aligned}\tag{51}$$

where, recalling that we set $\Phi=\log\phi$ and $\Psi=\log\psi$,

$$\alpha+\beta_1-\beta_2=\frac{1}{2}D\Phi+\frac{1}{2}D\Psi+\frac{b_2a_{12}-b_1(a_{22}+1)+ib_1a_{21}-ib_2(a_{11}+1)}{2\det(A+I)}.$$

That is,

$$\begin{aligned}\alpha &= \frac{2a_{11}(1+a_{22})-(a_{12}+a_{21})a_{12}+i\left[(a_{12}+a_{21})+a_{11}(a_{12}-a_{21})\right]}{4\det(A+I)}\Phi_x\\ &+\frac{(a_{12}+a_{21})-a_{22}(a_{12}-a_{21})+i\left[2a_{22}(1+a_{11})-(a_{12}+a_{21})a_{21}\right]}{4\det(A+I)}\Phi_y\\ &+\frac{2a_{11}(1+a_{22})-(a_{12}+a_{21})a_{12}+i\left[(a_{12}+a_{21})+a_{11}(a_{12}-a_{21})\right]}{4\det(A+I)}\Psi_x\\ &+\frac{(a_{12}+a_{21})-a_{22}(a_{12}-a_{21})+i\left[2a_{22}(1+a_{11})-(a_{12}+a_{21})a_{21}\right]}{4\det(A+I)}\Psi_y\end{aligned}\tag{52}$$

$$\beta_j=\frac{-W_{j,1}(a_{22}+1)+W_{j,2}a_{12}-iW_{j,2}(a_{11}+1)+iW_{j,1}a_{21}}{2\det(A+I)}\qquad\text{for } j=1,2.\tag{53}$$

It follows from the boundedness of A described by (2) in combination with Corollary 1, that for any $\varepsilon > 0$, there exists $t > 2$ such that

$$||\alpha||_{L^t\left(B_{\rho(7/5)}\right)} \le C\left(\sqrt{M} + K_1 + K_2\right)^{1+\varepsilon}.$$

The boundedness of A along with the assumptions on W_1 and W_2 implies that

$$||\beta||_{L^q\left(B_{\rho(7/5)}\right)} \le C\left(K_1 + K_2\right),$$

where $q = \min\{q_1, q_2\}$. We now apply the similarity principle given in Lemma 13 and Corollary 3 to conclude that any solution to (51) in $B_{\rho(7/5)}$ is a function of the form

$$w(z) = f(z)\, g(z),$$

with

$$D_w f = 0 \;\text{ in }\; B_{\rho(7/5)},$$

and for a.e. $z \in B_{\rho(7/5)}$,

$$\exp\left[-C\left(||\alpha||_{L^t\left(B_{\rho(7/5)}\right)} + ||\beta||_{L^q\left(B_{\rho(7/5)}\right)}\right)\right] \le |g(z)| \le \exp\left[C\left(||\alpha||_{L^t\left(B_{\rho(7/5)}\right)} + ||\beta||_{L^q\left(B_{\rho(7/5)}\right)}\right)\right].$$

That is,

$$\exp\left[-C\left(\sqrt{M} + K_1 + K_2\right)^{1+\varepsilon}\right] \le |g(z)| \le \exp\left[C\left(\sqrt{M} + K_1 + K_2\right)^{1+\varepsilon}\right] \;\text{ in } B_{\rho(7/5)}, \tag{54}$$

where we have used the bounds on α and β from above. By Corollary 2, the Hadamard three-quasi-circle theorem, applied to the operator D_w,

$$||f||_{L^\infty\left(Q_{s_1}\right)} \le \left(||f||_{L^\infty\left(Q_{s/4}\right)}\right)^{\theta}\left(||f||_{L^\infty\left(Q_{s_2}\right)}\right)^{1-\theta},$$

where $s < s_1 < s_2 < \frac{7}{5}$ and

$$\theta = \frac{\log\left(s_2/s_1\right)}{\log\left(4s_2/s\right)}.$$

Let $r/4 = \rho(s/4)$ and $r_2 = \rho(s_2)$ so that $Q_{s/4} \subset B_{r/4}$ and $Q_{s_2} \subset B_{r_2}$. We choose $r_3 \in (r_2, \rho(7/5))$ so that $r_3 - r_2 \sim 1$ and $\rho(7/5) - r_3 \sim 1$. Since f is a solution to an elliptic equation (see Lemma 10), then standard interior estimates for elliptic equations (see, for example, [12, Theorem 4.1]) imply that

$$||f||_{L^\infty\left(Q_{s_1}\right)} \le C\left(r^{-2}\,||f||_{L^1\left(B_{r/2}\right)}\right)^{\theta}\left(||f||_{L^1\left(B_{r_3}\right)}\right)^{1-\theta}, \tag{55}$$

where C is an absolute constant. Substituting $f = wg^{-1}$ into (55) and applying (54), we see that

$$||w||_{L^\infty\left(Q_{s_1}\right)} \le \exp\left[C\left(\sqrt{M} + K_1 + K_2\right)^{1+\varepsilon}\right]\left(r^{-2}\,||w||_{L^1\left(B_{r/2}\right)}\right)^{\theta}\left(||w||_{L^1\left(B_{r_3}\right)}\right)^{1-\theta}. \tag{56}$$

Since $w = \phi\psi v + i\tilde{v} = \phi u + i\tilde{v}$, then

$$|\phi u| \le |w| \le |\phi u| + |\tilde{v}|.$$

An application of (23) and Lemma 14 with $\kappa = 2$ shows that

$$||w||_{L^1\left(B_{r/2}\right)} \le ||\phi u||_{L^1\left(B_{r/2}\right)} + ||\tilde{v}||_{L^1\left(B_{r/2}\right)} \le \exp\left[c\left(\sqrt{M} + K_1 + K_2\right)\right]||u||_{L^2(B_r)}.$$

We similarly conclude that

$$||w||_{L^1(B_{r_3})} \le \exp\left[c\left(\sqrt{M}+K_1+K_2\right)\right]||u||_{L^2(B_d)} \le \exp\left[C\left(\sqrt{M}+K_1+K_2\right)\right],$$

where we have applied (13) in the second inequality. Upon setting $s_1 = 1$ in (56) and using the bounds established above, we have

$$||u||_{L^\infty(Q_1)} \le \exp\left[C\left(\sqrt{M}+K_1+K_2\right)^{1+\varepsilon}\right]\left(r^{-2}||u||_{L^2(B_r)}\right)^\theta.$$

Now we define

$$b = \sigma(1) \tag{57}$$

so that $B_b \subset Q_1$. Since $||u||_{L^\infty(Q_1)} \ge ||u||_{L^\infty(B_b)} \ge 1$ by (14), after rearranging we have

$$||u||_{L^2(B_r)} \ge r^2 \exp\left[-\frac{C\left(\sqrt{M}+K_1+K_2\right)^{1+\varepsilon}}{\theta}\right].$$

It follows from the definition of θ that

$$||u||_{L^2(B_r)} \ge r^{C\left(\sqrt{M}+K_1+K_2\right)^{1+\varepsilon}},$$

and the conclusion of Theorem 2 follows.

6. Unique continuation at infinity estimates

We follow the approach of Bourgain and Kenig from [4] and show how Theorem 1 follows from Theorem 2.

Let u be a solution to (7) in $\mathbb{R}^2$. Recall that $b = \sigma(1)$ and $d = \rho(7/5) + \frac{2}{5}$. Choose $z_0 \in \mathbb{R}^2$ and set $|z_0| = bR$. Define $u_R(z) = u(z_0 + Rz)$, $A_R(z) = A(z_0 + Rz)$, $W_{i,R}(z) = RW_i(z_0 + Rz)$ for $i = 1, 2$, and $V_R(z) = R^2 V(z_0 + Rz)$. Notice that for any $r > 0$,

$$\begin{aligned}
||V_R||_{L^p(B_r(0))} &= \left(\int_{B_r(0)} |V_R(z)|^p\, dz\right)^{\frac{1}{p}} = \left(\int_{B_r(0)} \left|R^2 V(z_0+Rz)\right|^p dz\right)^{\frac{1}{p}} \\
&= R^{2-\frac{2}{p}}\left(\int_{B_r(0)} |V(z_0+Rz)|^p\, d(Rz)\right)^{\frac{1}{p}} = R^{2-\frac{2}{p}}||V||_{L^p(B_{rR}(z_0))}
\end{aligned}$$

and

$$||W_{i,R}||_{L^q(B_r(0))} = R^{1-\frac{2}{q_i}}\left(\int_{B_r(0)} |W_i(z_0+Rz)|^{q_i}\, d(Rz)\right)^{\frac{1}{q_i}} = R^{1-\frac{2}{q_i}}||W_i||_{L^{q_i}(B_{rR}(z_0))}.$$

It follows that $||V_R||_{L^p(B_d(0))} \le A_0 R^{2-\frac{2}{p}}$ and $||W_{i,R}||_{L^q(B_d(0))} \le A_i R^{1-\frac{2}{q_i}}$ for $i = 1, 2$. Moreover,

$$\begin{aligned}
&-\operatorname{div}\left[A_R(z)\nabla u_R(z) + W_{1,R}(z)\, u_R(z)\right] + W_{2,R}(z)\cdot\nabla u_R(z) + V_R(z)\, u_R(z) \\
&= R^2\left\{-\operatorname{div}\left[A(z_0+Rz)\nabla u(z_0+Rz) + W_1(z_0+Rz)\, u(z_0+Rz)\right]\right\} \\
&+ R^2\left[W_2(z_0+Rz)\cdot\nabla u(z_0+Rz) + V(z_0+Rz)\, u(z_0+Rz)\right] = 0.
\end{aligned}$$

Hence, u_R satisfies a scaled version of (7) in B_d. By assumption (8),

$$||u_R||_{L^\infty(B_d)} = ||u||_{L^\infty(B_{Rd}(z_0))} \le \exp\left[C_0 (b+d)^\beta R^\beta\right].$$

Since $\beta = \max\left\{1-\frac{1}{p}, 1-\frac{2}{q_1}, 1-\frac{2}{q_2}\right\}$, if we choose sufficiently large $\hat{C}$ depending on C_0, b, d, β, and A_0, A_1 or A_2, we have that either $C_0 (b+d)^\beta R^\beta \le \hat{C}\sqrt{A_0} R^{1-\frac{1}{p}}$, $C_0 (b+d)^\beta R^\beta \le \hat{C} A_1 R^{1-\frac{2}{q_1}}$, or $C_0 (b+d)^\beta R^\beta \le \hat{C} A_2 R^{1-\frac{2}{q_2}}$. Therefore,

$$||u_R||_{L^\infty(B_d)} \le \exp\left[\hat{C}\left(\sqrt{A_0} R^{1-\frac{1}{p}} + A_1 R^{1-\frac{2}{q_1}} + A_2 R^{1-\frac{2}{q_2}}\right)\right].$$

Note that for $\widetilde{z_0} := -\dfrac{z_0}{R}$, $|\widetilde{z_0}| = b$ and $|u_R(\widetilde{z_0})| = |u(0)| \ge 1$ by (9), so that $||u_R||_{L^\infty(B_b)} \ge 1$. Thus, if R is sufficiently large, then we may apply Theorem 2 to u_R with $M = A_0 R^{2-\frac{2}{p}}$, $K_i = A_i R^{1-\frac{2}{q_i}}$ for $i = 1,2$, and $C_0 = \hat{C}$ to get

$$\begin{aligned}
||u||_{L^\infty(B_1(z_0))} =& ||u_R||_{L^\infty\left(B_{1/R}(0)\right)} \\
\ge& (1/R)^{C\left(\sqrt{A_0} R^{1-\frac{1}{p}} + A_1 R^{1-\frac{2}{q_1}} + A_2 R^{1-\frac{2}{q_2}}\right)^{1+\varepsilon}} \\
=& \exp\left[-C\left(\sqrt{A_0} R^{1-\frac{1}{p}} + A_1 R^{1-\frac{2}{q_1}} + A_2 R^{1-\frac{2}{q_2}}\right)^{1+\varepsilon} \log R\right].
\end{aligned}$$

Since $\beta = \max\left\{1-\frac{1}{p}, 1-\frac{2}{q_1}, 1-\frac{2}{q_2}\right\}$, then

$$||u||_{L^\infty(B_1(x_0))} \ge \exp\left[-C R^{\beta(1+\varepsilon)} \log R\right].$$

As $|x_0| = bR$, the conclusion of the theorem follows.

Appendix A. Maximum principles

Here we prove the maximum principles that are used in the proof of Lemma 4. In particular, we generalize some of the standard theorems from Chapter 8 of [10] to extend to elliptic operators with singular lower order terms. To this end, we employ roughly the same techniques, but with applications of Hölder and Sobolev inequalities to accommodate for the unbounded potential functions.

We assume that $\Omega \subset \mathbb{R}^2$ is open, connected and bounded with a C^1 boundary. Assume that A satisfies (1) and (2), while V, W_1, and W_2 satisfy (3) – (4). Define

$$\mathscr{L}^* := -\operatorname{div}\left(A^T \nabla + W_2\right) + W_1 \cdot \nabla + V.$$

Theorem 4 (cf. Theorem 8.1 in [10]). *Let $u \in W^{1,2}(\Omega)$ weakly satisfy $\mathscr{L}^* u \le 0$ in Ω. Assume that* (12) *holds. Then*

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+.$$

Proof. Let $v \in W_0^{1,2}(\Omega)$ be a non-negative function for which $uv \ge 0$ in Ω. Since $\frac{2p}{p-1} > 2$, then the Sobolev inequality implies that $u, v \in L^{\frac{2p}{p-1}}(\Omega)$. The Hölder inequality gives that $uv \in L^{p'}(\Omega)$. As $\frac{2q_i}{q_i-1} > 2$ for each i, the Sobolev inequality implies that $u, v \in L^{\frac{2q_i}{q_i-2}}(\Omega)$ and we may similarly conclude that $uv \in L^{q_i'}(\Omega)$. Note that $D(uv) = Du\,v + u\,Dv$. Another application of the Hölder inequality in combination with the boundary information implies $uv \in W_0^{1,q_1'}(\Omega) \cap W_0^{1,q_2'}(\Omega) \cap L^{p'}(\Omega)$, so we may use it as a test function. Since $\mathscr{L}^* u \le 0$, it follows from the definition that

$$\int_\Omega A^T \nabla u \cdot \nabla v + W_2\, u \cdot \nabla v + W_1 \cdot \nabla u\, v + V\, u\, v \le 0.$$

Rearranging and using (12), we see that

$$\int_\Omega A^T \nabla u \cdot \nabla v + (W_1 - W_2) \cdot \nabla u\, v \le -\int_\Omega V\, u\, v + W_2 \cdot \nabla (uv) \le 0.$$

Therefore,

$$\int_\Omega A^T \nabla u \cdot \nabla v \le \int_\Omega (W_2 - W_1) \cdot \nabla u\, v \le \sum_{i=1}^2 ||W_i||_{L^{q_i}(\Omega)} \left(\int_\Omega |\nabla u\, v|^{q_i'}\right)^{\frac{1}{q_i'}}.$$

In the case where $||W_1||_{L^{q_1}(\Omega)} = 0$ and $||W_2||_{L^{q_2}(\Omega)} = 0$, set $l = \sup\limits_{\partial\Omega} u^+$ and define $v = \max\{u - l, 0\} = (u-l)^+$. The conclusion is then immediate. Otherwise, choose k so that $l \le k \le \sup\limits_\Omega u$ and set $v = (u-k)^+$. (If no such k exists, then we are finished.) We have that $v \in W_0^{1,2}(\Omega)$ and

$$Dv = \begin{cases} Du & u > k \\ 0 & u \le k. \end{cases}$$

It follows from the last line of inequalities that

$$\int_\Omega A^T \nabla v \cdot \nabla v \le \sum_{i=1}^2 ||W_i||_{L^{q_i}(\Omega)} \left(\int_\Gamma |Dv\, v|^{q_i'}\right)^{\frac{1}{q_i'}}.$$

where $\Gamma = \operatorname{supp} Dv \subset \operatorname{supp} v$. The ellipticity condition in combination with a Hölder inequality then gives that

$$||Dv||^2_{L^2(\Omega)} \le \lambda^{-1} ||Dv||_{L^2(\Omega)} \sum_{i=1}^2 ||W_i||_{L^{q_i}(\Omega)} ||v||_{L^{\frac{2q_i}{q_i-2}}(\Gamma)}$$

or

$$||Dv||_{L^2(\Omega)} \le \lambda^{-1} \sum_{i=1}^2 ||W_i||_{L^{q_i}(\Omega)} ||v||_{L^{\frac{2q_i}{q_i-2}}(\Gamma)}.$$

Now we apply the Sobolev inequality with some $2^* > \max\left\{\frac{2q_1}{q_1-2}, \frac{2q_2}{q_2-2}\right\} > 2$ and the Hölder inequality to see that

$$C\,||v||_{L^{2^*}(\Omega)} \le ||Dv||_{L^2(\Omega)} \le \lambda^{-1} ||v||_{L^{2^*}(\Omega)} \sum_{i=1}^2 ||W_i||_{L^{q_i}(\Omega)} |\operatorname{supp} Dv|^{\frac{q_i-2}{2q_i} - \frac{1}{2^*}}.$$

In particular, with $Q > 0$ chosen so that $\max\left\{|\operatorname{supp} Dv|^{\frac{q_i-2}{2q_i} - \frac{1}{2^*}}\right\}_{i=1}^2 = |\operatorname{supp} Dv|^Q$,

$$|\operatorname{supp} Dv| \ge \left(\frac{C\lambda}{||W_1||_{L^{q_1}(\Omega)} + ||W_2||_{L^{q_2}(\Omega)}}\right)^{\frac{1}{Q}}.$$

Since this inequality is independent of k, it also holds as k tends to $\sup_{\Omega} u$. This means that the function u must attain its supremum in Ω on a set of positive measure, where at the same time $Du = 0$. This contradiction implies that $\sup_{\Omega} u \leq l$, as required. $\square$

Corollary 4. *Let $u \in W^{1,2}(\Omega)$ weakly satisfy $-\operatorname{div}\left(A^T \nabla u\right) + W_1 \cdot \nabla u = 0$ in Ω. Then for a.e. $z \in \Omega$,*

$$\inf_{\partial\Omega} u \leq u(z) \leq \sup_{\partial\Omega} u.$$

Note that here W_1 does not need to satisfy the sign condition.

To prove this corollary, we follow the same approach from above and set $v = \max\left\{u - \sup_{\partial\Omega} u, 0\right\}$ and $v = \max\left\{\inf_{\partial\Omega} u - u, 0\right\}$. In this case, we don't require a sign condition on the test functions.

Appendix B. Green's functions

The purpose of this appendix is to establish a representation formula for solutions to non-homogeneous uniformly elliptic equations with vanishing Dirichlet boundary data. To this end, we mimic the main technique presented in [5], which is based on the ideas in [13] and [11]. Since we only require such results for reasonably nice, bounded domains (balls), we assume throughout that $\Omega \subset \mathbb{R}^2$ is open, bounded, and connected. Finally, we point out that the bounds given for the Green's functions are not sharp. As is well known, pointwise logarithmic bounds for Green's functions in the plane are the best possible. However, since we work using the methods of [11], [13], and [5], and seek integrability properties for the Green's function instead of sharp pointwise bounds, our estimates will have power bounds.

We consider second-order, uniformly elliptic, bounded operators of divergence form with one first order term. We use coercivity, the Caccioppoli inequality, and De Giorgi-Nash-Moser theory to establish existence, uniqueness, and a priori estimates for the Dirichlet Green's functions.

Notation and properties of solutions. Let $\Omega \subset \mathbb{R}^2$ be open, bounded, and connected. In contrast to the main body of the article where we use the notation $z = (x, y)$ to denote a point in $\mathbb{R}^2$, here we let x, y, etc. denote points in $\mathbb{R}^2$.

For any $x \in \Omega$, $r > 0$, we define $\Omega_r(x) := \Omega \cap B_r(x)$ and $\Sigma_r(x) := \partial\Omega \cap B_r(x)$. Let $C_c^\infty(\Omega)$ denote the set of all infinitely differentiable functions with compact support in Ω.

For future reference, we mention that for $\Omega, U \subset \mathbb{R}^2$ open and connected, the assumption

$$u \in W^{1,2}(\Omega), \quad u = 0 \text{ on } U \cap \partial\Omega,$$

is always meant in the weak sense of

$$u \in W^{1,2}(\Omega) \text{ and } u\xi \in W_0^{1,2}(\Omega) \text{ for any } \xi \in C_c^\infty(U). \tag{B.1}$$

This definition of (weakly) vanishing on the boundary is independent of the choice of U. Indeed, suppose V is another open and connected subset of $\mathbb{R}^2$ such that $V \cap \partial\Omega = U \cap \partial\Omega$ and let $\xi \in C_c^\infty(V)$. Choose $\psi \in C_c^\infty(U \cap V)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on the support of ξ in some neighborhood of the boundary. Then $\xi(1-\psi)|_\Omega \in C_c^\infty(\Omega)$, so that $u\xi(1-\psi) \in W_0^{1,2}(\Omega)$. Additionally, $\xi\psi \in C_c^\infty(U)$, so by (B.1), $u\xi\psi \in W_0^{1,2}(\Omega)$. Therefore, $u\xi = u\xi\psi + u\xi(1-\psi) \in W_0^{1,2}(\Omega)$, as desired.

Let $A = \left(a_{ij}\right)_{i,j=1}^{2}$ be bounded, measurable coefficients defined on Ω. We assume that A satisfies an ellipticity condition described by (1) and the boundedness assumption given in (2). Choose W so that (4) and (6) hold with $q_i = q$. The non-homogeneous second-order operator is

$$\widetilde{L} = -\operatorname{div}(A\nabla) + W \cdot \nabla \tag{B.2}$$

and the adjoint operator to $\widetilde{L}$ is given by

$$\widetilde{L}^* = -\operatorname{div}\left(A^T\nabla + W\right). \tag{B.3}$$

All operators are understood in the sense of distributions on Ω. Specifically, for every $u \in W^{1,2}(\Omega)$ and $v \in C_c^\infty(\Omega)$, we use the naturally associated bilinear form and write the action of the functional $\widetilde{L}u$ on v as

$$(\widetilde{L}u, v) = B[u,v] = \int_\Omega A\nabla u \cdot \nabla v + W \cdot \nabla u\, v. \tag{B.4}$$

It is not hard to check that for such u, v and for the coefficients as described above, the bilinear form above is well-defined and finite. Similarly, $B^*[\cdot,\cdot]$ denotes the bilinear operator associated to $\widetilde{L}^*$, given by

$$(\widetilde{L}^*u, v) = B^*[u,v] = \int A^T\nabla u \cdot \nabla v + W\, u \cdot \nabla v. \tag{B.5}$$

Clearly,

$$B[v,u] = B^*[u,v]. \tag{B.6}$$

For any distribution $\mathbf{F} = (f, G)$ on Ω and u as above, we always understand $\widetilde{L}u = \mathbf{F}$ on Ω in the weak sense, that is, as $B[u,v] = \mathbf{F}(v)$ for all $v \in C_c^\infty(\Omega)$. Typically f and G will be elements of some $L^p(\Omega)$ spaces, so the action of $\mathbf{F}$ on v is then simply $\displaystyle\int fv + G \cdot Dv$. The identity $\widetilde{L}^*u = \mathbf{F}$ is interpreted similarly.

Remark. Assumptions (2) and (4) in combination with Hölder and Sobolev inequalities imply that there exists $\Lambda > 0$ so that for every $u, v \in W_0^{1,2}(\Omega)$,

$$|B[u,v]| \le \Lambda ||u||_{W^{1,2}(\Omega)} ||v||_{W^{1,2}(\Omega)}. \tag{B.7}$$

Since (6) holds, then (1) and the Poincaré inequality implies that there exists $\gamma > 0$ so that for every $u \in W_0^{1,2}(\Omega)$

$$B[u,u] \ge \gamma ||u||^2_{W^{1,2}(\Omega)}. \tag{B.8}$$

Now we describe the important properties of solutions to either $\widetilde{L}u = 0$ or $\widetilde{L}u = \mathbf{F}$ that will be employed in the constructions below. We will use the following version of Moser (boundary) boundedness.

Lemma 15. [5, Lemma 5.1] *Let $\Omega \subset \mathbb{R}^2$ be open and connected. Let $u \in W^{1,2}(\Omega_{2R})$ satisfy $u = 0$ along Σ_{2R}. Let $f \in L^\ell(\Omega_R)$ for some $\ell \in (1, \infty]$, $G \in L^m(\Omega_R)$ for some $m \in (2, \infty]$ and assume that $\widetilde{L}u \le -\operatorname{div} G + f$ in Ω_R weakly in the sense that for any $\varphi \in W_0^{1,2}(\Omega_R)$ such that $\varphi \ge 0$ in Ω_R, we have*

$$B[u,\varphi] \le \int G \cdot \nabla\varphi + f\varphi.$$

Then $u^+ \in L^\infty_{loc}(\Omega_R)$ *and for any* $r < R$, $s > 0$,

$$\sup_{\Omega_r} u^+ \le \frac{C}{(R-r)^{\frac{2}{s}}} \left|\left|u^+\right|\right|_{L^s(\Omega_R)} + c_s \left[R^{2-\frac{2}{\ell}} \left|\left|f\right|\right|_{L^\ell(\Omega_R)} + R^{1-\frac{2}{m}} \left|\left|G\right|\right|_{L^m(\Omega_R)}\right], \tag{B.9}$$

where $C = C\left(q, s, \ell, m, \gamma, \Lambda, ||W||_{L^q(\Omega_R)}\right)$ *and* c_s *depends only on s. Note that all of the constants are independent of R.*

Remark. Because of assumption (6), the conclusion of Lemma 15 also holds for the operator $\widetilde{L}^*$.

The following Caccioppoli inequality will be used in our constructions.

Lemma 16. [5, Lemma 4.1] *If* $u \in W^{1,2}_0(\Omega)$ *is a weak solution to* $\widetilde{L}u = 0$ *in* Ω *and* $\zeta \in C^\infty(\mathbb{R}^2)$, *then*

$$\int |Du|^2 \zeta^2 \le C \int |u|^2 |D\zeta|^2, \tag{B.10}$$

where C is a constant that depends on γ, Λ, q, $||W||_{L^q(\Omega)}$, *but C is independent of the sets on which* ζ *and* $D\zeta$ *are supported.*

We also rely on a lemma regarding the Hölder continuity of solutions.

Lemma 17. [5, Lemma 6.6] *Let* $u \in W^{1,2}(B_{2R_0})$ *be a solution in the sense that* $B[u,\varphi] = 0$ *for any* $\varphi \in W^{1,2}_0(B_{R_0})$. *Then there exists* $\eta \in (0,1)$, *such that for any* $R \le R_0$, *if* $x, y \in B_{R/2}$

$$|u(x) - u(y)| \le C_{R_0} \left(\frac{|x-y|}{R}\right)^\eta \left(\fint_{B_R} |u|^{2^*}\right)^{\frac{1}{2^*}}. \tag{B.11}$$

Green's functions. This subsection resembles the work done in [13] and [5]. We use the properties of our operator as well as the properties of solutions to $\widetilde{L}u = \mathbf{F}$ or $\widetilde{L}^*u = \mathbf{F}$ described above to establish existence, uniqueness, and a collection of a priori estimates for the Dirichlet Green's function associated to $\Omega \subset \mathbb{R}^2$. We follow closely the arguments in [13] and [5], adapting to $n = 2$. As previously mentioned, our estimates are not sharp since we do not obtain logarithmic bounds for the Green's functions.

First, we clarify the meaning of the Green's function.

Definition 4. *Let* Ω *be an open, connected, bounded subset of* $\mathbb{R}^2$. *We say that the function* $\Gamma(x,y)$ *defined on the set* $\{(x,y) \in \Omega \times \Omega : x \ne y\}$ *is the* ***Green's function*** *of* $\widetilde{L}$ *if it satisfies the following properties:*

1) $\Gamma(\cdot,y)$ *is locally integrable and* $\widetilde{L}\Gamma(\cdot,y) = \delta_y I$ *for all* $y \in \Omega$ *in the sense that for every* $\phi \in C^\infty_c(\Omega)$,

$$B[\Gamma(\cdot,y), \phi] = \phi(y). \tag{B.12}$$

2) *For all* $y \in \Omega$ *and* $r > 0$, $\Gamma(\cdot,y) \in W^{1,2}(\Omega \setminus \Omega_r(y))$. *In addition,* $\Gamma(\cdot,y)$ *vanishes on* $\partial\Omega$ *in the sense that for every* $\zeta \in C^\infty_c(\Omega)$ *satisfying* $\zeta \equiv 1$ *on* $B_r(y)$ *for some* $r > 0$, *we have*

$$(1-\zeta)\Gamma(\cdot,y) \in W^{1,2}_0(\Omega \setminus \Omega_r(y)). \tag{B.13}$$

3) *For some* $\ell_0 \in (1,\infty]$ *and* $m_0 \in (2,\infty]$, *and any* $f \in L^{\ell_0}(\Omega)$, $G \in L^{m_0}(\Omega)$, *the function u given by*

$$u(y) = \int_\Omega \left[\Gamma(x,y) f(x) + D_x\Gamma(x,y) \cdot G(x)\right] dx \tag{B.14}$$

belongs to $W_0^{1,2}(\Omega)$ *and satisfies* $\widetilde{L}u = f - \operatorname{div} G$ *in the sense that for every* $\phi \in C_c^\infty(\Omega)$*,*

$$B[u,\phi] = \int_\Omega f\phi + G \cdot D\phi. \tag{B.15}$$

We say that the function $\Gamma(x,y)$ *is the* ***continuous Green's function*** *if it satisfies the conditions above and is also continuous.*

We show here that there is at most one Green's function. In general, we mean uniqueness in the sense of Lebesgue, i.e. almost everywhere uniqueness. However, when we refer to the continuous Green's function, we mean true pointwise equivalence.

Assume that Γ and $\widetilde{\Gamma}$ are Green's functions satisfying Definition 4. Then, for all $f \in L^\infty(\Omega)$, the functions u and $\widetilde{u}$ given by

$$u(y) = \int_\Omega \Gamma(x,y) f(x)\,dx, \quad \widetilde{u}(y) = \int_\Omega \widetilde{\Gamma}(x,y) f(x)\,dx$$

satisfy

$$\widetilde{L}^*(u - \widetilde{u}) = 0 \quad \text{in } \Omega$$

and $u - \widetilde{u} \in W_0^{1,2}(\Omega)$. By uniqueness of solutions ensured by the Lax-Milgram lemma, $u - \widetilde{u} \equiv 0$. Thus, for a.e. $x \in \Omega$,

$$\int_\Omega \left[\Gamma(x,y) - \widetilde{\Gamma}(x,y)\right] f(x)\,dx = 0, \quad \forall f \in L^\infty(\Omega).$$

Therefore, $\Gamma = \widetilde{\Gamma}$ a.e. in $\{x \neq y\}$. If we further assume that Γ and $\widetilde{\Gamma}$ are continuous Green's functions, then we conclude that $\Gamma \equiv \widetilde{\Gamma}$ in $\{x \neq y\}$.

Theorem 5. *Let* Ω *be an open, connected, bounded subset of* $\mathbb{R}^2$*. Then there exists a unique continuous Green's function* $\Gamma(x,y)$*, defined in* $\{x,y \in \Omega, x \neq y\}$*, that satisfies Definition 4. We have* $\Gamma(x,y) = \Gamma^*(y,x)$*, where* Γ^* *is the unique continuous Green's function associated to* $\widetilde{L}^*$*. Furthermore,* $\Gamma(x,y)$ *satisfies the following estimates:*

$$||\Gamma(\cdot,y)||_{W^{1,2}(\Omega\setminus\Omega_r(y))} + ||\Gamma(x,\cdot)||_{W^{1,2}(\Omega\setminus\Omega_r(x))} \le Cr^{-\varepsilon}, \qquad \forall r > 0, \tag{B.16}$$

$$||\Gamma(\cdot,y)||_{L^s(\Omega_r(y))} + ||\Gamma(x,\cdot)||_{L^s(\Omega_r(x))} \le C_s r^{-\varepsilon+\frac{2}{s}}, \qquad \forall r > 0, \quad \forall s \in [1,\infty), \tag{B.17}$$

$$||D\Gamma(\cdot,y)||_{L^s(\Omega_r(y))} + ||D\Gamma(x,\cdot)||_{L^s(\Omega_r(x))} \le C_s r^{-1-\varepsilon+\frac{2}{s}}, \qquad \forall r > 0, \quad \forall s \in [1,2), \tag{B.18}$$

$$|\{x \in \Omega : |\Gamma(x,y)| > \tau\}| + |\{y \in \Omega : |\Gamma(x,y)| > \tau\}| \le C\tau^{-\frac{2}{\varepsilon}}, \qquad \forall \tau > 0, \tag{B.19}$$

$$|\{x \in \Omega : |D_x\Gamma(x,y)| > \tau\}| + \left|\left\{y \in \Omega : \left|D_y\Gamma(x,y)\right| > \tau\right\}\right| \le C\tau^{-\frac{2}{1+\varepsilon}}, \qquad \forall \tau > 0, \tag{B.20}$$

$$|\Gamma(x,y)| \le C|x-y|^{-\varepsilon} \qquad \forall x \neq y, \tag{B.21}$$

where in each case, $\varepsilon > 0$ *is some arbitrarily small number that may vary from line to line. Moreover, each constant depends on* γ*,* Λ*,* ε*, and the constants from* (B.9) *and* (B.10)*, and each* C_s *depends additionally on s. Moreover, for any* $0 < R \le R_0$*,*

$$|\Gamma(x,y) - \Gamma(z,y)| \le C_{R_0} C \left(\frac{|x-z|}{R}\right)^\eta R^{-\varepsilon}, \tag{B.22}$$

whenever $|x-z|<\frac{R}{2}$ *and*

$$|\Gamma(x,y)-\Gamma(x,z)|\leq C_{R_0}C\left(\frac{|y-z|}{R}\right)^{\eta}R^{-\varepsilon}, \tag{B.23}$$

whenever $|y-z|<\frac{R}{2}$, *where* C_{R_0} *and* $\eta=\eta(R_0)$ *are the same as in* (B.11).

Proof of Theorem 5. Let $u\in W_0^{1,2}(\Omega)$. Fix $y\in\Omega$, $\rho>0$, and consider the linear functional

$$u\mapsto \fint_{B_\rho(y)}u.$$

By the Hölder inequality and Sobolev embedding with $2^*\in(2,\infty)$,

$$\begin{aligned}\left|\fint_{B_\rho(y)}u\right| &\leq \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}|u|\leq |B_\rho(y)|^{-\frac{1}{2^*}}\left(\int_\Omega |u|^{2^*}\right)^{\frac{1}{2^*}}\leq c\,|B_\rho(y)|^{-\frac{1}{2^*}}\left(\int_\Omega |Du|^2\right)^{\frac{1}{2}}\\ &\leq C\rho^{-\frac{2}{2^*}}\,||u||_{W^{1,2}(\Omega)}.\end{aligned}\tag{B.24}$$

Therefore, the functional is bounded on $W_0^{1,2}(\Omega)$, and by the Lax-Milgram theorem there exists a unique $\Gamma^\rho=\Gamma^\rho_y=\Gamma^\rho(\cdot,y)\in W_0^{1,2}(\Omega)$ satisfying

$$B[\Gamma^\rho,u]=\fint_{B_\rho(y)}u=\frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}u,\quad \forall u\in W_0^{1,2}(\Omega). \tag{B.25}$$

By the coercivity of A given by (B.8) along with (B.24), we obtain,

$$\gamma||\Gamma^\rho||^2_{W^{1,2}(\Omega)}\leq B[\Gamma^\rho,\Gamma^\rho]=\left|\fint_{B_\rho(y)}\Gamma^\rho\right|\leq C\rho^{-\frac{2}{2^*}}\,||\Gamma^\rho||_{W^{1,2}(\Omega)}$$

so that for any $\varepsilon\in(0,1)$,

$$||D\Gamma^\rho||_{L^2(\Omega)}\leq C\rho^{-\varepsilon}. \tag{B.26}$$

For some $\ell_0\in(1,\infty]$ and $m_0\in(2,\infty]$, let $f\in L^{\ell_0}(\Omega)$ and $G\in L^{m_0}(\Omega)$. For any $\ell\in(1,\ell_0]$ and any $m\in(1,m_0]$, it is clear that that $f\in L^\ell(\Omega)$ and $G\in L^m(\Omega)$ since Ω is bounded. Consider the linear functionals

$$\begin{aligned}W_0^{1,2}(\Omega)\ni w&\mapsto\int_\Omega fw\\ W_0^{1,2}(\Omega)\ni w&\mapsto\int_\Omega G\cdot Dw.\end{aligned}$$

The first functional is bounded on $W_0^{1,2}(\Omega)$ since for every $w\in W_0^{1,2}(\Omega)$ and every $\ell\in(1,\ell_0]$,

$$\left|\int_\Omega fw\right|\leq ||f||_{L^\ell(\Omega)}\,||w||_{L^{2^*}(\Omega)}\,|\operatorname{supp}f|^{1-\frac{1}{2^*}-\frac{1}{\ell}}\leq c\,||f||_{L^\ell(\Omega)}\,|\operatorname{supp}f|^{1-\frac{1}{2^*}-\frac{1}{\ell}}\,||Dw||_{L^2(\Omega)}, \tag{B.27}$$

where we have again used Sobolev embedding with some $2^*\in(2,\infty)$. Similarly, we see that the second functional is also bounded on $W_0^{1,2}(\Omega)$ since for every $m\in(2,m_0]$

$$\left|\int_\Omega G\cdot Dw\right|\leq ||G||_{L^m(\Omega)}\,|\operatorname{supp}G|^{\frac{1}{2}-\frac{1}{m}}\,||Dw||_{L^2(\Omega)}. \tag{B.28}$$

Once again, by Lax-Milgram, we obtain $u_1, u_2 \in W_0^{1,2}(\Omega)$ such that

$$B^*[u_1, w] = \int_\Omega f w, \quad \forall w \in W_0^{1,2}(\Omega) \tag{B.29}$$

and

$$B^*[u_2, w] = \int_\Omega G \cdot Dw, \quad \forall w \in W_0^{1,2}(\Omega). \tag{B.30}$$

Set $w = u_1$ in (B.29) and use the coercivity assumption, (B.8), for B^* along with (B.27) to get

$$||Du_1||_{L^2(\Omega)} \le C ||f||_{L^\ell(\Omega)} |\operatorname{supp} f|^{1-\frac{1}{2^*}-\frac{1}{\ell}}. \tag{B.31}$$

With $w = u_2$ in (B.30), we similarly obtain from (B.28) that

$$||Du_2||_{L^2(\Omega)} \le C ||G||_{L^m(\Omega)} |\operatorname{supp} G|^{\frac{1}{2}-\frac{1}{m}}. \tag{B.32}$$

Also, if we take $w = \Gamma^\rho$ in (B.29) and (B.30), we get

$$\int_\Omega f \Gamma^\rho = B^*[u_1, \Gamma^\rho] = B[\Gamma^\rho, u_1] = \fint_{B_\rho(y)} u_1, \tag{B.33}$$

and

$$\int_\Omega G \cdot D\Gamma^\rho = B^*[u_2, \Gamma^\rho] = B[\Gamma^\rho, u_2] = \fint_{B_\rho(y)} u_2. \tag{B.34}$$

In particular, with $u := u_1 + u_2$, we see that

$$\int_\Omega f \Gamma^\rho + G \cdot D\Gamma^\rho = \fint_{B_\rho(y)} u. \tag{B.35}$$

Now assume that f and G are supported in $\Omega_r(y)$, for some $r > 0$. Let u_1, u_2 be as in (B.29), (B.30), respectively. Since $u_1, u_2 \in W_0^{1,2}(\Omega)$, then $u_1, u_2 \in W^{1,2}(\Omega_{2r})$ and $u_1, u_2 = 0$ on Σ_{2r} so that Lemma 15 is applicable. Then, by (B.9) with some $s = 2^* \in (2, \infty)$

$$||u_1||^2_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C \left(r^{-\frac{4}{2^*}} ||u_1||^2_{L^{2^*}(\Omega_r(y))} + r^{4-\frac{4}{\ell}} ||f||^2_{L^\ell(\Omega_r(y))} \right)$$

and

$$||u_2||^2_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C \left(r^{-\frac{4}{2^*}} ||u_2||^2_{L^{2^*}(\Omega_r(y))} + r^{2-\frac{4}{m}} ||G||^2_{L^m(\Omega_r(y))} \right).$$

By Sobolev embedding and (B.31) with $\operatorname{supp} f \subset \Omega_r(y)$,

$$||u_1||^2_{L^{2^*}(\Omega_r(y))} \le ||u_1||^2_{L^{2^*}(\Omega)} \le C ||Du_1||^2_{L^2(\Omega)} \le C ||f||^2_{L^\ell(\Omega)} |\Omega_r(y)|^{2-\frac{2}{2^*}-\frac{2}{\ell}} \le C r^{4-\frac{4}{2^*}-\frac{4}{\ell}} ||f||^2_{L^\ell(\Omega)}.$$

Combining the previous two inequalities for u_1, we see that

$$||u_1||^2_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C \left(1 + r^{-\frac{8}{2^*}}\right) r^{4-\frac{4}{\ell}} ||f||^2_{L^\ell(\Omega_r(y))}.$$

For any $\ell \in (1, \ell_0]$, choose $2^* \in (2, \infty)$ so that $\frac{2}{2^*} < 1 - \frac{1}{\ell}$. Since Ω is bounded, then so too is r, and we have

$$||u_1||_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C r^{2-\frac{2}{\ell}-\frac{4}{2^*}} ||f||_{L^\ell(\Omega)} = C r^{2-\frac{2}{\ell}-\frac{4}{2^*}} ||f||_{L^\ell(\Omega_r(y))}. \tag{B.36}$$

Mimicking the argument with u_2, G and (B.32), we see that

$$||u_2||^2_{L^\infty\left(\Omega_{r/2}(y)\right)} \le C \left(1 + r^{-\frac{4}{2^*}}\right) r^{2-\frac{4}{m}} ||G||^2_{L^m(\Omega_r(y))}.$$

Now for any $m \in (2, m_0]$, choose $2^* \in (2,\infty)$ so that $\frac{2}{2^*} < 1 - \frac{2}{m}$ and we conclude that

$$||u_2||_{L^\infty\left(\Omega_{r/2}(y)\right)} \le Cr^{1-\frac{2}{m}-\frac{2}{2^*}} ||G||_{L^m(\Omega)} = Cr^{1-\frac{2}{m}-\frac{2}{2^*}} ||G||_{L^m(\Omega_r(y))}. \tag{B.37}$$

By (B.33) and (B.36), if $\rho \le r/2$, we have that for every $\ell \in (1, \ell_0]$,

$$\left| \int_{\Omega_r(y)} f\Gamma^\rho \right| = \left| \int_\Omega f\Gamma^\rho \right| \le \fint_{B_\rho(y)} |u_1| \le ||u_1||_{L^\infty(B_\rho(y))} \le ||u_1||_{L^\infty(\Omega_{r/2}(y))} \le Cr^{2-\frac{2}{\ell}-\frac{4}{2^*}} ||f||_{L^\ell(\Omega_r(y))}.$$

By duality, since we can take $\ell_0 = \infty$, this implies that for $r > 0$,

$$||\Gamma^\rho||_{L^s(\Omega_r(y))} \le Cr^{\frac{2}{s}-\varepsilon}, \quad \text{for all } \rho \le \frac{r}{2}, \quad \forall s \in [1,\infty). \tag{B.38}$$

We similarly conclude that

$$||D\Gamma^\rho||_{L^s(\Omega_r(y))} \le Cr^{\frac{2}{s}-1-\varepsilon}, \quad \text{for all } \rho \le \frac{r}{2}, \quad \forall s \in [1,2). \tag{B.39}$$

Note that in both cases, $\varepsilon \in (0,1)$ is chosen so that the power on r is positive.

Fix $x \ne y$ and set $r := \frac{4}{3}|x-y|$. For $\rho \le r/2$, Γ^ρ is a weak solution to $\widetilde{L}\Gamma^\rho = 0$ in $\Omega_{r/4}(x)$. Moreover, since $\Gamma^\rho \in W_0^{1,2}(\Omega)$, then $\Gamma^\rho \in W^{1,2}\left(\Omega_{r/2}(x)\right)$ and $\Gamma^\rho = 0$ on $\Sigma_{r/2}(x)$, so we may use Lemma 15. Thus, applying (B.9) and (B.38) with $s = 1$, we get for a.e. $x \in \Omega$ as above,

$$|\Gamma^\rho(x)| \le Cr^{-2} ||\Gamma^\rho||_{L^1\left(\Omega_{r/4}(x)\right)} \le Cr^{-2} ||\Gamma^\rho||_{L^1(\Omega_r(y))} \le Cr^{-\varepsilon} \approx |x-y|^{-\varepsilon}. \tag{B.40}$$

Now, for any $r > 0$ and $\rho \le r/2$, let ζ be a cut-off function such that

$$\zeta \in C^\infty(\mathbb{R}^n), \quad 0 \le \zeta \le 1, \quad \zeta \equiv 1 \text{ outside } B_r(y), \quad \zeta \equiv 0 \text{ in } B_{r/2}(y), \quad \text{and } |D\zeta| \le C/r. \tag{B.41}$$

Then the Caccioppoli inequality of Lemma 16 implies that

$$\int_\Omega \zeta^2 |D\Gamma^\rho|^2 \le C\int_\Omega |D\zeta|^2 |\Gamma^\rho|^2 \le Cr^{-2} \int_{\Omega_r(y)\setminus\Omega_{r/2}(y)} |\Gamma^\rho|^2, \quad \forall \rho \le \frac{r}{2}. \tag{B.42}$$

Combining (B.42) and (B.40), we have for all $r > 0$ and ζ as above,

$$\begin{aligned} \int_\Omega |D(\zeta\Gamma^\rho)|^2 &\le 2\int_\Omega \zeta^2 |D\Gamma^\rho|^2 + 2\int_\Omega |D\zeta|^2 |\Gamma^\rho|^2 \\ &\le Cr^{-2} \int_{\Omega_r(y)\setminus\Omega_{r/2}(y)} |\Gamma^\rho|^2 \le Cr^{-2\varepsilon}, \quad \forall \rho \le \frac{r}{2}. \end{aligned} \tag{B.43}$$

It follows from Sobolev embedding with arbitrary $2^* \in (2,\infty)$ and (B.43) that for $r > 0$,

$$\int_{\Omega\setminus\Omega_r(y)} |\Gamma^\rho|^{2^*} \le \int_\Omega |\zeta\Gamma^\rho|^{2^*} \le c\left(\int_\Omega |D(\zeta\Gamma^\rho)|^2\right)^{\frac{2^*}{2}} \le Cr^{-2^*\varepsilon}, \quad \forall \rho \le \frac{r}{2}.$$

On the other hand, if $\rho > \frac{r}{2}$, then (B.26) implies that

$$\int_{\Omega\setminus\Omega_r(y)} |\Gamma^\rho|^{2^*} \le \int_\Omega |\Gamma^\rho|^{2^*} \le c\left(\int_\Omega |D\Gamma^\rho|^2\right)^{\frac{2^*}{2}} \le Cr^{-2^*\varepsilon}.$$

Therefore, combining the previous two results, we have that for any $\varepsilon \in (0,1)$

$$\int_{\Omega\setminus\Omega_r(y)} |\Gamma^\rho|^{2^*} \le Cr^{-2^*\varepsilon}, \quad \forall r, \rho > 0. \tag{B.44}$$

For any $\varepsilon \in (0,1)$, $2^* \in (2,\infty)$, fix $\tau > 0$. Let $A_\tau = \{x \in \Omega : |\Gamma^\rho| > \tau\}$ and set $r = \tau^{-\frac{2^*}{2+2^*\varepsilon}}$. Then, using (B.44), we see that if $\rho > 0$,

$$|A_\tau \setminus \Omega_r(y)| \leq \tau^{-2^*} \int_{A_\tau \setminus \Omega_r(y)} |\Gamma^\rho|^{2^*} \leq C\tau^{-2^*} r^{-2^*\varepsilon} = C\tau^{-\frac{2^* 2}{2+2^*\varepsilon}}.$$

Since $|A_\tau \cap \Omega_r(y)| \leq |\Omega_r(y)| \leq Cr^2 = C\tau^{-\frac{2^* 2}{2+2^*\varepsilon}}$, we have

$$|\{x \in \Omega : |\Gamma^\rho(x)| > \tau\}| \leq C\tau^{-\frac{2^* 2}{2+2^*\varepsilon}} \quad \forall \rho > 0. \tag{B.45}$$

Fix $r > 0$ and let ζ be as in (B.41). Then (B.43) gives

$$\int_{\Omega \setminus \Omega_r(y)} |D\Gamma^\rho|^2 \leq Cr^{-2\varepsilon}, \quad \forall r > 0, \quad \forall \rho \leq \frac{r}{2}.$$

Now, if $\rho > \frac{r}{2}$, we have from (B.26) that

$$\int_{\Omega \setminus \Omega_r(y)} |D\Gamma^\rho|^2 \leq \int_\Omega |D\Gamma^\rho|^2 \leq C\rho^{-2\varepsilon} \leq Cr^{-2\varepsilon}.$$

Combining the previous two results yields

$$\int_{\Omega \setminus \Omega_r(y)} |D\Gamma^\rho|^2 \leq Cr^{-2\varepsilon}, \quad \forall r, \rho > 0. \tag{B.46}$$

Fix $\tau > 0$. Let $A_\tau = \{x \in \Omega : |D\Gamma^\rho| > \tau\}$ and set $r = \tau^{-\frac{1}{1+\varepsilon}}$. Then, using (B.46), we see that if $\rho > 0$,

$$|A_\tau \setminus \Omega_r(y)| \leq \tau^{-2} \int_{A_\tau \setminus \Omega_r(y)} |D\Gamma^\rho|^2 \leq C\tau^{-2} r^{-2\varepsilon} = C\tau^{-\frac{2}{1+\varepsilon}}.$$

Since $|A_\tau \cap \Omega_r(y)| \leq Cr^2 = C\tau^{-\frac{2}{1+\varepsilon}}$, then

$$|\{x \in \Omega : |D\Gamma^\rho(x)| > \tau\}| \leq C\tau^{-\frac{2}{1+\varepsilon}} \quad \forall \rho > 0. \tag{B.47}$$

For any $\sigma > 0$ and $s > 0$, we have

$$\int_{\Omega_r(y)} |D\Gamma^\rho|^s \leq \sigma^s |\Omega_r(y)| + \int_{\{|D\Gamma^\rho| > \sigma\}} |D\Gamma^\rho|^s.$$

By (B.47), for $s \in \left(0, \frac{2}{1+\varepsilon}\right)$ and $\rho > 0$,

$$\begin{aligned}\int_{\{|D\Gamma^\rho| > \sigma\}} |D\Gamma^\rho|^s &= \int_0^\infty s\tau^{s-1} |\{|D\Gamma^\rho| > \max\{\tau, \sigma\}\}| \, d\tau \\ &\leq C\sigma^{-\frac{2}{1+\varepsilon}} \int_0^\sigma s\tau^{s-1} \, d\tau + C\int_\sigma^\infty s\tau^{s-1-\frac{2}{1+\varepsilon}} \, d\tau = C\left(1 - \frac{s}{s - \frac{2}{1+\varepsilon}}\right) \sigma^{s-\frac{2}{1+\varepsilon}}.\end{aligned}$$

Therefore, taking $\sigma = r^{-(1+\varepsilon)}$, we conclude that

$$\int_{\Omega_r(y)} |D\Gamma^\rho|^s \leq C_s r^{-s(1+\varepsilon)+2}, \quad \forall r, \rho > 0, \quad \forall s \in \left(0, \tfrac{2}{1+\varepsilon}\right). \tag{B.48}$$

Now we repeat the process for Γ^ρ, using (B.45) in place of (B.47). For any $\sigma > 0$ and $s > 0$, we have

$$\int_{\Omega_r(y)} |\Gamma^\rho|^s \le \sigma^s |\Omega_r(y)| + \int_{\{|\Gamma^\rho| > \sigma\}} |\Gamma^\rho|^s.$$

By (B.45), for $s \in \left(0, \frac{2^* 2}{2+2^*\varepsilon}\right)$ and $\rho > 0$,

$$\begin{aligned}
\int_{\{|\Gamma^\rho|>\sigma\}} |\Gamma^\rho|^s &= \int_0^\infty s\tau^{s-1} \left|\{|\Gamma^\rho| > \max\{\tau,\sigma\}\}\right| d\tau \\
&\le C\sigma^{-\frac{2^*2}{2+2^*\varepsilon}} \int_0^\sigma s\tau^{s-1}\, d\tau + C\int_\sigma^\infty s\tau^{s-1-\frac{2^*2}{2+2^*\varepsilon}}\, d\tau \\
&= C\left(1 - \frac{s}{s - \frac{2^*2}{2+2^*\varepsilon}}\right)\sigma^{s-\frac{2^*2}{2+2^*\varepsilon}}.
\end{aligned}$$

Taking $\sigma = r^{-\frac{2+2^*\varepsilon}{2^*}}$, we conclude that

$$\int_{\Omega_r(y)} |\Gamma^\rho|^s \le C_s r^{-s\frac{2+2^*\varepsilon}{2^*}+2}, \quad \forall r, \rho > 0, \quad \forall s \in \left(0, \frac{2^*2}{2+2^*\varepsilon}\right). \tag{B.49}$$

Fix $s \in [1,2)$ and $\tilde{s} \in [1,\infty)$. There exists $\varepsilon \in (0,1)$ and $2^* \in (2,\infty)$ so that $s < \frac{2}{1+\varepsilon}$ and $\tilde{s} < \frac{2^*2}{2+2^*\varepsilon}$. It follows from (B.48) and (B.49) that for any $r > 0$

$$\|\Gamma^\rho\|_{W^{1,s}(\Omega_r(y))} \le C(r) \text{ and } \|\Gamma^\rho\|_{L^{\tilde{s}}(\Omega_r(y))} \le C(r) \quad \text{uniformly in } \rho. \tag{B.50}$$

Therefore, (using diagonalization) we can show that there exists a sequence $\{\rho_\mu\}_{\mu=1}^\infty$ tending to 0 and a function $\Gamma = \Gamma_y = \Gamma(\cdot, y)$ such that

$$\Gamma^{\rho_\mu} \rightharpoonup \Gamma \quad \text{in } W^{1,s}(\Omega_r(y)) \text{ and in } L^{\tilde{s}}(\Omega_r(y)), \text{ for all } r > 0. \tag{B.51}$$

Furthermore, for fixed $r_0 < r$, (B.44) and (B.46) and that Ω is bounded imply uniform bounds on Γ^{ρ_μ} in $W^{1,2}(\Omega \setminus \Omega_{r_0}(y))$ for small ρ_μ. Thus, there exists a subsequence of $\{\rho_\mu\}$ (which we will not rename) and a function $\widetilde{\Gamma} = \widetilde{\Gamma}_y = \widetilde{\Gamma}(\cdot, y)$ such that

$$\Gamma^{\rho_\mu} \rightharpoonup \widetilde{\Gamma} \quad \text{in } W^{1,2}(\Omega \setminus \Omega_{r_0}(y)).$$

Since $\Gamma \equiv \widetilde{\Gamma}$ on $\Omega_r(y) \setminus \Omega_{r_0}(y)$, we can extend Γ to the entire Ω by setting $\Gamma = \widetilde{\Gamma}$ on $\Omega \setminus \Omega_r(y)$. For ease of notation, we call the extended function Γ. Applying the diagonalization process again, we conclude that there exists a sequence $\rho_\mu \to 0$ and a function Γ on Ω such that for every $s \in [1,2)$ and $\tilde{s} \in [1,\infty)$,

$$\Gamma^{\rho_\mu} \rightharpoonup \Gamma \quad \text{in } W^{1,s}(\Omega_r(y)) \text{ and in } L^{\tilde{s}}(\Omega_r(y)), \tag{B.52}$$

and

$$\Gamma^{\rho_\mu} \rightharpoonup \Gamma \quad \text{in } W^{1,2}(\Omega \setminus \Omega_{r_0}(y)), \tag{B.53}$$

for all $0 < r_0 < r$.

Let $\phi \in C_c^\infty(\Omega)$ and $r > 0$. Choose $\eta \in C_c^\infty(B_r(y))$ to be a cutoff function so that $\eta \equiv 1$ in $B_{r/2}(y)$. We write $\phi = \eta\phi + (1-\eta)\phi$. By (B.25) and the definition of B,

$$\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \eta\phi = \lim_{\mu\to\infty} B[\Gamma_y^{\rho_\mu}, \eta\phi] = \lim_{\mu\to\infty} \int_\Omega A\nabla\Gamma_y^{\rho_\mu} \cdot \nabla(\eta\phi) + W \cdot \nabla\Gamma_y^{\rho_\mu}\, \eta\phi.$$

Note that $\eta\phi$ and $D(\eta\phi)$ belong to $C_c^\infty(\Omega_r(y))$. From this and the boundedness of A given by (2), it follows that there exists a $s' > 2$ such that each $a_{ij}D_i(\eta\phi)$ belongs to $L^{s'}(\Omega_r(y))$. Since $W \in L^q(\Omega)$ for some $q \in (2,\infty]$, then $W\eta\phi \in L^q(\Omega_r(y))$. Therefore, by (B.52),

$$\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \eta\phi = \int_\Omega A\nabla\Gamma_y \cdot \nabla(\eta\phi) + W\cdot\nabla\Gamma_y\,\eta\phi = B[\Gamma_y, \eta\phi]. \tag{B.54}$$

Another application of (B.25) shows that

$$\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} (1-\eta)\,\phi = \lim_{\mu\to\infty} \int_\Omega A\nabla\Gamma_y^{\rho_\mu} \cdot \nabla[(1-\eta)\,\phi] + W\cdot\nabla\Gamma_y^{\rho_\mu}\,(1-\eta)\,\phi.$$

Since $\phi \in C_c^\infty(\Omega)$ and $\eta \in C_c^\infty(B_r(y))$, then $(1-\eta)\,\phi$ and $D[(1-\eta)\,\phi]$ belong to $C_c^\infty(\Omega\setminus B_{r/2}(y))$. In combination with (2), this implies that each $a_{ij}D_i[(1-\eta)\,\phi]$ belongs to $L^2\left(\Omega\setminus B_{r/2}(y)\right)$. Hölder's inequality and that Ω is bounded implies that $W(1-\eta)\,\phi$ belongs to $L^2\left(\Omega\setminus B_{r/2}(y)\right)$ as well. Therefore, it follows from (B.53) that

$$\lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} (1-\eta)\,\phi = \int_\Omega A\nabla\Gamma_y \cdot \nabla[(1-\eta)\,\phi] + W\cdot\nabla\Gamma_y\,(1-\eta)\,\phi = B[\Gamma_y, (1-\eta)\,\phi]. \tag{B.55}$$

Upon combining (B.54) and (B.55), we see that for any $\phi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\phi(y) = \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \phi &= \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \eta\phi + \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} (1-\eta)\,\phi \\ &= B[\Gamma_y, \eta\phi] + B[\Gamma_y, (1-\eta)\,\phi] = B[\Gamma_y, \phi].\end{aligned}$$

That is, for any $\phi \in C_c^\infty(\Omega)$,

$$B[\Gamma_y, \phi] = \phi(y)$$

and Γ satisfies property (B.12) in the definition of the Green's function.

As before, for $\ell_0 \in (1,\infty]$ and $m_0 \in (2,\infty]$, we take $f \in L^{\ell_0}(\Omega)$, $G \in L^{m_0}(\Omega)$ and let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be the unique weak solutions to $\widetilde{L}^* u_1 = f$ and $\widetilde{L}^* u_2 = -\operatorname{div} G$. That is, u_1 and $u_2 \in W_0^{1,2}(\Omega)$ satisfy (B.29) and (B.30), respectively, so that with $u := u_1 + u_2$,

$$B^*[u, w] = \int_\Omega wf + Dw\cdot G, \quad \forall w \in W_0^{1,2}(\Omega).$$

Then for a.e. $y \in \Omega$,

$$u(y) = \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} u = \lim_{\mu\to\infty} B\left[\Gamma_y^{\rho_\mu}, u\right] = \lim_{\mu\to\infty} B^*\left[u, \Gamma_y^{\rho_\mu}\right] = \lim_{\mu\to\infty} \int_\Omega \Gamma^{\rho_\mu} f + D\Gamma^{\rho_\mu}\cdot G \tag{B.56}$$

where we have used (B.33) - (B.35).

Let $\eta \in C_c^\infty(B_r(y))$ be as defined in the previous paragraph. Then $\eta f \in L^{\ell_0}(B_r(y))$. Since Ω is bounded, then $f \in L^\ell(\Omega)$ for some $\ell \in (1,2)$ and it follows that $(1-\eta)\,f \in L^\ell\left(\Omega\setminus B_{r/2}(y)\right)$. Equation (B.53) in combination with a Sobolev inequality implies that for all $0 < r_0 < r$,

$$\Gamma^{\rho_\mu} \rightharpoonup \Gamma \quad \text{in } L^{\ell'}(\Omega\setminus\Omega_{r_0}(y)),$$

where $\ell' \in (2,\infty)$ denotes the Hölder conjugate to ℓ. Consequently, using the property above and (B.52) with $\tilde{s} = \ell_0'$ shows that

$$\begin{aligned}\lim_{\mu\to\infty}\int_\Omega \Gamma^{\rho_\mu} f &= \lim_{\mu\to\infty}\int_{B_r(y)} \Gamma^{\rho_\mu}\eta f + \lim_{\mu\to\infty}\int_{\Omega\setminus B_{r/2}(y)} \Gamma^{\rho_\mu}(1-\eta) f \\ &= \int_{B_r(y)} \Gamma\eta f + \int_{\Omega\setminus B_{r/2}(y)} \Gamma(1-\eta) f = \int_\Omega \Gamma f.\end{aligned}$$

Since $m_0 > 2$, then $m_0' \in (1,2)$ and then according to (B.52) we can pair ηG with $D\Gamma^{\rho_\mu}$ in $B_r(y)$ and take the limit. As Ω is bounded, then $G \in L^2(\Omega)$ so that $(1-\eta)G \in L^2\left(\Omega\setminus B_{r/2}(y)\right)$. With the aid of (B.52) with $s = m_0'$ and (B.53), we see that

$$\begin{aligned}\lim_{\mu\to\infty}\int_\Omega D\Gamma^{\rho_\mu}\cdot G &= \lim_{\mu\to\infty}\int_{B_r(y)} D\Gamma^{\rho_\mu}\cdot\eta G + \lim_{\mu\to\infty}\int_{\Omega\setminus B_{r/2}(y)} D\Gamma^{\rho_\mu}\cdot(1-\eta) G \\ &= \int_{B_r(y)} D\Gamma\cdot\eta G + \int_{\Omega\setminus B_{r/2}(y)} D\Gamma\cdot(1-\eta) G = \int_\Omega D\Gamma\cdot G.\end{aligned}$$

Combining the last two equations with (B.56) gives (B.14). Property (B.15) follows as well.

The first part of each of the estimates (B.12)–(B.16) follow almost directly by passage to the limit and recalling that we use the notation $\Gamma = \Gamma_y = \Gamma(\cdot,y)$. Indeed, for any $r > 0$ and any $g \in L^\infty(\Omega_r(y))$, (B.49) implies that for any $s \in [1,\infty)$

$$\left|\int_\Omega \Gamma g\right| = \lim_{\mu\to\infty}\left|\int_\Omega \Gamma^{\rho_\mu} g\right| \le C_s r^{-\varepsilon+\frac{2}{s}}\,||g||_{L^{s'}(\Omega_r(y))},$$

where $\varepsilon > 0$ is arbitrarily small and s' is the Hölder conjugate exponent of s. By duality, we obtain that for every $s \in [1,\infty)$ and $r > 0$,

$$||\Gamma(\cdot,y)||_{L^s(\Omega_r(y))} \le C_s r^{-\varepsilon+\frac{2}{s}},$$

that is, the first part of (B.17) holds. A similar argument using (B.48), (B.44) and (B.46) yields the first parts of (B.18) and (B.16), respectively. Now, as in the proofs of (B.45) and (B.47), the first part of (B.16) gives the first parts of (B.19) and (B.20).

Passing to the proof of (B.21), fix $x \ne y$. For a.e. $x \in \Omega$, the Lebesgue differentiation theorem implies that

$$\Gamma(x) = \lim_{\delta\to0^+} \fint_{\Omega_\delta(x)} \Gamma = \lim_{\delta\to0^+}\frac{1}{|\Omega_\delta|}\int \Gamma\chi_{\Omega_\delta(x)},$$

where χ denotes an indicator function. Assuming as we may that $2\delta \le \min\{d_x, |x-y|\}$, it follows that $\chi_{\Omega_\delta(x)} = \chi_{B_\delta(x)} \in L^{2^*}(\Omega\setminus\Omega_\delta(y))$ for any $2^* \in (2,\infty)$, where $d_x = \operatorname{dist}(x,\partial\Omega)$. Therefore, (B.52) implies that

$$\frac{1}{|B_\delta|}\int \Gamma\chi_{B_\delta(x)} = \lim_{\mu\to\infty}\frac{1}{|B_\delta|}\int \Gamma^{\rho_\mu}\chi_{B_\delta(x)} = \lim_{\mu\to\infty}\fint_{B_\delta(x)} \Gamma^{\rho_\mu}.$$

If $\rho_\mu \le \frac{1}{3}|x-y|$, $\rho_\mu < d_y$, then (B.40) implies that for a.e. $z \in B_\delta(x)$

$$|\Gamma^{\rho_\mu}(z)| \le C|z-y|^{-\varepsilon},$$

where C is independent of ρ_μ. Since $|z-y| > \frac{1}{2}|x-y|$ for every $z \in B_\delta(x) \subset B_{|x-y|/2}(x)$, then

$$||\Gamma^{\rho_\mu}||_{L^\infty(B_\delta(x))} \le C|x-y|^{-\varepsilon}.$$

By combining with the observations above, we see that for a.e. $x \in \Omega$,

$$\Gamma(x,y) = \lim_{\delta\to 0^+} \frac{1}{|\Omega_\delta|}\int \Gamma\chi_{\Omega_\delta(x)} = \lim_{\delta\to 0^+}\lim_{\mu\to\infty} \fint_{B_\delta(x)} \Gamma^{\rho_\mu} \le \lim_{\delta\to 0^+}\lim_{\mu\to\infty} C|x-y|^{-\varepsilon} = C|x-y|^{-\varepsilon},$$

which is (B.21).

Now we have to prove that $\Gamma(\cdot,y) = 0$ on $\partial\Omega$ in the sense that for all $\zeta \in C_c^\infty(\Omega)$ satisfying $\zeta \equiv 1$ on $B_r(y)$ for some $r > 0$, equation (B.13) holds. By Mazur's lemma, $W_0^{1,2}(\Omega)$ is weakly closed in $W^{1,2}(\Omega)$. Therefore, since $(1-\zeta)\Gamma^{\rho_\mu} = \Gamma^{\rho_\mu} - \zeta\Gamma^{\rho_\mu} \in W_0^{1,2}(\Omega)$ for all $\rho_\mu > 0$, it suffices for (B.13) to show that

$$(1-\zeta)\Gamma^{\rho_\mu} \rightharpoonup (1-\zeta)\Gamma \quad \text{in } W^{1,2}(\Omega). \tag{B.57}$$

Since $(1-\zeta) \equiv 0$ on $B_r(y)$, the result (B.57) follows from (B.53). Indeed,

$$\begin{aligned}\int_\Omega (1-\zeta)\Gamma(\cdot,y)\phi &= \int_\Omega \Gamma(\cdot,y)(1-\zeta)\phi = \lim_{\mu\to\infty}\int_\Omega \Gamma^{\rho_\mu}(\cdot,y)(1-\zeta)\phi \\ &= \lim_{\mu\to\infty}\int_\Omega (1-\zeta)\Gamma^{\rho_\mu}(\cdot,y)\phi, \quad \forall\phi\in L^2(\Omega), \quad \text{and}\end{aligned}$$

$$\begin{aligned}\int_\Omega D[(1-\zeta)\Gamma(\cdot,y)]\cdot\psi &= -\int_\Omega \Gamma(\cdot,y)D\zeta\cdot\psi + \int_\Omega D\Gamma(\cdot,y)\cdot(1-\zeta)\psi \\ &= -\lim_{\mu\to\infty}\int_\Omega \Gamma^{\rho_\mu}(\cdot,y)D\zeta\cdot\psi + \lim_{\mu\to\infty}\int_\Omega D\Gamma^{\rho_\mu}(\cdot,y)\cdot(1-\zeta)\psi \\ &= \lim_{\mu\to\infty}\int_\Omega D[(1-\zeta)\Gamma^{\rho_\mu}(\cdot,y)]\cdot\psi, \quad \forall\psi\in L^2(\Omega)^2,\end{aligned}$$

so that (B.13) holds. Since $\Gamma(x,y)$ satisfies (B.12) – (B.15), it is the unique Green's function associated to $\widetilde{L}$.

Fix $x, y \in \Omega$ and $0 < R \le R_0 < |x-y|$. Then $\widetilde{L}\Gamma(\cdot,y) = 0$ on $B_{R_0}(x)$. Therefore, by Hölder continuity of solutions described by (B.11) and the pointwise bound (B.21), whenever $|x-z| < \frac{R}{2}$ we have

$$|\Gamma(x,y) - \Gamma(z,y)| \le C_{R_0}\left(\frac{|x-z|}{R}\right)^\eta C\,||\Gamma(\cdot,y)||_{L^\infty(B_R(x))} \le C_{R_0}C\left(\frac{|x-z|}{R}\right)^\eta R^{-\varepsilon}.$$

This is the Hölder continuity of $\Gamma(\cdot,y)$ described by (B.22).

Using the pointwise bound on Γ^ρ in place of those for Γ, a similar statement holds for Γ^ρ with $\rho \le \frac{3}{8}|x-y|$, and it follows that for any compact set $K \Subset \Omega\setminus\{y\}$, the sequence $\{\Gamma^{\rho_\mu}(\cdot,y)\}_{\mu=1}^\infty$ is equicontinuous on K. Furthermore, for any such $K \Subset \Omega\setminus\{y\}$, there are constants $C_K < \infty$ and $\rho_K > 0$ such that for all $\rho < \rho_K$,

$$||\Gamma^\rho(\cdot,y)||_{L^\infty(K)} \le C_K.$$

Passing to a subsequence if necessary, we have that for any such compact $K \Subset \Omega\setminus\{y\}$,

$$\Gamma^{\rho_\mu}(\cdot,y) \to \Gamma(\cdot,y) \tag{B.58}$$

uniformly on K.

We now aim to show

$$\Gamma(x,y) = \Gamma^*(y,x),$$

where Γ^* is the Green's function associated to $\widetilde{L}^*$. Let $\widehat{\Gamma}^\sigma = \widehat{\Gamma}_x^\sigma = \widehat{\Gamma}^\sigma(x,\cdot)$ denote the averaged function associated to $\widetilde{L}^*$ at the point $x \in \Omega$. That is, we follow the procedure from above that was used to construct Γ_y^ρ, except that we work with the adjoint operator $\widetilde{L}^*$ and consider the function in terms of the variable y centred at the point $x \in \Omega$. The resulting function is $\widehat{\Gamma}^\sigma$.

By the same arguments used for Γ^ρ, we obtain a sequence $\{\sigma_\nu\}_{\nu=1}^\infty$, $\sigma_\nu \to 0$, such that $\widehat{\Gamma}^{\sigma_\nu}(\cdot,x) = \widehat{\Gamma}_x^{\sigma_\nu}$ converges to $\Gamma^*(\cdot,x)$ uniformly on compact subsets of $\Omega \setminus \{x\}$, where $\Gamma^*(\cdot,x)$ is a Green's function for $\widetilde{L}^*$ that satisfies the properties analogous to those for $\Gamma(\cdot,y)$. In particular, $\Gamma^*(\cdot,x)$ is Hölder continuous.

By (B.25), for ρ_μ and σ_ν sufficiently small,

$$\fint_{B_{\rho_\mu}(y)} \widehat{\Gamma}^{\sigma_\nu}(\cdot,x) = B\left[\Gamma_y^{\rho_\mu}, \widehat{\Gamma}_x^{\sigma_\nu}\right] = B^*\left[\widehat{\Gamma}_x^{\sigma_\nu}, \Gamma_y^{\rho_\mu}\right] = \fint_{B_{\sigma_\nu}(x)} \Gamma^{\rho_\mu}(\cdot,y). \tag{B.59}$$

Define

$$g_{\mu\nu} := \fint_{B_{\rho_\mu}(y)} \widehat{\Gamma}^{\sigma_\nu}(\cdot,x) = \fint_{B_{\sigma_\nu}(x)} \Gamma^{\rho_\mu}(\cdot,y).$$

By continuity of $\Gamma^{\rho_\mu}(\cdot,y)$, it follows that for any $x \neq y \in \Omega$,

$$\lim_{\nu\to\infty} g_{\mu\nu} = \lim_{\nu\to\infty} \fint_{B_{\rho_\mu}(y)} \widehat{\Gamma}^{\sigma_\nu}(\cdot,x) = \Gamma^{\rho_\mu}(x,y),$$

so that by (B.58),

$$\lim_{\mu\to\infty}\lim_{\nu\to\infty} g_{\mu\nu} = \lim_{\mu\to\infty} \Gamma^{\rho_\mu}(x,y) = \Gamma(x,y).$$

But by weak convergence in $W^{1,s}(B_r(y))$, i.e., (B.52),

$$\lim_{\nu\to\infty} g_{\mu\nu} = \lim_{\nu\to\infty} \fint_{B_{\rho_\mu}(y)} \widehat{\Gamma}^{\sigma_\nu}(\cdot,x) = \fint_{B_{\rho_\mu}(y)} \Gamma^*(\cdot,x),$$

and it follows then by continuity of $\Gamma^*(\cdot,x)$ that

$$\lim_{\mu\to\infty}\lim_{\nu\to\infty} g_{\mu\nu} = \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \Gamma^*(\cdot,x) = \Gamma^*(y,x).$$

Therefore, for all $x \neq y$,

$$\Gamma(x,y) = \Gamma^*(y,x). \tag{B.60}$$

Consequently, all the estimates which hold for $\Gamma(\cdot,y)$ hold analogously for $\Gamma(x,\cdot)$ and the proof is complete. $\square$

Remark. We have seen that there is a subsequence $\{\rho_\mu\}_{\mu=1}^\infty$, $\rho_\mu \to 0$, such that $\Gamma^{\rho_\mu}(x,y) \to \Gamma(x,y)$ for all $x \in \Omega \setminus \{y\}$. In fact, a stronger result can be proved. By (B.59),

$$\Gamma^\rho(x,y) = \lim_{\nu\to\infty} \fint_{B_{\sigma_\nu}(x)} \Gamma^\rho(\cdot,y) = \lim_{\nu\to\infty} \fint_{B_\rho(y)} \widehat{\Gamma}^{\sigma_\nu}(\cdot,x) = \fint_{B_\rho(y)} \Gamma^*(\cdot,x).$$

By (B.60), this gives

$$\Gamma^\rho(x,y) = \fint_{B_\rho(y)} \Gamma(x,z)\,dz.$$

By continuity, for all $x \neq y$,

$$\lim_{\rho \to 0} \Gamma^{\rho}(x,y) = \Gamma(x,y).$$

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