Three-region inequalities for the second order elliptic equation with discontinuous coefficients and size estimate

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Abstract

In this paper, we would like to derive a quantitative uniqueness estimate, the three-region inequality, for the second order elliptic equation with jump discontinuous coefficients. The derivation of the inequality relies on the Carleman estimate proved in our previous work [5]. We then apply the three-region inequality to study the size estimate problem with one boundary measurement.

1 Introduction

In this work we aim to study the size estimate problem with one measurement when the background conductivity has jump interfaces. A typical application of this study is to estimate the size of a cancerous tumor inside an organ by the electric impedance tomography (EIT). In this case, considering discontinuous medium is typical, for instance, the conductivities of heart, liver, intestines are 0.70 (S/m), 0.10 (S/m), 0.03 (S/m), respectively. Previous works on this problem assumed that the conductivity of the studied body is Lipschitz continuous, see, for example, [3, 4]. The first result on the size estimate problem with a discontinuous background conductivity was given in [17], where only the two dimensional case was considered. In this paper, we will study the problem in dimension $n \geq 2$.

The main ingredients of our method are quantitative uniqueness estimates for

$$\operatorname{div}(A\nabla u) = 0 \quad \Omega \subset \mathbb{R}^n. \tag{1.1}$$

Those estimates are well-known when A is Lipschitz continuous. The derivation of the estimates is based on the Carleman estimate or the frequency function method. For $n = 2$ and $A \in L^\infty$, quantitative uniqueness estimates are obtained via the connection

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between (1.1) and quasiregular mappings. This is the method used in [17]. For $n \geq 3$, the connection with quasiregular mappings is not true. Hence we return to the old method – the Carleman estimate, to derive quantitative uniqueness estimates when A is discontinuous. Precisely, when A has a $C^{1,1}$ interface and is Lipschitz away from the interface, a Carleman estimate was obtained in [5] (see [10, 11, 12] for related results). Here we will derive three-region inequalities using this Carleman estimate. The three-region inequality provides us a way to propagate "smallness" across the interface (see also [11] for similar estimates). Relying on the three-region inequality, we then derive bounds of the size of an inclusion with one boundary measurement. For other results on the size estimate, we mention [1] for the isotropic elasticity, [14, 15, 16] for the isotropic/anisotropic thin plate, [7, 6] for the shallow shell.

2 The Carleman estimate

In this section, we would like to describe the Carleman estimate derived in [5]. We first denote $H_\pm = \chi_{\mathbb{R}^n_\pm}$ where $\mathbb{R}^n_\pm = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \gtrless 0\}$ and $\chi_{\mathbb{R}^n_\pm}$ is the characteristic function of $\mathbb{R}^n_\pm$. Let $u_\pm \in C^\infty(\mathbb{R}^n)$ and define

$$u = H_+u_+ + H_-u_- = \sum_\pm H_\pm u_\pm,$$

hereafter, $\sum_\pm a_\pm = a_+ + a_-$, and

$$\mathcal{L}(x,y,\partial)u := \sum_\pm H_\pm \mathrm{div}_{x,y}(A_\pm(x,y)\nabla_{x,y}u_\pm), \tag{2.1}$$

where

$$A_\pm(x,y) = \{a^\pm_{ij}(x,y)\}^n_{i,j=1}, \quad x \in \mathbb{R}^{n-1}, y \in \mathbb{R} \tag{2.2}$$

is a Lipschitz symmetric matrix-valued function satisfying, for given constants $\lambda_0 \in (0,1]$, $M_0 > 0$,

$$\lambda_0|z|^2 \leq A_\pm(x,y)z \cdot z \leq \lambda_0^{-1}|z|^2, \, \forall (x,y) \in \mathbb{R}^n, \, \forall\, z \in \mathbb{R}^n \tag{2.3}$$

and

$$|A_\pm(x',y') - A_\pm(x,y)| \leq M_0(|x'-x| + |y'-y|). \tag{2.4}$$

We write

$$h_0(x) := u_+(x,0) - u_-(x,0), \;\forall\, x \in \mathbb{R}^{n-1}, \tag{2.5}$$

$$h_1(x) := A_+(x,0)\nabla_{x,y}u_+(x,0)\cdot\nu - A_-(x,0)\nabla_{x,y}u_-(x,0)\cdot\nu, \;\forall\, x \in \mathbb{R}^{n-1}, \tag{2.6}$$

where $\nu = -e_n$.

For a function $h \in L^2(\mathbb{R}^n)$, we define

$$\hat{h}(\xi,y) = \int_{\mathbb{R}^{n-1}} h(x,y)e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^{n-1}.$$

As usual $H^{1/2}(\mathbb{R}^{n-1})$ denotes the space of the functions $f \in L^2(\mathbb{R}^{n-1})$ satisfying

$$\int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi < \infty,$$

with the norm

$$||f||^2_{H^{1/2}(\mathbb{R}^{n-1})} = \int_{\mathbb{R}^{n-1}} (1+|\xi|^2)^{1/2} |\hat{f}(\xi)|^2 d\xi. \tag{2.7}$$

Moreover we define

$$[f]_{1/2,\mathbb{R}^{n-1}} = \left[\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x)-f(y)|^2}{|x-y|^n} dydx \right]^{1/2},$$

and recall that there is a positive constant C, depending only on n, such that

$$C^{-1} \int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi \le [f]^2_{1/2,\mathbb{R}^{n-1}} \le C \int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi,$$

so that the norm (2.7) is equivalent to the norm $||f||_{L^2(\mathbb{R}^{n-1})} + [f]_{1/2,\mathbb{R}^{n-1}}$. From now on, we use the letters $C, C_0, C_1, \cdots$ to denote constants (depending on λ_0, M_0, n). The value of the constants may change from line to line, but it is always greater than 1. We will denote by $B_r(x)$ the $(n-1)$-ball centered at $x \in \mathbb{R}^{n-1}$ with radius $r > 0$. Whenever $x = 0$ we denote $B_r = B_r(0)$.

Theorem 2.1 *Let u and $A_\pm(x,y)$ satisfy (2.1)-(2.6). There exist $L, \beta, \delta_0, r_0, \tau_0$ positive constants, with $r_0 \le 1$, depending on λ_0, M_0, n, such that if $\alpha_+ > L\alpha_-$, $\delta \le \delta_0$ and $\tau \ge \tau_0$, then*

$$\begin{aligned}
&\sum_\pm \sum_{|k|=0}^{2} \tau^{3-2|k|} \int_{\mathbb{R}^n_\pm} |D^k u_\pm|^2 e^{2\tau\phi_{\delta,\pm}(x,y)} dxdy + \sum_\pm \sum_{|k|=0}^{1} \tau^{3-2|k|} \int_{\mathbb{R}^{n-1}} |D^k u_\pm(x,0)|^2 e^{2\phi_\delta(x,0)} dx \\
&\quad + \sum_\pm \tau^2 [e^{\tau\phi_\delta(\cdot,0)} u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_\pm [D(e^{\tau\phi_{\delta,\pm}} u_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\
&\le C \Bigg(\sum_\pm \int_{\mathbb{R}^n_\pm} |\mathcal{L}(x,y,\partial)(u_\pm)|^2 \, e^{2\tau\phi_{\delta,\pm}(x,y)} dxdy + [e^{\tau\phi_\delta(\cdot,0)} h_1]^2_{1/2,\mathbb{R}^{n-1}} \\
&\quad + [\nabla_x (e^{\tau\phi_\delta} h_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \tau^3 \int_{\mathbb{R}^{n-1}} |h_0|^2 e^{2\tau\phi_\delta(x,0)} dx + \tau \int_{\mathbb{R}^{n-1}} |h_1|^2 e^{2\tau\phi_\delta(x,0)} dx \Bigg).
\end{aligned} \tag{2.8}$$

where $u = H_+ u_+ + H_- u_-$, $u_\pm \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} u \subset B_{\delta/2} \times [-\delta r_0, \delta r_0]$, and $\phi_{\delta,\pm}(x,y)$ is given by

$$\phi_{\delta,\pm}(x,y) = \begin{cases} \dfrac{\alpha_+ y}{\delta} + \dfrac{\beta y^2}{2\delta^2} - \dfrac{|x|^2}{2\delta}, & y \ge 0, \\ \dfrac{\alpha_- y}{\delta} + \dfrac{\beta y^2}{2\delta^2} - \dfrac{|x|^2}{2\delta}, & y < 0, \end{cases} \tag{2.9}$$

and $\phi_\delta(x,0) = \phi_{\delta,+}(x,0) = \phi_{\delta,-}(x,0)$.

Remark 2.2 *It is clear that* (2.8) *remains valid if can add lower order terms* $\sum_{\pm} H_{\pm}\left(W\cdot\nabla_{x,y}u_{\pm}+Vu_{\pm}\right)$*, where* W, V *are bounded functions, to the operator* $\mathcal{L}$*. That is, one can substitute*

$$\mathcal{L}(x,y,\partial)u=\sum_{\pm} H_{\pm}\mathrm{div}_{x,y}(A_{\pm}(x,y)\nabla_{x,y}u_{\pm})+\sum_{\pm} H_{\pm}\left(W\cdot\nabla_{x,y}u_{\pm}+Vu_{\pm}\right) \quad (2.10)$$

in (2.8).

3 Three-region inequalities

Based on the Carleman estimate given in Theorem 2.1, we will derive three-region inequalities across the interface $y=0$. Here we consider $u=H_+u_+ + H_-u_-$ satisfying

$$\mathcal{L}(x,y,\partial)u=0 \quad \text{in} \quad \mathbb{R}^n,$$

where $\mathcal{L}$ is given in (2.10) and

$$\|W\|_{L^\infty(\mathbb{R}^n)}+\|V\|_{L^\infty(\mathbb{R}^n)}\le \lambda_0^{-1}.$$

Fix any $\delta\le\delta_0$, where δ_0 is given in Theorem 2.1.

Theorem 3.1 *Let* u *and* $A_{\pm}(x,y)$ *satisfy* (2.1)-(2.6) *with* $h_0=h_1=0$*. Then there exist* C *and* R*, depending only on* λ_0, M_0, n*, such that if* $0<R_1,R_2\le R$*, then*

$$\int_{U_2}|u|^2dx\le (e^{\tau_0 R_2}+CR_1^{-4})\left(\int_{U_1}|u|^2dxdy\right)^{\frac{R_2}{2R_1+3R_2}}\left(\int_{U_3}|u|^2dxdy\right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}, \quad (3.1)$$

where τ_0 *is the constant derived in Theorem 2.1,*

$$U_1=\{z\ge -4R_2,\ \frac{R_1}{8a}<y<\frac{R_1}{a}\},$$
$$U_2=\{-R_2\le z\le \frac{R_1}{2a},\ y<\frac{R_1}{8a}\},$$
$$U_3=\{z\ge -4R_2,\ y<\frac{R_1}{a}\},$$

$a=\alpha_+/\delta$*, and*

$$z(x,y)=\frac{\alpha_- y}{\delta}+\frac{\beta y^2}{2\delta^2}-\frac{|x|^2}{2\delta}. \quad (3.2)$$

Proof. To apply the estimate (2.8), u needs to satisfy the support condition. Also, we can choose α_+ and α_- in Theorem 2.1 such that $\alpha_+>\alpha_-$. We can choose $r\le r_0$ satisfying

$$r\le \min\left\{\frac{13\alpha_-}{8\beta},\frac{2\delta}{19\alpha_-+8\beta}\right\}. \quad (3.3)$$

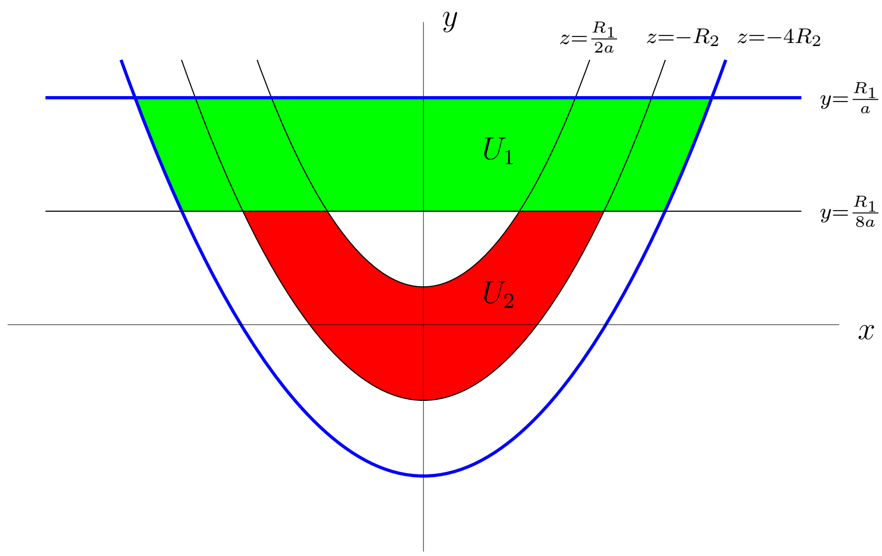

Figure 1: U_1 and U_2 are shown in green and red, respectively. U_3 is the region enclosed by blue boundaries.

Note that the choices of δ, r also depend on λ_0, M_0, n. We then set

$$R = \frac{\alpha_- r}{16}.$$

It follows from (3.3) that

$$R \leq \frac{13\alpha_-^2}{128\beta}. \tag{3.4}$$

Given $0 < R_1 < R_2 \leq R$. Let $\vartheta_1(t) \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq \vartheta_1(t) \leq 1$ and

$$\vartheta_1(t) = \begin{cases} 1, & t > -2R_2, \\ 0, & t \leq -3R_2. \end{cases}$$

Also, define $\vartheta_2(y) \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \vartheta_2(y) \leq 1$ and

$$\vartheta_2(y) = \begin{cases} 0, & y \geq \dfrac{R_1}{2a}, \\ 1, & y < \dfrac{R_1}{4a}. \end{cases}$$

Finally, we define $\vartheta(x,y) = \vartheta_1(z(x,y))\vartheta_2(y)$, where z is defined by (3.2).

We now check the support condition for ϑ. From its definition, we can see that $\mathrm{supp}\vartheta$ is contained in

$$\begin{cases} z(x,y) = \dfrac{\alpha_- y}{\delta} + \dfrac{\beta y^2}{2\delta^2} - \dfrac{|x|^2}{2\delta} > -3R_2, \\ y < \dfrac{R_1}{2a}. \end{cases} \tag{3.5}$$

In view of the relation

$$\alpha_+ > \alpha_- \quad \text{and} \quad a = \frac{\alpha_+}{\delta},$$

we have that

$$\frac{R_1}{2a} < \frac{\delta}{2\alpha_-} \cdot R_1 < \frac{\delta}{\alpha_-} \cdot \frac{\alpha_- r}{16} < \delta r,$$

i.e., $y < \delta r \le \delta r_0$. Next, we observe that

$$-3R_2 > -3R = -\frac{3\alpha_- r}{16} > \frac{\alpha_-}{\delta}(-\delta r) + \frac{\beta}{2\delta^2}(-\delta r)^2,$$

which gives $-\delta r < y$ due to (3.3). Consequently, we verify that $|y| < \delta r$. One the other hand, from the first condition of (3.5) and (3.3), we see that

$$\frac{|x|^2}{2\delta} < 3R_2 + \frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} \le \frac{3\alpha_- r}{16} + \frac{\alpha_-}{\delta} \cdot \delta r + \frac{\beta}{2\delta^2} \cdot \delta^2 r^2 \le \frac{\delta}{8},$$

which gives $|x| < \delta/2$.

Since $h_0 = 0$, we have that

$$\vartheta(x,0)u_+(x,0) - \vartheta(x,0)u_-(x,0) = 0, \ \forall\, x \in \mathbb{R}^{n-1}. \tag{3.6}$$

Applying (2.8) to ϑu and using (3.6) yields

$$\begin{aligned} &\sum_{\pm} \sum_{|k|=0}^{2} \tau^{3-2|k|} \int_{\mathbf{R}^n_\pm} |D^k(\vartheta u_\pm)|^2 e^{2\tau\phi_{\delta,\pm}(x,y)} dxdy \\ \le& C \sum_{\pm} \int_{\mathbf{R}^n_\pm} |\mathcal{L}(x,y,\partial)(\vartheta u_\pm)|^2 \, e^{2\tau\phi_{\delta,\pm}(x,y)} dxdy \\ &+ C\tau \int_{\mathbf{R}^{n-1}} |A_+(x,0)\nabla_{x,y}(\vartheta u_+(x,0)) \cdot \nu - A_-(x,0)\nabla_{x,y}(\vartheta u_-)(x,0) \cdot \nu|^2 e^{2\tau\phi_\delta(x,0)} dx \\ &+ C[e^{\tau\phi_\delta(\cdot,0)}\big(A_+(x,0)\nabla_{x,y}(\vartheta u_+)(x,0) \cdot \nu - A_-(x,0)\nabla_{x,y}(\vartheta u_-)(x,0) \cdot \nu\big)]^2_{1/2,\mathbf{R}^{n-1}}. \end{aligned} \tag{3.7}$$

We now observe that $\nabla_{x,y}\vartheta_1(z) = \vartheta_1'(z)\nabla_{x,y} z = \vartheta_1'(z)(-\frac{x}{\delta}, \frac{\alpha_-}{\delta} + \frac{\beta y}{\delta^2})$ and it is nonzero only when

$$-3R_2 < z < -2R_2.$$

Therefore, when $y = 0$, we have

$$2R_2 < \frac{|x|^2}{2\delta} < 3R_2.$$

Thus, we can see that

$$|\nabla_{x,y}\vartheta(x,0)|^2 \le CR_2^{-2}\left(\frac{6R_2}{\delta} + \frac{\alpha_-^2}{\delta^2}\right) \le CR_2^{-2}. \tag{3.8}$$

By $h_0(x) = h_1(x) = 0$, (3.8), and the easy estimate of [5, Proposition 4.2], it is not hard to estimate

$$
\begin{aligned}
&\tau \int_{\mathbf{R}^{n-1}} |A_+(x,0)\nabla_{x,y}(\vartheta u_+(x,0)) \cdot \nu - A_-(x,0)\nabla_{x,y}(\vartheta u_-)(x,0) \cdot \nu|^2 e^{2\tau\phi_\delta(x,0)} dx \\
&\quad + [e^{\tau\phi_\delta(\cdot,0)}\big(A_+(x,0)\nabla_{x,y}(\vartheta u_+)(x,0) \cdot \nu - A_-(x,0)\nabla_{x,y}(\vartheta u_-)(x,0) \cdot \nu\big)]^2_{1/2,\mathbf{R}^{n-1}} \\
&\le C R_2^{-2} e^{-4\tau R_2} \left(\tau \int_{\{\sqrt{4\delta R_2} \le |x| \le \sqrt{6\delta R_2}\}} |u_+(x,0)|^2 dx + [u_+(x,0)]^2_{1/2,\{\sqrt{4\delta R_2} \le |x| \le \sqrt{6\delta R_2}\}} \right) \\
&\quad + C\tau^2 R_2^{-3} e^{-4\tau R_2} \int_{\{\sqrt{4\delta R_2} \le |x| \le \sqrt{6\delta R_2}\}} |u_+(x,0)|^2 dx \\
&\le C\tau^2 R_2^{-3} e^{-4\tau R_2} E,
\end{aligned}
\tag{3.9}
$$

where

$$
E = \int_{\{\sqrt{4\delta R_2} \le |x| \le \sqrt{6\delta R_2}\}} |u_+(x,0)|^2 dx + [u_+(x,0)]^2_{1/2,\{\sqrt{4\delta R_2} \le |x| \le \sqrt{6\delta R_2}\}}.
$$

Expanding $\mathcal{L}(x,y,\partial)(\vartheta u_\pm)$ and considering the set where $D\vartheta \ne 0$, we can estimate

$$
\begin{aligned}
&\sum_{\pm} \sum_{|k|=0}^{1} \tau^{3-2|k|} \int_{\{-2R_2 \le z \le \frac{R_1}{2a},\, y < \frac{R_1}{4a}\}} |D^k u_\pm|^2 e^{2\tau\phi_{\delta,\pm}(x,y)} dxdy \\
&\le C \sum_{\pm} \sum_{|k|=0}^{1} R_2^{2(|k|-2)} \int_{\{-3R_2 \le z \le -2R_2,\, y < \frac{R_1}{2a}\}} |D^k u_\pm|^2 e^{2\tau\phi_{\delta,\pm}(x,y)} dxdy \\
&\quad + C \sum_{|k|=0}^{1} R_1^{2(|k|-2)} \int_{\{-3R_2 \le z,\, \frac{R_1}{4a} < y < \frac{R_1}{2a}\}} |D^k u_+|^2 e^{2\tau\phi_{\delta,+}(x,y)} dxdy \\
&\quad + C\tau^2 R_2^{-3} e^{-4\tau R_2} E \\
&\le C \sum_{\pm} \sum_{|k|=0}^{1} R_2^{2(|k|-2)} e^{-4\tau R_2} e^{2\tau \frac{(\alpha_+ - \alpha_-)}{\delta} \frac{R_1}{4a}} \int_{\{-3R_2 \le z \le -2R_2,\, y < \frac{R_1}{4a}\}} |D^k u_\pm|^2 dxdy \\
&\quad + \sum_{|k|=0}^{1} R_1^{2(|k|-2)} e^{2\tau \frac{\alpha_+}{\delta} \frac{R_1}{2a}} e^{2\tau \frac{\beta}{2\delta^2} (\frac{R_1}{2a})^2} \int_{\{z \ge -3R_2,\, \frac{R_1}{4a} < y < \frac{R_1}{2a}\}} |D^k u_+|^2 dxdy \\
&\quad + C\tau^2 R_2^{-3} e^{-4\tau R_2} E.
\end{aligned}
\tag{3.10}
$$

Let us denote $U_1=\{z\geq -4R_2,\ \frac{R_1}{8a}<y<\frac{R_1}{a}\}$, $U_2=\{-R_2\leq z\leq \frac{R_1}{2a},\ y<\frac{R_1}{8a}\}$. From (3.10) and interior estimates (Caccioppoli's type inequality), we can derive that

$$
\begin{aligned}
&\tau^3 e^{-2\tau R_2}\int_{U_2}|u|^2dxdy\\
\leq&\,\tau^3 e^{-2\tau R_2}\int_{\{-R_2\leq z\leq \frac{R_1}{2a},\, y<\frac{R_1}{8a}\}}|u|^2dxdy\\
\leq&\sum_{\pm}\tau^3\int_{\{-2R_2\leq z\leq \frac{R_1}{2a},\, y<\frac{R_1}{4a}\}}|u_\pm|^2e^{2\tau\phi_{\delta,\pm}(x,y)}dxdy\\
\leq&\, C\sum_{\pm}\sum_{|k|=0}^{1}R_2^{2(|k|-2)}e^{-4\tau R_2}e^{2\tau\frac{(\alpha_+-\alpha_-)}{\delta}\frac{R_1}{4a}}\int_{\{-3R_2\leq z\leq -2R_2,\, y<\frac{R_1}{4a}\}}|D^ku_\pm|^2dxdy\\
&+\sum_{|k|=0}^{1}R_1^{2(|k|-2)}e^{2\tau\frac{\alpha_+}{\delta}\frac{R_1}{2a}}e^{2\tau\frac{\beta}{2\delta^2}(\frac{R_1}{2a})^2}\int_{\{z\geq -3R_2,\ \frac{R_1}{4a}<y<\frac{R_1}{2a}\}}|D^ku_+|^2dxdy\\
&+C\tau^2R_2^{-3}e^{-4\tau R_2}E\\
\leq&\, CR_1^{-4}e^{-3\tau R_2}\int_{\{-4R_2\leq z\leq -R_2,\, y<\frac{R_1}{a}\}}|u|^2dxdy+C\tau^2R_2^{-3}e^{-4\tau R_2}E\\
&+CR_1^{-4}e^{(1+\frac{\beta R_1}{4\alpha_-^2})\tau R_1}\int_{\{z\geq -4R_2,\ \frac{R_1}{8a}<y<\frac{R_1}{a}\}}|u|^2dxdy\\
\leq& CR_1^{-4}\left(e^{2\tau R_1}\int_{U_1}|u|^2dxdy+\tau^2e^{-3\tau R_2}F\right),
\end{aligned}
\tag{3.11}
$$

where

$$
F=\int_{\{z\geq -4R_2,\, y<\frac{R_1}{a}\}}|u|^2dxdy
$$

and we used the inequality $\frac{\beta R_1}{4\alpha_-^2}<1$ due to (3.4).

Dividing $\tau^3e^{-2\tau R_2}$ on both sides of (3.11) implies that

$$
\int_{U_2}|u|^2dxdy\leq CR_1^{-4}\left(e^{2\tau(R_1+R_2)}\int_{U_1}|u|^2dxdy+e^{-\tau R_2}F\right). \tag{3.12}
$$

Now, we consider two cases. If $\int_{U_1}|u|^2dxdy\neq 0$ and

$$
e^{2\tau_0(R_1+R_2)}\int_{U_1}|u|^2dxdy<e^{-\tau_0R_2}F,
$$

then we can pick a $\tau>\tau_0$ such that

$$
e^{2\tau(R_1+R_2)}\int_{U_1}|u|^2dxdy=e^{-\tau R_2}F.
$$

Using such τ, we obtain from (3.12) that

$$\begin{aligned}\int_{U_2}|u|^2dxdy &\le CR_1^{-4}e^{2\tau(R_1+R_2)}\int_{U_1}|u|^2dxdy\\ &= CR_1^{-4}\left(\int_{U_1}|u|^2dxdy\right)^{\frac{R_2}{2R_1+3R_2}}(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}.\end{aligned}\tag{3.13}$$

If $\int_{U_1}|u|^2dxdy = 0$, then letting $\tau\to\infty$ in (3.12) we have $\int_{U_2}|u|^2dxdy = 0$ as well. The three-regions inequality (3.1) obviously holds.

On the other hand, if

$$e^{-\tau_0R_2}F \le e^{2\tau_0(R_1+R_2)}\int_{U_1}|u|^2dxdy,$$

then we have

$$\begin{aligned}\int_{U_2}|u|^2dx &\le (F)^{\frac{R_2}{2R_1+3R_2}}(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}\\ &\le \exp(\tau_0R_2)\left(\int_{U_1}|u|^2dxdy\right)^{\frac{R_2}{2R_1+3R_2}}(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}.\end{aligned}\tag{3.14}$$

Putting together (3.13), (3.14), we arrive at

$$\int_{U_2}|u|^2dx \le (\exp(\tau_0R_2)+CR_1^{-4})\left(\int_{U_1}|u|^2dxdy\right)^{\frac{R_2}{2R_1+3R_2}}(F)^{\frac{2R_1+2R_2}{2R_1+3R_2}}.\tag{3.15}$$

□

4 Size estimate

We will apply the three-region inequality (3.1) to estimate the size of embedded inclusion in this section. Here we denote Ω a bounded open set in $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary $\partial\Omega$ with constants s_0, L_0, where $0<\alpha\le 1$. Assume that Σ is a C^2 hypersurface with constants r_0, K_0 satisfying

$$\mathrm{dist}(\Sigma,\partial\Omega)\ge d_0\tag{4.1}$$

for some $d_0>0$. We divide Ω into three sets, namely,

$$\Omega=\Omega_+\cup\Sigma\cup\Omega_-$$

where $\Omega_\pm$ are open subsets. Note that $\overline{\Omega}_- = \partial\Omega\cup\Sigma$ and $\partial\Omega_+=\Sigma$. We also define

$$\Omega_h=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>h\}.$$

Definition 4.1 [$C^{1,\alpha}$ regularity] We say that Σ is C^2 with constants r_0, K_0 if for any $P \in \Sigma$ there exists a rigid transformation of coordinates under which $P = 0$ and

$$\Omega_\pm \cap B(0, r_0) = \{(x, y) \in B(0, r_0) \subset \mathbb{R}^n : y \gtrless \psi(x)\},$$

where ψ is a C^2 function on $B_{r_0}(0)$ satisfying $\psi(0) = 0$ and

$$\|\psi\|_{C^2(B_{r_0}(0))} \leq K_0.$$

The definition of $C^{1,\alpha}$ boundary is similar. Note that $B(a, r)$ stands for the n-ball centered at a with radius $r > 0$. We remind the reader that $B_r(a)$ denotes the $(n-1)$-ball centered at a with radius $r > 0$.

Assume that $A_\pm = \{a_{ij}^\pm(x, y)\}_{i,j=1}^n$ satisfy (2.3) and (2.4). Let us define $H_\pm = \chi_{\Omega_\pm}$, $A = H_+A_+ + H_-A_-$, $u = H_+u_+ + H_-u_-$. We now consider the conductivity equation

$$\operatorname{div}(A\nabla u) = 0 \quad \text{in} \quad \Omega. \tag{4.2}$$

It is not hard to check that u satisfies homogeneous transmission conditions (2.5), (2.6) (with $h_0 = h_1 = 0$), where in this case ν is the outer normal of Σ. For $\phi \in H^{1/2}(\partial\Omega)$, let u solve (4.2) and satisfy the boundary value $u = \phi$ on $\partial\Omega$.

Next we assume that D is a measurable subset of Ω. Suppose that $\hat{A}$ is a symmetric $n \times n$ matrix with $L^\infty(\Omega)$ entries. In addition, we assume that there exist $\eta > 0, \zeta > 1$ such that

$$(1 + \eta)A \leq \hat{A} \leq \zeta A \quad \text{a.e. in} \quad \Omega \tag{4.3}$$

or $\eta > 0, 0 < \zeta < 1$ such that

$$\zeta A \leq \hat{A} \leq (1 - \eta)A \quad \text{a.e. in} \quad \Omega. \tag{4.4}$$

Now let $v = H_+v_+ + H_-v_-$ be the solution of

$$\begin{cases} \operatorname{div}((A\chi_{\Omega\setminus\bar{D}} + \hat{A}\chi_D)\nabla v) = 0 \quad \text{in} \quad \Omega, \\ v = \phi \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.5}$$

The inverse problem considered here is to estimate $|D|$ by the knowledge of $\{\phi, A\nabla v \cdot \nu|_{\partial\Omega}\}$. In this work we would like to consider the most interesting case where

$$\bar{D} \subseteq \bar{\Omega}_+. \tag{4.6}$$

In practice, one could think of Ω_+ being an organ and D being a tumor. The aim is to estimate the size of D by measuring one pair of voltage and current on the surface of the body.

We denote W_0 and W the powers required to maintain the voltage ϕ on $\partial\Omega$ when the inclusion D is absent or present. It is easy to see that

$$W_0 = \int_{\partial\Omega} \phi A\nabla u \cdot \nu = \int_\Omega A\nabla u \cdot \nabla u$$

and

$$W=\int_{\partial\Omega}\phi(A\chi_{\Omega\setminus\bar{D}}+\hat{A}\chi_D)\nabla v\cdot\nu=\int_{\Omega}(A\chi_{\Omega\setminus\bar{D}}+\hat{A}\chi_D)\nabla v\cdot\nabla v.$$

The size of D will be estimate by the power gap $W-W_0$. To begin, we recall the following energy inequalities proved in [4].

Lemma 4.1 *[4, Lemma 2.1] Assume that A satisfies the ellipticity condition (2.3). If either (4.3) or (4.4) holds, then*

$$C_1\int_D|\nabla u|^2\le|W_0-W|\le C_2\int_D|\nabla u|^2, \tag{4.7}$$

where C_1, C_2 are constants depending only on λ, η, and ζ.

The derivation of bounds on $|D|$ will be based on (4.7) and the following Lipschitz propagation of smallness for u.

Proposition 4.1 *(Lipschitz propagation of smallness) Let $u\in H^1(\Omega)$ be the solution of (4.2) with Dirichlet data ϕ. For any $B(x,\rho)\subset\Omega_+$, we have that*

$$\int_{B(x,\rho)}|\nabla u|^2\ge C\int_{\Omega}|\nabla u|^2, \tag{4.8}$$

where C depends on $\Omega_\pm$, d_0, λ_0, M_0, r_0, K_0, s_0, L_0, α, α', ρ, and

$$\frac{\|\phi-\phi_0\|_{C^{1,\alpha'}(\partial\Omega)}}{\|\phi-\phi_0\|_{H^{1/2}(\partial\Omega)}},$$

for $\phi_0=|\partial\Omega|^{-1}\int_{\partial\Omega}\phi$. Here α' satisfies $0<\alpha'<\frac{\alpha}{(\alpha+1)n}$.

Before proving Proposition 4.1, we need to adjust the three-region inequality (3.1) for the C^2 interface Σ. Let $0\in\Sigma$ and the coordinate transform $(x',y')=T(x,y)=(x,y-\psi(x))$ for $x\in B_{s_0}(0)$. Denote $\tilde{U}=T(B(0,s_0))$ and $\tilde{\mathcal{A}}_\pm=\{\tilde{a}^\pm_{i,j}\}^n_{i,j=1}$ the coefficients of $A_\pm$ in the new coordinates (x',y'). It is easy to see that $\tilde{\mathcal{A}}_\pm$ satisfies (2.3) and (2.4) with possible different constants $\tilde{\lambda}_0, \tilde{M}_0$, depending on λ_0, M_0, r_0, K_0. Then there exist C and $\tilde{R}$, depending on $\tilde{\lambda}_0, \tilde{M}_0, n$, such that for

$$0<R_1<R_2\le\tilde{R} \tag{4.9}$$

and U_1, U_2, U_3 defined as in Theorem 3.1, we have that $U_3\subset\tilde{U}$ (so U_1, U_2 are contained in $\tilde{U}$ as well) and (3.1) holds. Now let $\tilde{U}_j=T^{-1}(U_j)$, $j=1,2,3$, then (3.1) becomes

$$\int_{\tilde{U}_2}|u|^2dxdy\le C\left(\int_{\tilde{U}_1}|u|^2dxdy\right)^{\frac{R_2}{2R_1+3R_2}}\left(\int_{\tilde{U}_3}|u|^2dxdy\right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}, \tag{4.10}$$

where C depends on $\lambda_0, M_0, r_0, K_0, n, R_1, R_2$. Furthermore, by Caccioppoli's inequality and generalized Poincaré's inequality (see (3.8) in [2]), we obtain from (4.10) that

$$\int_{\tilde{U}_2} |\nabla u|^2 dxdy \leq C \left(\int_{\tilde{U}_1} |\nabla u|^2 dxdy \right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{\tilde{U}_3} |\nabla u|^2 dxdy \right)^{\frac{2R_1+2R_2}{2R_1+3R_2}} \tag{4.11}$$

with a possibly different constant C.

Since A_+ (respectively A_-) is Lipschitz in Ω_+ (respectively Ω_-), the following three-sphere inequality is well-known. Let $u_\pm$ be a solution to $\mathrm{div}(A_\pm \nabla u_\pm) = 0$ in $\Omega_\pm$. Then for $B(x_0, \bar{r}) \subset \Omega_+$ (or $B(x_0, \bar{r}) \subset \Omega_-$) and $0 < r_1 < r_2 < r_3 < \bar{r}$, we have that

$$\int_{B(x_0,r_2)} |\nabla u_\pm|^2 dxdy \leq C \left(\int_{B(x_0,r_1)} |\nabla u_\pm|^2 dxdy \right)^{\theta} \left(\int_{B(x_0,r_3)} |\nabla u_\pm|^2 dxdy \right)^{1-\theta}, \tag{4.12}$$

where $0 < \theta < 1$ and C depend on $\lambda_0, M_0, n, r_1/r_3, r_2/r_3$.

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. It suffices to study the case where ρ is small. Since $\Sigma \in C^2$, it satisfies both the uniform interior and exterior sphere properties, i.e., there exists $a_0 > 0$ such that for all $z \in \Sigma$, there exist balls $B \subset \Omega_+$ and $B' \subset \Omega_-$ of radius a_0 such that $\overline{B} \cap \Sigma = \overline{B}' \cap \Sigma = \{z\}$. Next let ν_z be the unit normal at $z \in \Sigma$ pointing into Ω_+ (inwards) and $L = \{z + t\nu_z \subset \mathbb{R}^n : t \in [\rho_0, -3\rho_0]\}$. We then fix R_1, R_2 satisfying (4.9) and choose $\rho_0 > 0$ so that

$$S_z = \cup_{y \in L} B(y, \rho_0) \subset \tilde{U}_2.$$

Denote $\kappa = R_2/(2R_1 + 3R_2)$. Note that we move the construction of the three-region inequality from 0 to z.

Let $x \in \Omega_+$ and consider $B(x, \rho) \subset \Omega_+$, where $\rho \leq \min\{a_0, \rho_0\}$. For any $y \in \Omega_{2\rho}$, we discuss three cases.

(i) Let $y \in \Omega_{+,\rho}$, then by (4.12) and the chain of balls argument, we have that

$$\frac{\int_{B(y,\rho)} |\nabla u|^2}{\int_\Omega |\nabla u|^2} \leq C \left(\frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_\Omega |\nabla u|^2} \right)^{\theta^{N_1}}, \tag{4.13}$$

where N_1 depends on Ω_+ and ρ.

(ii) Let $y \in \{y \in \overline{\Omega}_+ : \mathrm{dist}(y, \Sigma) \leq \rho\} \cup \{y \in \Omega_- : \mathrm{dist}(y, \Sigma) \leq 3\rho\}$, then $B(y, \rho) \subset S_z$ for some $z \in \Sigma$. Note that $\tilde{U}_1 \subset \Omega_{+,\rho}$ (taking ρ even smaller if necessary). We then apply (4.13) iteratively to estimate

$$\frac{\int_{\tilde{U}_1} |\nabla u|^2}{\int_\Omega |\nabla u|^2} \leq C \left(\frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_\Omega |\nabla u|^2} \right)^{\theta^{N_1}}, \tag{4.14}$$

where C depends on $\tilde{U}_1$ and ρ. Combining estimates (4.14) and (4.11) yields

$$\frac{\int_{B(y,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\le C\left(\frac{\int_{B(x,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\right)^{\kappa\theta^{N_1}}. \tag{4.15}$$

(iii) Finally, we consider the case where $y\in\Omega_-\cap\Omega_{2\rho}$ and $\operatorname{dist}(y,\Sigma)>3\rho$. We observe that if $y_*=z+(-3\rho)\nu_z$, then (4.15) implies

$$\frac{\int_{B(y_*,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\le C\left(\frac{\int_{B(x,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\right)^{\kappa\theta^{N_1}}. \tag{4.16}$$

Again using (4.12) and the chain of balls argument (starting with (4.16)), we obtain that

$$\frac{\int_{B(y,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\le C\left(\frac{\int_{B(x,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\right)^{\kappa\theta^{N_1}\theta^{N_2}}. \tag{4.17}$$

Putting together (4.13), (4.15), and (4.17) gives

$$\frac{\int_{B(y,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\le C\left(\frac{\int_{B(x,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\right)^{s} \tag{4.18}$$

for all $y\in\Omega_{2\rho}$, where $0<s<1$ and C depends on $\lambda_0,M_0,n,r_0,K_0,\rho,\Omega_\pm$.

In view of (4.18) and covering $\Omega_{3\rho}$ with balls of radius ρ, we have that

$$\frac{\int_{\Omega_{3\rho}}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\le C\left(\frac{\int_{B(x,\rho)}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\right)^{s}. \tag{4.19}$$

Note that $u-\phi_0$ is the solution to (4.2) with Dirichlet boundary value $\phi-\phi_0$. By Corollary 1.3 in [13], we have that

$$\|\nabla u\|^2_{L^\infty(\Omega)}=\|\nabla(u-\phi_0)\|^2_{L^\infty(\Omega)}\le C\|\phi-\phi_0\|^2_{C^{1,\alpha'}(\partial\Omega)}$$

with $0<\alpha'<\frac{\alpha}{(\alpha+1)n}$, which implies

$$\int_{\Omega\setminus\Omega_{3\rho}}|\nabla u|^2\le C|\Omega\setminus\Omega_{5\rho}|\|\phi-\phi_0\|^2_{C^{1,\alpha'}(\partial\Omega)}\le C\rho\|\phi-\phi_0\|^2_{C^{1,\alpha'}(\partial\Omega)}. \tag{4.20}$$

Here we have used $|\Omega\backslash\Omega_{5\rho}|\lesssim\rho$ proved in [3]. Using the Poincaré inequality, we have

$$\|\phi-\phi_0\|^2_{H^{1/2}(\partial\Omega)}\le C\|u-\phi_0\|^2_{H^1(\Omega)}\le C\|\nabla u\|^2_{L^2(\Omega)}.$$

Combining this and (4.20), we see that if ρ is small enough depending on $\Omega_\pm$, d_0, λ_0, M_0, r_0, K_0, s_0, L_0, α, α', ρ, and $\|\phi-\phi_0\|_{C^{1,\alpha'}(\partial\Omega)}/\|\phi-\phi_0\|_{H^{1/2}(\partial\Omega)}$, then

$$\frac{\|\nabla u\|^2_{L^2(\Omega_{3\rho})}}{\|\nabla u\|^2_{L^2(\Omega)}}\ge\frac{1}{2}.$$

The proposition follows from this and (4.19). $\square$

We now have enough tools to derive bounds on $|D|$.

Theorem 4.2 *Suppose that the assumptions of this section hold.*

(i) *If, moreover, there exists $h > 0$ such that*

$$|D_h| \geq \frac{1}{2}|D| \quad \textit{(fatness condition)}. \tag{4.21}$$

Then there exist constants $K_1, K_2 > 0$ depending only on $\Omega_\pm$, d_0, h, λ_0, M_0, r_0, K_0, s_0, L_0, α, α', and $\|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)}/\|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)}$, such that

$$K_1 \left| \frac{W_0 - W}{W_0} \right| \leq |D| \leq K_2 \left| \frac{W_0 - W}{W_0} \right|.$$

(ii) *For a general inclusion D contained strictly in Ω_+, we assume that there exists $d_1 > 0$ such that*

$$\operatorname{dist}(D, \partial\Omega_+) \geq d_1.$$

Then there exist constants K_1, K_2', $p > 1$, depending only on $\Omega_\pm$, d_0, d_1, h, λ_0, M_0, r_0, K_0, s_0, L_0, α, α', and $\|\phi - \phi_0\|_{C^{1,\alpha'}(\partial\Omega)}/\|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)}$, such that

$$K_1 \left| \frac{W_0 - W}{W_0} \right| \leq |D| \leq K_2' \left| \frac{W_0 - W}{W_0} \right|^{\frac{1}{p}}. \tag{4.22}$$

Proof. The proof follows closely the arguments in [4] and [17]. The lower bound can be obtained by basic estimates. Let $c = \frac{1}{|\Omega_{d/4}|} \int_{\Omega_{d/4}} u$. By the gradient estimate of [13, Theorem 1.1], the interior estimate of [9, Theorem 8.17] and the Poincaré inequality for the domain $\Omega_{d/4}$, we have

$$\|\nabla u\|_{L^\infty(\Omega_{d/2})} \leq C\|u - c\|_{L^\infty(\Omega_{d/3})} \leq C\|u - c\|_{L^2(\Omega_{d/4})} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

From this, the trivial estimate $\|\nabla u\|^2_{L^2(D)} \leq C|D|\|\nabla u\|^2_{L^\infty(\Omega_{d/2})}$ and the second inequality of (4.7), the lower bound follows.

Next, we prove the upper bounds.

(i) Let $\rho = \frac{h}{4}$ and cover D_h with internally nonoverlapping closed squares $\{Q_k\}_{k=1}^J$ of side length 2ρ. It is clear that $Q_k \subset D$, hence

$$\begin{aligned}\int_D |\nabla u|^2 dx \geq \int_{\cup_{k=1}^J Q_k} |\nabla u|^2 dx \geq \frac{|D_h|}{\rho^2} \min_k \int_{Q_k} |\nabla u|^2 dx. \\ \geq \frac{C|D|}{\rho^2} \int_\Omega |\nabla u|^2 dx.\end{aligned}$$

Here we have used Proposition 4.1 and the fatness condition at the last inequality. The upper bound of $|D|$ follows from this and the first inequality of (4.7).

(ii) To prove the upper bound without the fatness condition, we need the fact that $|\nabla u|^2$ is an A_p weight which an easy consequence of the doubling condition for ∇u. It turns out when D is strictly contained in Ω_+ where the coefficient A_+ is Lipschitz. The well-known theorem guarantees that $|\nabla u|^2$ is an A_p weight in Ω_+ (see [8] or [4]), i.e., for any $\bar{r} > 0$, there exists $B > 0$ and $p > 1$ such that

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}|\nabla u|^2\right)\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}|\nabla u|^{-\frac{2}{p-1}}\right)^{p-1} \le B$$

for any ball $B(a,r) \subset \Omega_{+,\bar{r}}$, where B and p depends on various constants listed in Proposition 4.1. To derive the upper bound of (4.22), we choose $\bar{r} = d_1/2$ and follow exactly the same lines as in the proof of Theorem 2.2 [4].

$\square$

Remark 4.3 *We point out that part (i) of Theorem 4.2 still holds if the assumption* (4.6) *is replaced by*

$$dist(D, \partial\Omega) \ge d_2 > 0.$$

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