Quantitative uniqueness estimates for the general second order elliptic equations

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Abstract

In this paper we study quantitative uniqueness estimates of solutions to general second order elliptic equations with magnetic and electric potentials. We derive lower bounds of decay rate at infinity for any nontrivial solution under some general assumptions. The lower bounds depend on asymptotic behaviors of magnetic and electric potentials. The proof is carried out by the Carleman method and bootstrapping arguments.

1 Introduction

In this paper we study the asymptotic behaviors of solutions to the general second order elliptic equation

$$Pv + W(x)\cdot\nabla v + V(x)v + q(x)v = 0 \quad \text{in} \quad \Omega := \mathbb{R}^n \setminus \bar{B}, \tag{1.1}$$

where B is a bounded set in Ω. Here $P(x,D)=\sum_{jk} a_{jk}(x)\partial_j\partial_k$ is uniformly elliptic, i.e., for some $\lambda_0>0$

$$\lambda_0|\xi|^2 \le \sum_{jk} a_{jk}(x)\xi_j\xi_k \le \lambda_0^{-1}|\xi|^2 \quad \forall\, x\in\Omega,\ \xi\in\mathbb{R}^n. \tag{1.2}$$

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and $a_{jk}(x)$ is Lipschitz continuous. We are interested in deriving lower bounds of the decay rate for any nontrivial solution v to (1.1) under certain a priori assumptions. This kind of problem was originally posed by Landis in the 60's [10]. He conjectured that if v is a bounded solution of

$$\Delta v + q(x)v = 0 \quad \text{in} \quad \mathbb{R}^n \tag{1.3}$$

with $\|q\|_{L^\infty} \le 1$ and $|v(x)| \le C\exp(-C|x|^{1+})$ for some constant C, then v is identically zero. This conjecture was disproved by Meshkov [13] who constructed a $q(x)$ and a nontrivial $v(x)$ with $|v(x)| \le C\exp(-C|x|^{4/3})$ satisfying (1.3). He also proved that if $|v(x)| \le C_k \exp(-k|x|^{4/3})$ for all $k > 0$ then $v \equiv 0$. Note that $q(x)$ and $v(x)$ constructed by Meshkov are complex valued. In 2005, Bourgain and Kenig [2] derived a quantitative version of Meshkov's result in their resolution of Anderson localization for the Bernoulli model. Precisely, they showed that if v is a bounded solution of $\Delta v + q(x)v = 0$ in $\mathbb{R}^n$ satisfying $\|q\|_{L^\infty} \le 1$ and $v(0) = 1$, then

$$\inf_{|x_0|=R} \sup_{B(x_0,1)} |v(x)| \ge C\exp(-R^{4/3}\log R).$$

In view of Meshkov's example, the exponent $4/3$ is optimal.

Recently, Davey [4] derived similar quantitative asymptotic estimates for (1.1) with $P = -\Delta$ and $q(x) = -E \in \mathbb{C}$, i.e.,

$$-\Delta v + W(x)\cdot\nabla v + V(x)v = Ev. \tag{1.4}$$

To describe her result, we define

$$I(x_0) = \int_{|y-x_0|<1} |v(y)|^2 dy$$

and

$$M(t) = \inf_{|x_0|=t} I(x_0).$$

Assume that $|V(x)| \lesssim \langle x\rangle^{-N}$ and $|W(x)| \lesssim \langle x\rangle^{-\tilde p}$, where $\langle x\rangle = \sqrt{1+|x|^2}$. Then it was shown that for any nontrivial bounded solution v of (1.4) with $v(0) = 1$, we have

$$M(t) \gtrsim \exp(-Ct^{\beta_0}(\log t)^{b(t)}), \tag{1.5}$$

where

$$\beta_0 = \max\{2 - 2\tilde p, \frac{4-2N}{3}, 1\}$$

and $b(t)$ is either a constant C or $C\log\log t$. Moreover, in [4], some Meshkov's type examples were constructed to ensure the optimality of (1.5). There are also some related qualitative results in [3], [6], [7], and [8]. Especially, in [3] and [8], the authors studied the Schrödinger equation with potential $-\Delta v + V(x)v = Eu$, where $|V(x)| \lesssim \langle x\rangle^{-N}$ with $0 < N < 1/2$ (in [3]) and $N \le 0$, $N > 1/2$ (in [8]). In addition to qualitative results, they also showed the optimality of β_0 (here $\beta_0 = \max\{\frac{4-2N}{3}, 1\}$). For the case of $N = 1/2$, the qualitative result was proved in [7].

In this work, we extend Davey's results to more general cases. Precisely, we consider the second order elliptic operator P with more general assumptions on the asymptotic behaviors of W, V, and q. The main theorem is stated as follows.

Theorem 1.1. *Let $v \in H^1_{loc}(\Omega)$ be a nontrivial solution of* (1.1) *satisfying*

$$|v(x)| \le \lambda |x|^{\alpha} \quad \textit{with} \quad \alpha \ge 0 \tag{1.6}$$

for some $\lambda > 0$. Assume that the ellipticity condition (1.2) *holds and for $\epsilon > 0$, $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathbb{R}$,*

$$\begin{cases} |W(x)| \le \lambda |x|^{-\kappa_1},\ |V(x)| \le \lambda |x|^{-\kappa_2}, \\ |\nabla a_{ij}(x)| \le \lambda |x|^{-1-\epsilon}, \\ |q(x)| \le \lambda |x|^{\kappa_3},\ |\nabla q(x)| \le \lambda |x|^{-\kappa_4}. \end{cases} \tag{1.7}$$

Denote $\kappa_0 = \max\{2 - 2\kappa_1, \frac{4-2\kappa_2}{3}, \frac{2+\kappa_3}{2}, \frac{3-\kappa_4}{2}\}$, $\kappa = \max\{\kappa_0, 1\}$. Then we have that

- *For $\kappa > 1$ (i.e., $\kappa_0 > 1$), there exist t_0 depending on λ_0, λ, ϵ and positive constants C, C' such that*

  $$M(t) \ge \exp\left(-Ct^{\kappa_0}(\log t)^{\gamma(t)}\right) \quad \textit{for} \quad t \ge t_0 \tag{1.8}$$

  with

  $$\gamma(t) = \frac{C'(\log t)(\log\log\log t)}{(\log\log t)^2},$$

  where $C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha)$ and $C' = C'(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \alpha)$.

- *For $\kappa = 1$ (i.e., $\kappa_0 \le 1$), there exists a positive constant C'' such that*

  $$M(t) \ge \exp\left(-C''t(\log t)^{\gamma(t)}\right) \quad \textit{for} \quad t \ge t_0, \tag{1.9}$$

where C'' depends on α, λ_0, λ, $\kappa_1, \kappa_2, \kappa_3$, ϵ,

$$\left|\log\left(\min\{\inf_{t_0^{\frac{1}{1+\epsilon}}<|x|<t_0}\int_{|y-x|<1}|v(y)|^2dy, 1\}\right)\right|.$$

Remark 1.2. 1. *The condition on the decay rate of ∇a_{ij} was also used by T Nguyen* [14] *in his proof of qualitative and quantitative Landis-Oleinik conjecture, the parabolic counterpart of Landis conjecture.*

2. *We have made general assumptions on the asymptotic behaviors of W, V, and q. They may grow in $|x|$. Our theorem provides quantitative uniqueness estimates for solutions of $-\Delta v + V(x)v = Ev$ with $|V| \lesssim |x|^m$, $m > 0$. Moreover, our method works for any $\kappa_0 \in \mathbb{R}$ including the case $\kappa_0 = 1$ that is missing in* [4].

Similar to the arguments in [2] and [4], a key ingredient to prove Theorem 1.1 is the Carleman type estimate. Before applying the Carleman estimate, v is shifted and rescaled appropriately. We will modify our ideas in [12]. Basically, we use the Carleman estimate for the shifted and rescaled solution on a ball of radius depending on $|x_0|$ (see the definition of Ω_t in Section 3). Note that in order to use the behaviors of coefficients in (1.7), the radius of the ball is sufficiently small. In fact, after undoing sifting and rescaling, any point in Ω_t is at least $|x_0|^{1-\delta(x_0)}$ distance from the origin, where $\delta(x_0) = (\log\log|x_0|)^2/\log|x_0|$ (see Section 3). We then apply the Carleman estimate to derive three-ball inequalities in which we can estimate the L^2 bound of the solution in a unit ball centered at $x_0/|x_0|^{\delta(x_0)}$ by the L^2 bound of the solution in a unit ball centered at x_0 up to certain power (see (3.21)). To obtain the desired estimates, we want to apply bootstrapping arguments based on a chain of balls similar to what we did in [12]. An bootstrapping step was also used in [4] to prove estimate (1.5). However, our method here is simpler than that of [4].

We now discuss the optimality of (1.8) and (1.9), at least, for some simple cases. It is readily seen that if $v(x) = \exp(-|x|^{1+\varepsilon})$ with $0 < \varepsilon \le 1/2$, then $\Delta v + q(x)v = 0$ in $\{|x| < 1\}^c$ with $|q(x)| \sim |x|^{2\varepsilon}$ and $|\nabla q(x)| \sim |x|^{2\varepsilon-1}$. In this case, we can see that $\kappa = 1 + \varepsilon$. So the exponent κ of (1.8) is optimal. This example also shows that if the first derivatives of the potential possess certain decaying property, we can break the 4/3 barrier in the case of bounded potentials. On the other hand, for $\varepsilon = 0$, we obtain that $v = \exp(-|x|)$

satisfies $\Delta v + q(x)v = 0$ in $\{|x| < 1\}^c$ with $q(x) = -1 + (n-1)|x|^{-1}$. Since we can write

$$(\log t)^{\gamma(t)} = t^{\frac{C \log\log\log t}{\log\log t}},$$

(1.9) is equivalent to

$$M(t) \geq \exp\left(-Ct^{1+o(1)}\right).$$

Thus, (1.9) is almost optimal.

This paper is organized as follows. In Section 2, we prove a Carleman estimate for the operator $P + q$, which plays an essential role in our proof. In Section 3, we begin to prove the main theorem, Theorem 1.1, by deriving three-ball inequalities for solutions of (1.1). In Section 4, we give detailed arguments of bootstrapping and complete the proof of Theorem 1.1.

2 Carleman estimate

In this section, we would like to derive a Carleman estimate for $P+q$ with q is C^1. Similar Carleman estimate for such operator with P being the Laplace-Beltrami operator was also derived in [1] using Donnelly and Fefferman's approach [5]. Since we are working in the Euclidean space, we give a more elementary proof motived by the ideas in [15]. To begin, we introduce polar coordinates in $\mathbb{R}^n \backslash \{0\}$ by setting $x = r\omega$, with $r = |x|$, $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. Furthermore, using new coordinate $t = \log r$, we can see that

$$\frac{\partial}{\partial x_j} = e^{-t}(\omega_j \partial_t + \Omega_j), \quad 1 \leq j \leq n,$$

where Ω_j is a vector field in S^{n-1}. We could check that the vector fields Ω_j satisfy

$$\sum_j \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_j \Omega_j \omega_j = n-1.$$

Since $r \to 0$ iff $t \to -\infty$, we are mainly interested in values of t near $-\infty$.

It is easy to see that

$$\frac{\partial^2}{\partial x_j \partial x_\ell} = e^{-2t}(\omega_j \partial_t - \omega_j + \Omega_j)(\omega_\ell \partial_t + \Omega_\ell), \quad 1 \leq j, \ell \leq n.$$

and, therefore, the Laplacian becomes

$$e^{2t}\Delta = \partial_t^2 + (n-2)\partial_t + \Delta_\omega, \tag{2.1}$$

where $\Delta_\omega = \Sigma_j \Omega_j^2$ denotes the Laplace-Beltrami operator on S^{n-1}. We recall that the eigenvalues of $-\Delta_\omega$ are $k(k+n-2), k \in \mathbb{N}$, and the corresponding eigenspaces are E_k, where E_k is the space of spherical harmonics of degree k. It follows that

$$\iint |\Delta_\omega v|^2 dt d\omega = \sum_{k\geq 0} k^2(k+n-2)^2 \iint |v_k|^2 dt d\omega \tag{2.2}$$

and

$$\sum_j \iint |\Omega_j v|^2 dt d\omega = \sum_{k\geq 0} k(k+n-2) \iint |v_k|^2 dt d\omega, \tag{2.3}$$

where v_k is the projection of v onto E_k and the integral $\iint f dt d\omega$ denotes $\int_{S^{n-1}} \int_{\mathbb{R}} f(t,w) dt d\omega$.

Our aim is to derive Carleman-type estimates with weights $\varphi_\beta = \varphi_\beta(x) = \exp(-\beta\tilde{\psi}(x))$, where $\beta > 0$ and $\tilde{\psi}(x) = \log|x| + \log((\log|x|)^2)$. For simplicity, we denote $\psi(t) = t + \log t^2$, i.e., $\tilde{\psi}(x) = \psi(\log|x|)$.

Lemma 2.1. *Assume that $P = \sum_{jk} a_{jk}(x)\partial_j\partial_k$ is a second order elliptic operator satisfying*

$$\lambda_0|\xi|^2 \leq \sum_{jk} a_{jk}(x)\xi_j\xi_k \leq \lambda_0^{-1}|\xi|^2 \quad \forall\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,$$

and

$$\|\nabla a_{jk}\|_{L^\infty(\mathbb{R}^n)} \leq L.$$

Then there exist a sufficiently small $r_1 = r_1(\lambda_0, L) > 0$ such that for all $u \in U_{r_1}$ and

$$\beta \geq (||q||_{L^\infty(\mathbb{R}^n)} + ||\nabla q||_{L^\infty(\mathbb{R}^n)})^{1/2},$$

we have

$$\begin{aligned} &\beta \int_{\mathbb{R}^\mathbf{n}} \varphi_\beta^2 (\log|x|)^{-2}|x|^{-n}(|x|^2|\nabla u|^2 + |u|^2)dx \\ &\leq C_0 \int_{\mathbb{R}^\mathbf{n}} \varphi_\beta^2 |x|^{-n}|x|^4|Pu + q(x)u|^2 dx, \end{aligned} \tag{2.4}$$

where $U_{r_1} = \{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \mathrm{supp}(u) \subset B_{r_1}\}$ and $C_0 = C_0(\lambda_0, L) > 0$.

Proof. If we set $u = e^{\beta\psi(t)}v$ and $P_\beta v = e^{-\beta\psi(t)}P(e^{\beta\psi(t)}v)$, then

$$\begin{aligned}
& e^{2t}P_\beta v + e^{2t}qv \\
= \;& \partial_t^2 v + b\partial_t v + Av + \Delta_\omega v + e^{2t}qv + \sum_{j+|\alpha|\leq 2} C_{j\alpha}(t,\omega)\partial_t^j\Omega^\alpha v \\
& + \sum_{j+|\alpha|\leq 1} C_{j\alpha}(t,\omega)\beta\psi'\partial_t^j\Omega^\alpha v + C_{20}(t,\omega)(\beta^2\psi'^2 + \beta\psi'')v \\
= \;& \partial_t^2 v + b\partial_t v + Av + \Delta_\omega v + e^{2t}qv + Sv + hv, \qquad (2.5)
\end{aligned}$$

where

$$\begin{cases}
A = \beta\psi'' + \beta^2(\psi')^2 + (n-2)\beta\psi' \\
\quad = (1+2t^{-1})^2\beta^2 + (n-2)\beta + 2(n-2)t^{-1}\beta - 2t^{-2}\beta, \\
b = 2\beta\psi' + n - 2 \\
\quad = 2\beta + 4\beta t^{-1} + n - 2, \\
S(v) = \sum_{j+|\alpha|\leq 2} C_{j\alpha}(t,\omega)\partial_t^j\Omega^\alpha v + C_{20}(t,\omega)(\beta^2\psi'^2 + \beta\psi'')v, \\
h(v) = \sum_{j+|\alpha|\leq 1} C_{j\alpha}(t,\omega)\beta\psi'\partial_t^j\Omega^\alpha v, \\
C_{j\alpha} = O(e^t), \partial_t C_{j\alpha} = O(e^t), \Omega^\alpha C_{j\alpha} = O(e^t).
\end{cases}$$

It is clear that (2.4) holds if for t near $-\infty$ we have

$$\begin{aligned}
& \iint |e^{2t}P_\beta v + e^{2t}qv|^2 dt d\omega \\
& \geq C\left\{\iint \beta t^{-2}|\partial_t v|^2 dt d\omega + \beta\sum_j \iint t^{-2}|\Omega_j v|^2 dt d\omega + \beta^3\iint t^{-2}|v|^2 dt d\omega\right\}.
\end{aligned} \qquad (2.6)$$

We remark that $\operatorname{supp}(u) \subset B_{r_1}$ with $r_1 < 1$. So $\operatorname{supp}(v(t,\omega)) \subset (-\infty, \log r_1) \times S^{n-1}$ with $\log r_1 < 0$. Thus, the integral domain for t is actually in $(-\infty, \log r_1)$ which does not contain 0. We obtain from (2.5) that

$$|e^{2t}P_\beta v + e^{2t}qv|^2 = |L(v)|^2 + 2b\partial_t v L(v) + 2h(v)L(v) + |b\partial_t v + h(v)|^2, \quad (2.7)$$

where $L(v) := \partial_t^2 v + Av + \Delta_\omega v + e^{2t} qv + S(v)$. Now we write

$$\begin{aligned}
& \iint 2b\partial_t v L(v) dtd\omega \\
= & \iint 2b\partial_t v \partial_t^2 v dtd\omega + \iint 2abv\partial_t v dtd\omega + \iint 2be^{2t} q\partial_t v v dtd\omega \\
& + \iint 2b\partial_t v \Delta_\omega v dtd\omega + \iint 2b\partial_t v S(v) dtd\omega.
\end{aligned} \tag{2.8}$$

By straightforward computations, we can get that

$$\iint 2b\partial_t v\partial_t^2 v dtd\omega = -\iint (\partial_t b)|\partial_t v|^2 dtd\omega, \tag{2.9}$$

$$\iint 2Abv\partial_t v dtd\omega = -\iint \partial_t(Ab)|v|^2 dtd\omega, \tag{2.10}$$

$$\iint 2b\partial_t v\Delta_\omega v dtd\omega = \sum_j \iint (\partial_t b)|\Omega_j v|^2 dtd\omega, \tag{2.11}$$

and

$$\iint 2be^{2t} qv\partial_t v dtd\omega = -\iint (\partial_t b) e^{2t} q|v|^2 dtd\omega - \iint \partial_t(e^{2t}q) b|v|^2 dtd\omega. \tag{2.12}$$

Combining (2.7) to (2.12) yields

$$\begin{aligned}
& \iint |e^{2t} P_\beta v + e^{2t} qv|^2 dtd\omega \\
= & \iint |L(v)|^2 dtd\omega + 4\beta \iint t^{-2} |\partial_t v|^2 dtd\omega \\
& + 12\beta^3 \iint t^{-2}(1 + O(t^{-1}))|v|^2 dtd\omega - 4\beta \sum_j \iint t^{-2} |\Omega_j v|^2 dtd\omega \\
& + 4\beta \iint t^{-2} e^{2t} q|v|^2 dtd\omega - \iint \partial_t(e^{2t}q) b|v|^2 dtd\omega + \iint 2b\partial_t v S(v) dtd\omega \\
& + \iint 2h(v)L(v) dtd\omega + \iint |b\partial_t v + h(v)|^2 dtd\omega.
\end{aligned} \tag{2.13}$$

Likewise, we write

$$|L(v)|^2 = |L(v) + 3\beta t^{-2} v|^2 - 6\beta t^{-2} vL(v) - (3\beta t^{-2})^2 |v|^2. \tag{2.14}$$

It is easy to check that

$$
\begin{aligned}
&-6\beta\iint t^{-2}vL(v)dtd\omega\\
=&-6\beta\iint t^{-2}v(\partial_t^2 v+av+\Delta_\omega v+e^{2t}qv+S(v))dtd\omega\\
=&-6\beta^3\iint t^{-2}(1+O(t^{-1}))|v|^2dtd\omega-6\beta\iint e^{2t}qt^{-2}|v|^2dtd\omega\\
&+6\beta\iint t^{-2}|\partial_t v|^2dtd\omega-12\beta\iint t^{-3}v\partial_t vdtd\omega+6\beta\sum_j\iint t^{-2}|\Omega_j v|^2dtd\omega\\
&-6\beta\iint t^{-2}vS(v)dtd\omega. \qquad (2.15)
\end{aligned}
$$

From (2.13)-(2.15), we have that for $t\le\tau$ (τ depends on λ_0, L)

$$
\begin{aligned}
&\iint|e^{2t}P_\beta v+e^{2t}qv|^2dtd\omega\\
\ge&\,\beta\iint t^{-2}|\partial_t v|^2dtd\omega+2\beta\sum_j\iint t^{-2}|\Omega_j v|^2dtd\omega+3\beta^3\iint t^{-2}|v|^2dtd\omega\\
&-2\beta\iint(q+\partial_t q)t^{-2}|v|^2dtd\omega+\iint 2b\partial_t vS(v)dtd\omega\\
&+\iint 2h(v)L(v)dtd\omega-6\beta\iint t^{-2}vS(v)dtd\omega. \qquad (2.16)
\end{aligned}
$$

Using integration by parts and choosing an even smaller τ, if necessary, we can see that $\iint 2b\partial_t vS(v)dtd\omega$, $\iint 2h(v)L(v)dtd\omega$, and $6\beta\iint t^{-2}vS(v)dtd\omega$ are bounded by the first three terms on the right side of (2.16). Therefore, taking

$$\beta\ge\sqrt{\|q\|_{L^\infty}+\|\nabla q\|_{L^\infty}},$$

(2.6) follows from (2.16). □

3 Proof of Theorem 1.1 – Part I: three-ball inequalities

We begin to prove Theorem 1.1 in this section. As in [2], [4], and [12], the solution of (1.1) is shifted and rescaled properly. Fixing x_0 with $|x_0|=t>>$

1, we define

$$\begin{cases} w(x) = u(atx + x_0),\ \tilde{a}_{jk}(x) = a_{jk}(atx + x_0),\ \tilde{P}(x, D) = \sum_{jk} \tilde{a}_{jk}(x)\partial_j\partial_k, \\ \tilde{W}(x) = (at)W(atx + x_0),\ \tilde{V}(x) = (at)^2 V(atx + x_0), \\ \tilde{q}(x) = (at)^2 q(atx + x_0), \end{cases}$$

where $a \geq 1/r_1$ will be determined later in the proof. Here r_1 is the constant appeared in Lemma 2.1. We now denote

$$\tilde{\Omega}_t := B_{\frac{1}{a} - \frac{1}{20at^\delta}}(0) = \{x : |x| < \frac{1}{a} - \frac{1}{20at^\delta}\}.$$

It follows from (1.1) that

$$\tilde{P}w + \tilde{W}(x) \cdot \nabla w + \tilde{V}(x)w + \tilde{q}(x)w = 0 \quad \text{in} \quad \tilde{\Omega}_t. \tag{3.1}$$

It is clear that $\tilde{a}_{jk}(x)$ satisfies (1.2) in Ω_t with same constant λ_0. Furthermore, in view of (1.7), we have that

$$\begin{cases} |\tilde{W}(x)| \leq 20\lambda a t^{1-\kappa_1+\kappa_1\delta},\ |\tilde{V}(x)| \leq 20\lambda a^2 t^{2-\kappa_2+\kappa_2\delta}, \\ |\nabla \tilde{a}_{ij}(x)| \leq 40\lambda a t^{(1+\epsilon)\delta-\epsilon}, \\ |\tilde{q}(x)| \leq 2\lambda a^2 t^{2+\kappa_3},\ |\nabla \tilde{q}(x)| \leq 20\lambda a^3 t^{3-\kappa_4+\kappa_4\delta}. \end{cases} \tag{3.2}$$

Unlike in [12], where δ is a fixed constant, here we take $\delta = \delta(t) = \frac{(\log\log t)^2}{\log t}$. We now choose an t_0 such that $\log t_0 \geq 1/r_1$ and

$$-\epsilon_0 := (1+\epsilon)\delta(t_0) - \epsilon < 0. \tag{3.3}$$

By setting $a = t_0^{\epsilon_0}$, one can see that $at^{(1+\epsilon)\delta(t)-\epsilon} \leq 1$ for all $t \geq t_0$. Let r_1 and C_0 be constants in Lemma 2.1 determined by λ_0 and $L = 40\lambda$. Then the Carleman estimate (2.4) can be applied to w in Ω_t for all $t \geq t_0$ with same r_1 and C_0. For simplicity, in this section, C denotes a general constant whose value may vary from line to line. Furthermore, it depends on λ_0, λ, and ϵ unless indicated otherwise.

Besides of the Carleman estimate (2.4), we also need the following interior estimate for solutions of (3.1) in our proof.

Lemma 3.1. *For any $0 < a_1 < a_2$ such that $B_{a_2} \subset \Omega_t$ for $t > 1$ and a large enough, let $X = B_{a_2} \backslash \bar{B}_{a_1}$ and $d(x)$ be the distant from $x \in X$ to $\mathbb{R}^n \backslash X$. Then we have*

$$
\begin{aligned}
& (1 + ||\tilde{W}||^2_{L^\infty(X)}) \int_X d(x)^2 |\nabla w|^2 dx \\
\leq\ & C \left(||\tilde{W}||^4_{L^\infty(X)} + ||\tilde{V}||^2_{L^\infty(X)} + ||\tilde{q}||^2_{L^\infty(X)} \right) \int_X |w|^2 dx \qquad (3.4)
\end{aligned}
$$

with $C = C(\lambda_0, L)$.

The lemma can be proved using similar arguments in [11]. We omit the details here.

Now we are ready to apply (2.4) to w solving (3.1). Before doing so, we need to introduce a suitable cut-off function. Let $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \chi(x) \leq 1$ and

$$
\chi(x) = \begin{cases} 0, & |x| \leq \frac{1}{8at}, \\ 1, & \frac{1}{4at} < |x| < \frac{1}{a} - \frac{3}{20at^\delta}, \\ 0, & |x| \geq \frac{1}{a} - \frac{2}{20at^\delta}. \end{cases}
$$

It is easy to see that for any multiindex α

$$
\begin{cases} |D^\alpha \chi| = O((at)^{|\alpha|}) & \text{if} \quad \frac{1}{8at} \leq |x| \leq \frac{1}{4at}, \\ |D^\alpha \chi| = O((at^\delta)^{|\alpha|}) & \text{if} \quad \frac{1}{a} - \frac{3}{20at^\delta} \leq |x| \leq \frac{1}{a} - \frac{2}{20at^\delta}. \end{cases} \qquad (3.5)
$$

To use (2.4), it suffices to take $\beta \geq \beta_1 = \sqrt{20\lambda a^3} t^{\kappa_0} t^{|\kappa_4|\delta/2}$. Thus, we have

$$
\begin{aligned}
& \int (\log |x|)^{-2} \varphi_\beta^2 |x|^{-n} (\beta |x|^2 |\nabla(\chi w)|^2 + \beta^3 |\chi w|^2) dx \\
\leq\ & C_0 \int \varphi_\beta^2 |x|^{-n} |x|^4 |\tilde{P}(x)(\chi w) + \tilde{q}(\chi w)|^2 dx. \qquad (3.6)
\end{aligned}
$$

Using equation (3.1), we obtain that

$$
\begin{aligned}
& \int_T (\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(\beta|x|^2|\nabla w|^2+\beta^3|w|^2)dx \\
\le & \int \varphi_\beta^2(\log|x|)^{-2}|x|^{-n}(\beta|x|^2\nabla(\chi w)|^2+\beta^3|\chi w|^2)dx \\
\le & \; C_0\int \varphi_\beta^2|x|^{-n}|x|^4|\chi(\tilde{W}(x)\cdot\nabla w+\tilde{V}(x)w)|^2dx \\
& +C_0\int \varphi_\beta^2|x|^{4-n}|[\tilde{P},\chi]w|^2,
\end{aligned}
\tag{3.7}
$$

where T denotes the domain $\{x : \frac{1}{4at} < |x| < \frac{1}{a} - \frac{3}{20at^\delta}\}$. To simplify the notations, we denote $Y = \{x : \frac{1}{8at} \le |x| \le \frac{1}{4at}\}$ and $Z = \{x : \frac{1}{a} - \frac{3}{20at^\delta} \le |x| \le \frac{1}{a} - \frac{2}{20at^\delta}\}$. By (3.2) and estimates (3.5), we deduce from (3.7) that

$$
\begin{aligned}
& \int_T (\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(\beta|x|^2|\nabla w|^2+\beta^3|w|^2)dx \\
\le & \; C'\int_T \varphi_\beta^2|x|^{-n}|x|^4(a^2t^{2-2\kappa_1+2\kappa_1\delta}|\nabla w|^2+a^4t^{4-2\kappa_2+2\kappa_2\delta}|w|^2)dx \\
& +C\int_Y \varphi_\beta^2|x|^{-n}|\tilde{U}|^2dx+C\|\tilde{W}\|_{L^\infty}^2\int_Z \varphi_\beta^2|x|^{-n}|x|^2|\nabla w|^2dx \\
& +C\|\tilde{V}\|_{L^\infty}^2\int_Z \varphi_\beta^2|x|^{-n}|w|^2dx,
\end{aligned}
\tag{3.8}
$$

where $|\tilde{U}(x)|^2 = |x|^2|\nabla w|^2+|w|^2$, $C' = C'(\lambda_0,\lambda)$, $C = C(\lambda_0,\lambda,a)$. From now on, $\|\cdot\|_{L^\infty}$ is taken over Ω_t.

Taking a larger t_0 (recall $a = t_0^{\epsilon_0}$), if necessary, we can obtain that $|x|^2(\log|x|)^2C' \le \frac{1}{2}$ for all $x \in T$. Additionally, we choose $\beta \ge \beta_2 := a^2t^{\kappa_0+\kappa_s\delta}$, where $\kappa_s = \max\{2|\kappa_1|, 2|\kappa_2|/3, |\kappa_4|/2\}$. then the first term on the right hand side of (3.8) can be absorbed by the left hand side of (3.8). With the choices described above, we obtain from (3.8) that

$$\begin{aligned}
&\beta^3 (b_1)^{-n} (\log b_1)^{-2} \varphi_\beta^2(b_1) \int_{\frac{1}{2at}<|x|<b_1} |w|^2 dx \\
\leq\ & \beta^3 \int_T (\log|x|)^{-2} \varphi_\beta^2 |x|^{-n} |w|^2 dx \\
\leq\ & C b_2^{-n} \varphi_\beta^2(b_2) \int_Y |\tilde{U}|^2 dx + C\|\tilde{W}\|^2_{L^\infty} b_3^{-n} \varphi_\beta^2(b_3) \int_Z |x|^2 |\nabla w|^2 dx \\
& + C\|\tilde{V}\|^2_{L^\infty} b_3^{-n} \varphi_\beta^2(b_3) \int_Z |w|^2 dx, \qquad (3.9)
\end{aligned}$$

where $b_1 = \frac{1}{a} - \frac{8}{20at^\delta}$, $b_2 = \frac{1}{8at}$ and $b_3 = \frac{1}{a} - \frac{3}{20at^\delta}$.

Using (3.4), we can control $|\tilde{U}|^2$ terms on the right hand side of (3.5). Indeed, let $X = Y_1 := \{x : \frac{1}{16at} \leq |x| \leq \frac{1}{2at}\}$, then we can see that

$$d(x) \geq C|x| \quad \text{for all} \quad x \in Y,$$

where C an absolute constant. Therefore, (3.4) implies

$$\begin{aligned}
&\int_Y |x|^2 |\nabla w|^2 dx \\
\leq\ & C \int_{Y_1} d(x)^2 |\nabla w|^2 dx \\
\leq\ & C(\|\tilde{W}\|^4_{L^\infty} + \|\tilde{V}\|^2_{L^\infty} + \|\tilde{q}\|^2_{L^\infty}) \int_{Y_1} |w|^2 dx, \qquad (3.10)
\end{aligned}$$

where $C = C(\lambda_0, \lambda, a)$. On the other hand, let $X = Z_1 := \{x : \frac{1}{2a} \leq |x| \leq \frac{1}{a} - \frac{1}{20at^\delta}\}$, then

$$d(x) \geq Ct^{-\delta}|x| \quad \text{for all} \quad x \in Z,$$

where C another absolute constant. Thus, it follows from (3.4) that

$$\begin{aligned}
&\int_Z |x|^2 |\nabla w|^2 dx \\
\leq\ & Ct^{2\delta} \int_{Z_1} d(x)^2 |w|^2 dx \\
\leq\ & Ct^{2\delta}(\|\tilde{W}\|^4_{L^\infty} + \|\tilde{V}\|^2_{L^\infty} + \|\tilde{q}\|^2_{L^\infty}) \int_{Z_1} |w|^2 dx. \qquad (3.11)
\end{aligned}$$

Here, C also depends on λ_0, λ. Combining (3.9), (3.10), and (3.11) leads to

$$\begin{aligned}
& b_1^{-2\beta-n}(\log b_1)^{-4\beta-2}\int_{\frac{1}{2at}<|x|<b_1}|w|^2dx \\
\leq\ & C''t^{p_1}b_2^{-n}\varphi_\beta^2(b_2)\int_{Y_1}|w|^2dx + C''t^{p_2}b_3^{-n}\varphi_\beta^2(b_3)\int_{Z_1}|w|^2dx, \quad (3.12)
\end{aligned}$$

where $C'' = C''(\lambda_0, \lambda, a)$, $p_1 = p_1(\kappa_1, \kappa_2, \kappa_3)$, $p_2 = p_2(\kappa_1, \kappa_2, \kappa_3)$. Notice that (3.12) holds for all $\beta \geq \beta_2$.

Changing $2\beta + n$ to β, (3.12) becomes

$$\begin{aligned}
& b_1^{-\beta}(\log b_1)^{-2\beta+2n-2}\int_{\frac{1}{2at}<|x|<b_1}|w|^2dx \\
\leq\ & C''t^{p_1}b_2^{-\beta}(\log b_2)^{-2\beta+2n}\int_{Y_1}|w|^2dx \\
& +C''t^{p_2}b_3^{-\beta}(\log b_3)^{-2\beta+2n}\int_{Z_1}|w|^2dx. \quad (3.13)
\end{aligned}$$

Recall that $\delta = \delta(t) = \frac{(\log\log t)^2}{\log t}$. By taking t_0 sufficiently large, if necessary, we can see that for $t \geq t_0$

$$\begin{cases} \frac{1}{a} - \frac{1}{at^\delta} + \frac{1}{at} \leq \frac{1}{a} - \frac{8}{20at^\delta}, \\ \frac{1}{a} - \frac{1}{at^\delta} - \frac{1}{at} \geq \frac{1}{2at}. \end{cases} \quad (3.14)$$

In view of (3.14), dividing $b_1^{-\beta}(\log b_1)^{-2\beta+2n-2}$ on the both sides of (3.13) and noting that we can let $\beta_2 \geq n-1$, i.e., $2\beta - 2n + 2 > 0$ for all $\beta \geq \beta_2$, we obtain that

$$
\begin{aligned}
& \int_{|x+\frac{b_4 x_0}{t}|<\frac{1}{at}} |w(x)|^2 dx \\
\leq\ & \int_{\frac{1}{2at}<|x|<b_1} |w(x)|^2 dx \\
\leq\ & C'' t^{p_1} (\log b_1)^2 (b_1/b_2)^\beta \int_{Y_1} |w|^2 dx \\
& + C'' t^{p_2} (b_1/b_3)^\beta (\log b_3)^2 [\log b_1/\log b_3]^{2\beta-2n+2} \int_{Z_1} |w|^2 dx \\
\leq\ & C'' t^{p_1} (\log b_1)^2 (8t)^\beta \int_{|x|<\frac{1}{at}} |w(x)|^2 dx \\
& + C'' t^{p_2} (\log b_3)^2 (b_1/b_5)^\beta \int_{Z_1} |w(x)|^2 dx, \qquad (3.15)
\end{aligned}
$$

where $b_4 = \frac{1}{a} - \frac{1}{at^\delta}$ and $b_5 = \frac{1}{a} - \frac{6}{20at^\delta}$. In deriving the third inequality above, we use the fact that if $a (= t_0^{\epsilon_0})$ is sufficiently large, then

$$0 \leq (\frac{b_5}{b_3})(\frac{\log b_1}{\log b_3})^2 \leq 1$$

for all $t \geq t_0$. From now on we fix a and hence t_0. It is helpful to remind that t_0 depends on λ_0 and λ. Having fixed constant a, $|\log b_1|$ and $|\log b_3|$ can be bounded by a positive constant. Thus, (3.15) is reduced to

$$
\begin{aligned}
\int_{|x+\frac{b_4 x_0}{t}|<\frac{1}{at}} |w(x)|^2 dx \ \leq\ & C t^{p_1} (8t)^\beta \int_{|x|<\frac{1}{at}} |w(x)|^2 dx \\
& + C t^{p_2} (b_1/b_5)^\beta \int_{Z_1} |w(x)|^2 dx, \qquad (3.16)
\end{aligned}
$$

where $C = C(\lambda_0, \lambda, \epsilon)$.

Using (3.16), (1.6), rescaling w back to v, and replacing β by $\beta - p_1$ (that is, taking $\beta \geq \beta_2 + p_1$), we have that

$$I(t^{1-\delta} y_0) \leq C (8t)^\beta I(t y_0) + C t^p \left(\frac{t^\delta}{t^\delta + 0.1} \right)^\beta, \qquad (3.17)$$

where $x_0 = t y_0$, $C = C(\lambda_0, \lambda, \kappa_1, \kappa_2, \kappa_3, \epsilon)$, and $p = p(\kappa_1, \kappa_2, \kappa_3, \alpha)$. For simplicity, by denoting

$$A(t) = \log 8t, \quad B(t) = \log \left(\frac{t^\delta + 0.1}{t^\delta} \right),$$

(3.17) becomes

$$I(t^{1-\delta}y_0) \le C\Big\{ \exp(\beta A(t)) I(ty_0) + t^p \exp(-\beta B(t)) \Big\}. \tag{3.18}$$

Now, we consider two cases. If

$$\exp(\beta_2 A(t)) I(ty_0) \ge t^p \exp(-\beta_2 B(t)),$$

then we have

$$I(x_0) = I(ty_0) \ge t^p \exp(-\beta_2(A(t)+B(t))) = t^p (8t)^{-\beta_2} \left(\frac{t^\delta + 0.1}{t^\delta}\right)^{-\beta_2},$$

that is

$$I(ty_0) \ge t^{-C\beta_2} = t^{-Ct^{\kappa_0+\kappa_s\delta}} \ge \exp(-Ct^{\kappa_0+\kappa_s\delta}\log t) \tag{3.19}$$

for all $t \ge t_0$, where $C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha)$. On the other hand, if

$$\exp(\beta_2 A(t)) I(ty_0) < t^p \exp(-\beta_2 B(t)),$$

then we can pick a $\tilde\beta > \beta_2$ such that

$$\exp(\tilde\beta A(t)) I(ty_0) = t^p \exp(-\tilde\beta B(t)). \tag{3.20}$$

Solving $\tilde\beta$ from (3.20) and using (3.18), we have that

$$\begin{aligned} I(t^{1-\delta}y_0) &\le C\exp(\tilde\beta A(t)) I(ty_0) \\ &= C\,(I(ty_0))^\tau\,(t^p)^{1-\tau} \\ &\le Ct^p\,(I(ty_0))^\tau, \end{aligned} \tag{3.21}$$

where $\tau = \frac{B(t)}{A(t)+B(t)}$. This estimate will serve as a building block in the bootstrapping step in the next section.

4 Proof of Theorem 1.1 – Part II: bootstrapping

In the previous section, we see that (3.19) gives us the desired estimate. However, we need to work harder to derive the wanted estimate from (3.21). We first observe that for $t^{\frac{2}{(\log\log t)^2}} \le \hat t \le t$ we have.

$$\begin{cases} \frac{1}{a} - \frac{1}{a\hat t^\delta} + \frac{1}{a\hat t} \le \frac{1}{a} - \frac{8}{20a\hat t^\delta}, \\ \frac{1}{a} - \frac{1}{a\hat t^\delta} - \frac{1}{a\hat t} \ge \frac{1}{2a\hat t}. \end{cases}$$

We now let $|x_0| = t$ with

$$t_0 \le t^{\frac{2}{(\log\log t)^2}}, \tag{4.1}$$

then we can write

$$t = \mu^{\left((1-\delta)^{-s}\right)} \tag{4.2}$$

for some positive integer s and

$$(t^{\frac{2}{(\log\log t)^2}})^{1-\delta} \le \mu \le t^{\frac{2}{(\log\log t)^2}}. \tag{4.3}$$

For simplicity, we define $d_j = \mu^{\left((1-\delta)^{-j}\right)}$ and $\tau_j = \frac{B(d_j)}{A(d_j)+B(d_j)}$ for $j = 1, 2\cdots s$. Define

$$\tilde{J} = \{1 \le j \le s : \exp(a^2 d_j^{\kappa_0+\kappa_s\delta} A(d_j)) I(d_j y_0) \ge (d_j)^p \exp(-a^2 d_j^{\kappa_0+\kappa_s\delta} B(d_j))\},$$

where $y_0 = x_0/t$ as before. Note that a is a fixed constant depending on λ_0, λ, ϵ. Now, we divide it into two cases. If $\tilde{J} = \emptyset$, we only need to consider (3.21). Using (3.21) iteratively starting from $t = d_1$, we obtain that

$$\begin{aligned} I(\mu y_0) &\le C(d_1^p)\,(I(d_1 y_0))^{\tau_1} \\ &\le C^s (d_1 d_2 \cdots d_s)^p\,(I(x_0))^{\tau_1\tau_2\cdots\tau_s}. \end{aligned} \tag{4.4}$$

It is easy to check that $s \le C \log t(\log\log\log t/(\log\log t)^2)$ for some absolute constant C. From (4.4) we have that

$$\begin{aligned} I(\mu y_0) &\le C^s t^{p/\delta}\,(I(x_0))^{\tau_1\tau_2\cdots\tau_s} \\ &\le t^{C\log t/(\log\log t)^2}\,(I(x_0))^{\tau_1\tau_2\cdots\tau_s}. \end{aligned} \tag{4.5}$$

Hereafter, $C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha)$, unless indicated otherwise. We now estimate

$$\frac{1}{\tau_j} = \frac{\log(8d_j) + \log(1 + 0.1d_j^{-\delta})}{\log(1 + 0.1d_j^{-\delta})} \le \frac{2\log(8d_j)}{\log(1 + 0.1d_j^{-\delta})} \le 40 d_j^{\delta} \log(d_j).$$

and thus

$$\begin{aligned} \frac{1}{\tau_1\tau_2\cdots\tau_s} &\le 40^s (\log t)^s (d_1 \cdots d_s)^{\delta} \\ &\le t\omega(t), \end{aligned} \tag{4.6}$$

where

$$\omega(t) = t^{\frac{C\log\log\log t}{\log\log t}}. \tag{4.7}$$

Raising both sides of (4.5) to the power $\frac{1}{\tau_1\tau_2\cdots\tau_s}$ and using (4.6), we obtain that

$$(\min\{I(\mu y_0),1\})^{t\omega(t)} \le t^{(C\log t/(\log\log t)^2)t\omega(t)} I(x_0),$$

i.e.,

$$I(x_0) \ge \exp(-Ct\omega(t))\,(\min\{I(\mu y_0),1\})^{t\omega(t)}\,. \tag{4.8}$$

Next, if $\tilde{J} \neq \emptyset$, let l be the largest integer in J. Estimate (3.19) implies

$$I(d_l y_0) \ge \exp(-Cd_l^{\kappa_0+\kappa_s\delta(d_l)}\log d_l) \ge \exp(-Ct^{\kappa_0+\kappa_s\delta(t)}\log t) \tag{4.9}$$

As in (4.4), iterating (3.21) starting from $t = d_{l+1}$ yields

$$\begin{aligned} I(d_l y_0) &\le C^{s-l}(d_{l+1}\cdots d_s)^p\,(I(x_0))^{\tau_{l+1}\cdots\tau_s} \\ &\le C^s(t/d_l)^{p/\delta}\,(I(x_0))^{\tau_{l+1}\cdots\tau_s} \\ &\le t^{C\log t/(\log\log t)^2}\,(I(x_0))^{\tau_{l+1}\cdots\tau_s}\,. \end{aligned} \tag{4.10}$$

It is enough to assume $I(d_l y_0) < 1$. Repeating the computations in (4.6), we can see that

$$\frac{1}{\tau_{l+1}\cdots\tau_s} \le (t/d_l)\omega(t). \tag{4.11}$$

Hence, combining (4.9), (4.10) and using (4.11), we get that

$$\exp(-Ct^{\kappa_0}\omega(t)) \le (I(x_0))\,. \tag{4.12}$$

Here $\omega(t)$ is given as in (4.7), but with $C = C(\lambda_0,\lambda,\epsilon,\kappa_1,\kappa_2,\kappa_3,\kappa_4,\alpha)$.

The last estimate (4.12) gives us the desired bound. We now focus on (4.8). In view of (4.3), if μ satisfies

$$(t^{\frac{2}{(\log\log t)^2}})^{1-\delta} \le \mu \le t_0,$$

then we are done. Note that $t_0^{\frac{1}{1+\epsilon}} \le (t^{\frac{2}{(\log\log t)^2}})^{1-\delta}$ due to (3.3) and (4.1). So we now consider $\mu > t_0$. We need another bootstrapping argument. Let us first rename $\tilde{t}_0 = t$, $\delta_1 = \delta(t)$, $s_1 = s$ (as in (4.2)). We then denote $\tilde{t}_1 = \mu$ and $\delta_2 = \delta_2(\tilde{t}_1) = \frac{(\log\log\tilde{t}_1)^2}{\log\tilde{t}_1}$. As before, we write

$$\tilde{t}_1 = \tilde{t}_2^{\left((1-\delta_2)^{-s_2}\right)}$$

for some positive integer s_2 and $(\tilde{t}_1^{\frac{2}{(\log\log \tilde{t}_1)^2}})^{1-\delta_2} \le \tilde{t}_2 \le \tilde{t}_1^{\frac{2}{(\log\log \tilde{t}_1)^2}}$. Inductively, we denote $\delta_k = \delta_k(\tilde{t}_{k-1}) = \frac{(\log\log(\tilde{t}_{k-1}))^2}{\log(\tilde{t}_{k-1})}$ and write

$$\tilde{t}_{k-1} = \tilde{t}_k^{\left((1-\delta_k)^{-s_k}\right)}$$

with positive constant s_k and $(\tilde{t}_{k-1}^{\frac{2}{(\log\log \tilde{t}_{k-1})^2}})^{1-\delta_k} \le \tilde{t}_k \le \tilde{t}_{k-1}^{\frac{2}{(\log\log \tilde{t}_{k-1})^2}}$ for $k = 1, 2, \cdots$. It is easily seen that there exists an m such that

$$t_0^{\frac{1}{1+\epsilon}} \le \tilde{t}_m \le t_0.$$

Indeed, we have $m \le \log t$.

Now we are ready to perform bootstrapping using either (4.8) or (4.12). It is enough to treat the case where we have (4.8) all the way until $\tilde{t}_m$, namely,

$$\begin{aligned} &I(x_0) \\ &\ge e^{-Ct\omega(t)} e^{-C\tilde{t}_1\omega(\tilde{t}_1)t\omega(t)} e^{-C\tilde{t}_2\omega(t_2)\tilde{t}_1\omega(\tilde{t}_1)t\omega(t)} \cdots e^{-C\tilde{t}_{m-1}\omega(\tilde{t}_{m-1})\cdots\tilde{t}_2\omega(t_2)\tilde{t}_1\omega(\tilde{t}_1)t\omega(t)} \\ &\quad \times \left(\min\{I(\tilde{t}_m y_0), 1\}\right)^{\tilde{t}_{m-1}\omega(\tilde{t}_{m-1})\cdots\tilde{t}_2\omega(t_2)\tilde{t}_1\omega(\tilde{t}_1)t\omega(t)}. \end{aligned} \tag{4.13}$$

In view of

$$\omega(\tilde{t}_k) \le \tilde{t}_k, \quad \tilde{t}_{k+1}^2 \le \tilde{t}_k \quad \text{for } 1 \le k \le (m-1),$$

we deduce that

$$\tilde{t}_{m-1}\omega(\tilde{t}_{m-1}) \cdots \tilde{t}_2\omega(t_2)\tilde{t}_1\omega(\tilde{t}_1)t\omega(t) \le \tilde{t}_1^4 t\omega(t).$$

Multiplying all terms in (4.13) implies

$$I(x_0) \ge \exp(-Ct\omega(t)), \tag{4.14}$$

where $C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha)$ and

$$\left|\log\left(\min\{\inf_{t_0^{\frac{1}{1+\epsilon}} < |x| < t_0} \int_{|y-x|<1} |v(y)|^2 dy, 1\}\right)\right|$$

and $\omega(t)$ is defined as in (4.7) with C depending on the same parameters.

On the other hand, we stop the bootstrapping process whenever (4.12) is satisfied. Similar computations give the following bound

$$I(x_0) \ge \exp(-C' t\omega(t)), \tag{4.15}$$

where $C' = C'(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha)$ and the constant C in $\omega(t)$ depends on $\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \alpha$. Notice that

$$\omega(t) = t^{\frac{C \log\log\log t}{\log\log t}} = (\log t)^{\frac{C(\log t)(\log\log\log t)}{(\log\log t)^2}}.$$

Therefore, (4.12) gives the estimate for $\kappa > 1$ and (4.12), (4.14), (4.15) lead to the estimate for $\kappa = 1$.

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