Carleman estimate for second order elliptic equations with Lipschitz leading coefficients and jumps at an interface

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Abstract

In this paper we prove a Carleman estimate for second order elliptic equations with a general anisotropic Lipschitz coefficients having a jump at an interface. Our approach does not rely on the techniques of microlocal analysis. We make use of the elementary method so that we are able to impose almost optimal assumptions on the coefficients and, consequently, the interface. It is possible that the framework can be applied to other cases.

Contents

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1 Introduction

Since T. Carleman's pioneer work [Car], Carleman estimates have been indispensable tools for proving the unique continuation property for partial differential equations. Recently, Carleman estimates have been successfully applied to study inverse problems, see for [Is], [KSU]. Most of Carleman estimates are proved under the assumption that the leading coefficients possess certain regularity. For example, for general second order elliptic operators, Carleman estimates were proved when the leading coefficients are at least Lipschitz [H], [H3]. The restriction of regularity on the leading coefficients also reflects the fact that the unique continuation may fail if the coefficients are only Hölder continuous in $\mathbb{R}^n$ with $n \geq 3$ (see examples constructed by Pliś [P] and [M]). In $\mathbb{R}^2$, the unique continuation property holds for $W^{1,2}$ solutions of second elliptic equations in either non-divergence or divergence forms with essentially bounded coefficients [BJS], [BN], [AM], [S]. It should be noted that the unique continuation property for the second order elliptic equations in the plane with essentially bounded coefficients is deduced from the theory of quasiregular mappings. No Carleman estimates are derived in this situation.

From discussions above, Carleman estimates for second order elliptic operators with general discontinuous coefficients are not likely to hold. However, when the discontinuities occur as jumps at an interface with homogeneous or non-homogeneous transmission conditions, one can still derive useful Carleman estimates. This is the main theme of the paper. There are some excellent works on this subject. We mention several closely related papers including Le Rousseau-Robbiano [LR1], [LR2], and Le Rousseau-Lerner [LL]. For the development of the problem and other related results, we refer the reader to the papers cited above and references therein. Our result is close to that of [LL], where the elliptic coefficient is a general anisotropic matrix-valued function. To put our paper in perspective, we would like to point out that the interface is assumed to be a C^∞ hypersurface in [LL] and the coefficients are C^∞ away from the interface. Here we prove a Carleman estimate near a flat interface from which it is easy to obtain under a standard change of coordinates a Carleman estimate for operator with leading coefficients which have a jump discontinuity at a $C^{1,1}$ interface and are Lipschitz continuous apart from such an interface (see Theorem 2.1 for a precise statement). The approach in [LL] is close to Calderón's seminal work on the uniqueness of Cauchy problem [Cal] as an application of singular integral operators (or pseudo-differential operators). Therefore, the regularity assumptions of [LL] are due to the use of calculus of pseudo-differential operators and the microlocal analysis techniques.

The aim here is to derive the Carleman estimate using more elementary methods. Our approach does not rely on the techniques of microlocal analysis, but rather on the straightforward Fourier transform. Thus we are able to relax the regularity assumptions on the coefficients and the interface. We first consider the simple case where the coefficients depend only on the normal variable. Taking advantage of the simple structure of coefficients, we are able to derive a Carleman estimate by elementary computations with the help of the Fourier transform on the tangential variables. To handle the general coefficients, we rely on some type of partition of unity. In Section 2 after the Theorem 2.1 we give a more detailed outline of our proof. Our approach is very general and is likely to be extended to other equations or systems.

2 Notations and statement of the main theorem

Define $H_\pm = \chi_{\mathbb{R}^n_\pm}$ where $\mathbb{R}^n_\pm = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} | y \gtrless 0\}$ and $\chi_{\mathbb{R}^n_\pm}$ is the characteristic function of $\mathbb{R}^n_\pm$. Let us stress that for a vector (x,y) of $\mathbb{R}^n$, we mean $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. In places we will use equivalently the symbols D, ∇, ∂ to denote the gradient of a function and we will add the index x or y to denote gradient in $\mathbb{R}^{n-1}$ and the derivative with respect to y respectively.

Let $u_\pm \in C^\infty(\mathbb{R}^n)$. We define

$$u = H_+ u_+ + H_- u_- = \sum_\pm H_\pm u_\pm,$$

hereafter, we denote $\sum_\pm a_\pm = a_+ + a_-$, and for $\mathbb{R}^{n-1} \times \mathbb{R}$

$$\mathcal{L}(x,y,\partial)u := \sum_\pm H_\pm \text{div}_{x,y}(A_\pm(x,y)\nabla_{x,y} u_\pm), \tag{2.1}$$

where

$$A_\pm(x,y) = \{a^\pm_{ij}(x,y)\}^n_{i,j=1}, \quad x \in \mathbb{R}^{n-1}, y \in \mathbb{R} \tag{2.2}$$

is a Lipschitz symmetric matrix-valued function satisfying, for given constants $\lambda_0 \in (0,1]$, $M_0 > 0$,

$$\lambda_0 |z|^2 \le A_\pm(x,y) z \cdot z \le \lambda_0^{-1} |z|^2, \, \forall (x,y) \in \mathbb{R}^n, \, \forall \, z \in \mathbb{R}^n \tag{2.3}$$

and

$$|A_\pm(x',y') - A_\pm(x,y)| \le M_0(|x'-x| + |y'-y|). \tag{2.4}$$

We write

$$h_0(x) := u_+(x,0) - u_-(x,0), \; \forall \, x \in \mathbb{R}^{n-1}, \tag{2.5}$$

$$h_1(x) := A_+(x,0)\nabla_{x,y} u_+(x,0) \cdot \nu - A_-(x,0)\nabla_{x,y} u_-(x,0) \cdot \nu, \; \forall \, x \in \mathbb{R}^{n-1}, \tag{2.6}$$

where $\nu = -e_n$.

Let us now introduce the weight function. Let φ be

$$\varphi(y) = \begin{cases} \varphi_+(y) := \alpha_+ y + \beta y^2/2, & y \ge 0, \\ \varphi_-(y) := \alpha_- y + \beta y^2/2, & y < 0, \end{cases} \tag{2.7}$$

where α_+, α_- and β are positive numbers which will be determined later. In what follows we denote by φ_+ and φ_- the restriction of the weight function φ to $[0, +\infty)$

and to $(-\infty, 0)$ respectively. We use similar notation for any other weight functions. For any $\varepsilon > 0$ let

$$\psi_\varepsilon(x,y) := \varphi(y) - \frac{\varepsilon}{2}|x|^2,$$

and let

$$\phi_\delta(x,y) := \psi_\delta\left(\delta^{-1}x, \delta^{-1}y\right). \tag{2.8}$$

For a function $h \in L^2(\mathbb{R}^n)$, we define

$$\hat{h}(\xi, y) = \int_{\mathbb{R}^{n-1}} h(x,y)e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^{n-1}.$$

As usual we denote by $H^{1/2}(\mathbb{R}^{n-1})$ the space of the functions $f \in L^2(\mathbb{R}^{n-1})$ satisfying

$$\int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi < \infty,$$

with the norm

$$||f||^2_{H^{1/2}(\mathbb{R}^{n-1})} = \int_{\mathbb{R}^{n-1}} (1+|\xi|^2)^{1/2}|\hat{f}(\xi)|^2 d\xi. \tag{2.9}$$

Moreover we define

$$[f]_{1/2,\mathbb{R}^{n-1}} = \left[\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|f(x)-f(y)|^2}{|x-y|^n} dydx\right]^{1/2},$$

and recall that there is a positive constant C, depending only on n, such that

$$C^{-1}\int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi \le [f]^2_{1/2,\mathbb{R}^{n-1}} \le C\int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi,$$

so that the norm (2.9) is equivalent to the norm $||f||_{L^2(\mathbb{R}^{n-1})} + [f]_{1/2,\mathbb{R}^{n-1}}$. We use the letters $C, C_0, C_1, \cdots$ to denote constants. The value of the constants may change from line to line, but it is always greater than 1.

We will denote by $B_r(x)$ the $(n-1)$-ball centered at $x \in \mathbb{R}^{n-1}$ with radius $r > 0$. Whenever $x = 0$ we denote $B_r = B_r(0)$.

Theorem 2.1 *Let u and $A_\pm(x,y)$ satisfy (2.1)-(2.6). There exist $\alpha_+, \alpha_-, \beta, \delta_0, r_0$ and C depending on λ_0, M_0 such that if $\delta \le \delta_0$ and $\tau \ge C$, then*

$$\begin{aligned}
&\sum_\pm\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^k u_\pm|^2 e^{2\tau\phi_{\delta,\pm}(x,y)}dxdy + \sum_\pm\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^k u_\pm(x,0)|^2 e^{2\phi_\delta(x,0)}dx \\
&+\sum_\pm \tau^2[e^{\tau\phi_\delta(\cdot,0)}u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_\pm [D(e^{\tau\phi_{\delta,\pm}}u_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\
\le& C\Bigg(\sum_\pm\int_{\mathbb{R}^n_\pm}|\mathcal{L}(x,y,\partial)(u_\pm)|^2\, e^{2\tau\phi_{\delta,\pm}(x,y)}dxdy + [e^{\tau\phi_\delta(\cdot,0)}h_1]^2_{1/2,\mathbb{R}^{n-1}} \\
&+[\nabla_x(e^{\tau\phi_\delta}h_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \tau^3\int_{\mathbb{R}^{n-1}}|h_0|^2e^{2\tau\phi_\delta(x,0)}dx + \tau\int_{\mathbb{R}^{n-1}}|h_1|^2e^{2\tau\phi_\delta(x,0)}dx\Bigg).
\end{aligned} \tag{2.10}$$

where $u = H_+u_+ + H_-u_-$, $u_\pm \in C^\infty(\mathbb{R}^n)$ *and* $\operatorname{supp} u \subset B_{\delta/2} \times [-\delta r_0, \delta r_0]$, *and* ϕ_δ *is given by* (2.8)*.*

Remark 2.2 *Estimate* (2.10) *is a local Carleman estimate near* $x_n = 0$*. As mentioned in the Introduction, by an easy change of coordinates, one can derive a local Carleman estimate near a* $C^{1,1}$ *interface from* (2.10)*.*

Remark 2.3 *Let us point out that the level sets*

$$\left\{(x,y) \in B_{\delta/2} \times (-\delta r_0, \delta r_0)) : \phi_\delta(x,y) = t\right\}$$

have approximately the shape of paraboloid and, in a neighborhood of $(0,0)$, $\partial_y \phi_\delta > 0$ *so that the gradient of* ϕ *points inward the halfspace* $\mathbb{R}^n_+$*. These features are crucial to derive from the Carleman estimate* (2.10) *a Hölder type smallness propagation estimate across the interface* $\{(x,0) : x \in \mathbb{R}^{n-1}\}$ *for weak solutions to the transmission problem*

$$\begin{cases} \mathcal{L}(x,y,\partial)u = \sum_\pm H_\pm b_\pm \cdot \nabla_{x,y} u_\pm + c_\pm u_\pm, \\ u_+(x,0) - u_+(x,0) = 0, \\ A_+(x,0)\nabla_{x,y}u_+(x,0) \cdot \nu - A_-(x,0)\nabla_{x,y}u_-(x,0) \cdot \nu = 0, \end{cases} \tag{2.11}$$

where $b_\pm \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *and* $c_\pm \in L^\infty(\mathbb{R}^n)$*. More precisely if the error of observation of* u *is known in an open set of* $\mathbb{R}^n_+$*, we can find a Hölder control of* u *in a bounded set of* $\mathbb{R}^n_-$*. For more details about such type of estimate we refer to [LR1, Sect. 3.1].*

The proof of Theorem 2.1 is divided into two steps as follows.

Step 1. We first consider the particular case of the leading matrices (2.2) independent of x and we prove (Theorem 3.1), for the corresponding operator $\mathcal{L}(y, \partial)$, a Carleman estimate with the weight function $\phi(x,y) = \varphi(y) + s\gamma \cdot x$, where s is a suitable small number and γ is an arbitrary unit vector of $\mathbb{R}^{n-1}$. The features of the leading matrices and of the weight function ϕ allow to factorize the Fourier transform of the conjugate of the operator $\mathcal{L}(y, \partial)u$ with respect to ϕ. So that we can follow, roughly speaking, at an elementary level the strategy of [LL] for the operator $\mathcal{L}(y, \partial)$. Nevertheless such an estimate has only a prepatory character to prove Theorem 2.1, because, due to the particular feature of the weight ϕ (i.e. linear with respect to x), the Carleman estimate obtained in Theorem 3.1 cannot yield to any kind of significant smallness propagation estimate across the interface.

Step 2. In the second we adapt the method described in [Tr, Ch. 4.1] to an operator with jump discontinuity. More precisely, we localize the operator (2.1) with respect to the x variable and we linearize the weight function, again with respect the x variable, and by the Carleman estimate obtained in the Step 1 we derive some local Carleman estimates. Subsequently we put together such local estimates by mean of the unity partition introduced in [Tr].

3 Step 1 - A Carleman estimate for leading coefficients depending on y only

In this section we consider the simple case of the leading matrices (2.2) independent of x. Moreover, the weight function that we consider is linear with respect to x variable, so that, as explained above, the Carleman estimates we get here are only preliminary to the one that we will get in the general case.

Assume that

$$A_{\pm}(y) = \{a_{ij}^{\pm}(y)\}_{i,j=1}^{n} \tag{3.1}$$

are symmetric matrix-valued functions satisfying (2.3) and (2.4), i.e.,

$$\lambda_0|z|^2 \leq A_{\pm}(y)z \cdot z \leq \lambda_0^{-1}|z|^2, \forall y \in \mathbb{R}, \forall\, z \in \mathbb{R}^n \tag{3.2}$$

$$|A_{\pm}(y') - A_{\pm}(y'')| \leq M_0|y' - y''|, \quad \forall\, y', y'' \in \mathbb{R}. \tag{3.3}$$

From (3.2), we have

$$a_{nn}^{\pm}(y) \geq \lambda_0 \quad \forall\, y \in \mathbb{R}. \tag{3.4}$$

In the present case the the differential operator (2.1) became

$$\mathcal{L}(y, \partial)u := \sum_{\pm} H_{\pm}\mathrm{div}_{x,y}(A_{\pm}(y)\nabla_{x,y}u_{\pm}), \tag{3.5}$$

where $u = \sum_{\pm} H_{\pm}u_{\pm}$, $u_{\pm} \in C^{\infty}(\mathbb{R}^n)$

We also set, for any $s \in [0, 1]$ and $\gamma \in \mathbb{R}^{n-1}$ with $|\gamma| \leq 1$

$$\phi(x, y) = \varphi(y) + s\gamma \cdot x = H_+\phi_+ + H_-\phi_-, \tag{3.6}$$

where φ is defined in (2.7).

Our aim here is to prove the following Carleman estimate.

Theorem 3.1 *There exist τ_0, s_0, r_0, C and β_0 depending only on λ_0, M_0, such that for $\tau \geq \tau_0$, $0 < s \leq s_0 < 1$, and for every $w = \sum_{\pm} H_{\pm}w_{\pm}$ with* $\mathrm{supp}\, w \subset$

$B_1 \times [-r_0, r_0]$, *we have that*

$$
\begin{aligned}
&\sum_{\pm}\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^k w_\pm|^2 e^{2\tau\phi_\pm}dxdy\\
&+\sum_{\pm}\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^k w(x,0)|^2 e^{2\tau\phi(x,0)}dx+\sum_{\pm}\tau^2[(e^{\tau\phi}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\
&+\sum_{\pm}[\partial_y(e^{\tau\phi_\pm}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\sum_{\pm}[\nabla_x(e^{\tau\phi}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\
\leq &C\Bigg(\int_{\mathbb{R}^n_\pm}|\mathcal{L}(y,\partial)w|^2 e^{2\tau\phi_\pm}dxdy+[\nabla_x\big(e^{\tau\phi(\cdot,0)}(w_+(\cdot,0)-w_-(\cdot,0))\big)]^2_{1/2,\mathbb{R}^{n-1}}\\
&+[e^{\tau\phi(\cdot,0)}(A_+(0)\nabla_{x,y}w_+(x,0)\cdot\nu-A_-(0)\nabla_{x,y}w_-(x,0)\cdot\nu)]^2_{1/2,\mathbb{R}^{n-1}}\\
&+\tau\int_{\mathbb{R}^{n-1}}e^{2\tau\phi(x,0)}|A_+(0)\nabla_{x,y}w_+(x,0)\cdot\nu-A_-(0)\nabla_{x,y}w_-(x,0)\cdot\nu|^2dx\\
&+\tau^3\int_{\mathbb{R}^{n-1}}e^{2\tau\phi(x,0)}|w_+(x,0)-w_-(x,0)|^2dx\Bigg),
\end{aligned}
\tag{3.7}
$$

with $\beta \geq \beta_0$ *and* $\alpha_\pm$ *properly chosen.*

3.1 Fourier transform of the conjugate operator and its factorization

To proceed further, we introduce some operators and find their properties. We use the notation $\partial_j = \partial_{x_j}$ for $1 \leq j \leq n-1$. Let us denote $B_\pm(y) = \{b^\pm_{jk}(y)\}_{j,k=1}^{n-1}$, the symmetric matrix satisfying, for $z = (z_1, \cdots, z_{n-1}, z_n) =: (z', z_n)$,

$$
B_\pm(y)z'\cdot z' = A_\pm(y)z\cdot z\ \Big|_{z_n=-\sum_{j=1}^{n-1}\frac{a^\pm_{nj}(y)z_j}{a^\pm_{nn}(y)}}\ . \tag{3.8}
$$

In view of (3.2) we have

$$
\lambda_1|z'|^2 \leq B_\pm(y)z'\cdot z' \leq \lambda_1^{-1}|z'|^2, \quad \forall\, y\in\mathbb{R},\ \forall\, z'\in\mathbb{R}^{n-1}, \tag{3.9}
$$

$\lambda_1 \leq \lambda_0$ depends only on λ_0.

Notice that

$$
b^\pm_{jk}(y) = a^\pm_{jk}(y) - \frac{a^\pm_{nj}(y)a^\pm_{nk}(y)}{a^\pm_{nn}(y)}, \quad j,k = 1,\cdots,n-1. \tag{3.10}
$$

We define the operator

$$
T_\pm(y,\partial_x)u_\pm := \sum_{j=1}^{n-1}\frac{a^\pm_{nj}(y)}{a^\pm_{nn}(y)}\partial_j u_\pm. \tag{3.11}
$$

It is easy to show, by direct calculations ([LL]), that

$$\operatorname{div}_{x,y}\big(A_\pm(y)\nabla_{x,y}u_\pm\big) = (\partial_y + T_\pm)a_{nn}^\pm(y)(\partial_y + T_\pm)u_\pm + \operatorname{div}_x\big(B_\pm(y)\nabla_x u_\pm\big). \quad (3.12)$$

Let $w = \sum_\pm H_\pm w_\pm$, where $w_\pm \in C^\infty(\mathbb{R}^n)$, and define

$$\theta_0(x) := w_+(x,0) - w_-(x,0) \quad \text{for } x \in \mathbb{R}^{n-1}, \quad (3.13)$$

$$\theta_1(x) := A_+(0)\nabla_{x,y}w_+(x,0)\cdot\nu - A_-(0)\nabla_{x,y}w_-(x,0)\cdot\nu \quad \text{for } x \in \mathbb{R}^{n-1}, \quad (3.14)$$

where $\nu = -e_n$. By straightforward calculations, we get

$$a_{nn}^+(y)\big(\partial_y + T_+(y,\partial_x)\big)w_+(x,y)\mid_{y=0} - a_{nn}^-(y)\big(\partial_y + T_-(y,\partial_x)\big)w_-(x,y)\mid_{y=0} = -\theta_1(x). \quad (3.15)$$

In order to derive the Carleman estimate (3.7), we investigate the conjugate operator of $\mathcal{L}(y,\partial)$ with $e^{\tau\phi}$ for ϕ given by (3.6). Let $v = e^{\tau\phi}w$ and $\tilde v = e^{-\tau s\gamma\cdot x}v$, then we have

$$w = e^{-\tau\phi}v = \sum_\pm H_\pm e^{-\tau\phi_\pm}v_\pm = \sum_\pm H_\pm e^{-\tau\varphi_\pm}\tilde v_\pm$$

and therefore

$$e^{\tau\phi}\mathcal{L}(y,\partial)(e^{-\tau\phi}v) = e^{\tau s\gamma\cdot x}e^{\tau\varphi}\mathcal{L}(y,\partial)(e^{-\tau\varphi}\tilde v).$$

It follows from (3.12) that

$$\begin{aligned} e^{\tau\varphi}\mathcal{L}(y,\partial)(e^{-\tau\varphi}\tilde v) =& \sum_\pm H_\pm\big[(\partial_y - \tau\varphi_\pm' + T_\pm)a_{nn}^\pm(y)(\partial_y - \tau\varphi_\pm' + T_\pm)\tilde v_\pm\big] \\ &+ \sum_\pm H_\pm \operatorname{div}_x\big(B_\pm(y)\nabla_x\tilde v_\pm\big), \end{aligned}$$

which leads to

$$\begin{aligned} & e^{\tau\phi}\mathcal{L}(y,\partial)(e^{-\tau\phi}v) = e^{\tau s\gamma\cdot x}e^{\tau\varphi}\mathcal{L}(y,\partial)(e^{-\tau\varphi}\tilde v) \\ & = e^{\tau s\gamma\cdot x}\sum_\pm H_\pm\big[(\partial_y - \tau\varphi_\pm' + T_\pm)a_{nn}^\pm(y)(\partial_y - \tau\varphi_\pm' + T_\pm)(e^{-\tau s\gamma\cdot x}v_\pm)\big] \\ & + e^{\tau s\gamma\cdot x}\sum_\pm H_\pm\operatorname{div}_x\big(B_\pm(y)\nabla_x(e^{-\tau s\gamma\cdot x}v_\pm)\big). \end{aligned} \quad (3.16)$$

By the definition of $T_\pm(y,\partial_x)$, we get that

$$\begin{aligned} T_\pm(y,\partial_x)(e^{-\tau s\gamma\cdot x}v_\pm) &= e^{-\tau s\gamma\cdot x}\sum_{j=1}^{n-1}\frac{a_{nj}^\pm(y)}{a_{nn}^\pm(y)}(\partial_j v_\pm - \tau s\gamma_j v_\pm) \\ &:= e^{-\tau s\gamma\cdot x}\,T_\pm(y,\partial_x - \tau s\gamma)v_\pm. \end{aligned}$$

To continue the computation, we observe that

$$\begin{aligned} & e^{\tau s\gamma\cdot x}\big[(\partial_y - \tau\varphi_\pm' + T_\pm(y,\partial_x))a_{nn}^\pm(y)(\partial_y - \tau\varphi_\pm' + T_\pm(y,\partial_x))(e^{-\tau s\gamma\cdot x}v_\pm)\big] \\ =& \big(\partial_y - \tau\varphi_\pm' + T_\pm(y,\partial_x - \tau s\gamma)\big)a_{nn}^\pm(y)\big(\partial_y - \tau\varphi_\pm' + T_\pm(y,\partial_x - \tau s\gamma)\big)v_\pm \end{aligned} \quad (3.17)$$

and

$$
\begin{aligned}
&e^{\tau s\gamma\cdot x}\mathrm{div}_x\big(B_\pm(y)\nabla_x(e^{-\tau s\gamma\cdot x}v_\pm)\big)\\
&\quad=\sum_{j,k=1}^{n-1} b_{jk}^\pm(y)\partial_{jk}^2 v_\pm - 2s\tau\sum_{j,k=1}^{n-1} b_{jk}^\pm(y)\partial_j v_\pm\gamma_k + s^2\tau^2\sum_{j,k=1}^{n-1} b_{jk}^\pm(y)\gamma_j\gamma_k v_\pm.
\end{aligned}
\tag{3.18}
$$

Combining (3.16), (3.17) and (3.18) yields

$$
\begin{aligned}
&e^{\tau\phi}\mathcal{L}(y,\partial)(e^{-\tau\phi}v)\\
=&\sum_\pm H_\pm\big[\big(\partial_y-\tau\varphi_\pm'+T_\pm(y,\partial_x-\tau s\gamma)\big)a_{nn}^\pm(y)\big(\partial_y-\tau\varphi_\pm'+T_\pm(y,\partial_x-\tau s\gamma)\big)v_\pm\big]\\
&+\sum_\pm H_\pm\big[\sum_{j,k=1}^{n-1} b_{jk}^\pm(y)\partial_{jk}^2 v_\pm - 2s\tau\sum_{j,k=1}^{n-1} b_{jk}^\pm(y)\partial_j v_\pm\gamma_k + s^2\tau^2\sum_{j,k=1}^{n-1} b_{jk}^\pm(y)\gamma_j\gamma_k v_\pm\big].
\end{aligned}
\tag{3.19}
$$

We now focus on the analysis of $e^{\tau\phi}\mathcal{L}(y,\partial)(e^{-\tau\phi}v)$. To simplify it, we introduce some notations:

$$
f(x,y)=e^{\tau\phi}\mathcal{L}(y,\partial)(e^{-\tau\phi}v),
\tag{3.20}
$$

$$
B_\pm(\xi,\gamma,y)=\sum_{j,k=1}^{n-1} b_{jk}^\pm(y)\xi_j\gamma_k,\quad \xi\in\mathbb{R}^{n-1},
\tag{3.21}
$$

$$
\zeta_\pm(\xi,y)=\frac{1}{a_{nn}^\pm(y)}\big[B_\pm(\xi,\xi,y)+2is\tau B_\pm(\xi,\gamma,y)-s^2\tau^2 B_\pm(\gamma,\gamma,y)\big],
\tag{3.22}
$$

and

$$
t_\pm(\xi,y)=\sum_{j=1}^{n-1}\frac{a_{nj}^\pm(y)}{a_{nn}^\pm(y)}\xi_j.
\tag{3.23}
$$

By (3.19), we have

$$
\hat f(\xi,y)=\sum_\pm H_\pm P_\pm\hat v_\pm,
\tag{3.24}
$$

where

$$
\begin{aligned}
P_\pm\hat v_\pm:=&\big(\partial_y-\tau\varphi_\pm'+it_\pm(\xi+i\tau s\gamma,y)\big)a_{nn}^\pm(y)\big(\partial_y-\tau\varphi_\pm'+it_\pm(\xi+i\tau s\gamma,y)\big)\hat v_\pm\\
&-a_{nn}^\pm(y)\zeta_\pm(\xi,y)\hat v_\pm.
\end{aligned}
\tag{3.25}
$$

Our aim is to estimate $f(x,y)$ or, equivalently, its Fourier transform $\hat f(\xi,y)$. In order to do this, we want to factorize the operators $P_\pm$. For any $z=a+ib$ with $(a,b)\neq(0,0)$, we define the square root of z,

$$
\sqrt{z}=\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}+i\frac{b}{\sqrt{2(a+\sqrt{a^2+b^2})}}.
$$

We remind that the square root $\sqrt{z}$ is defined with a cut along the negative real axis and note that $\Re\sqrt{z} \geq 0$. Thus, it needs extra care to estimate its derivative. Now we define two operators

$$E_\pm = \partial_y + it_\pm(\xi + i\tau s\gamma, y) - (\tau\varphi'_\pm + \sqrt{\zeta_\pm}), \tag{3.26}$$

$$F_\pm = \partial_y + it_\pm(\xi + i\tau s\gamma, y) - (\tau\varphi'_\pm - \sqrt{\zeta_\pm}). \tag{3.27}$$

With all the definitions given above, we obtain that

$$P_+\hat{v}_+ = E_+ a^+_{nn}(y) F_+\hat{v}_+ - \hat{v}_+\partial_y\big(a^+_{nn}(y)\sqrt{\zeta_+}\big), \tag{3.28}$$

$$P_-\hat{v}_- = F_- a^-_{nn}(y) E_-\hat{v}_- + \hat{v}_-\partial_y\big(a^-_{nn}(y)\sqrt{\zeta_-}\big). \tag{3.29}$$

Let us now introduce some other useful notations and estimates that will be intensively used in the sequel. After taking the Fourier transform, the terms on the interface (3.13) and (3.15), become

$$\eta_0(\xi) := \hat{v}_+(\xi,0) - \hat{v}_-(\xi,0) = \widehat{e^{\tau\phi(x,0)}\theta_0}(x) \tag{3.30}$$

and

$$\begin{aligned}\eta_1(\xi) &:= -\widehat{e^{\tau\phi(x,0)}\theta_1}(x)\\ &= a^+_{nn}(0)\big[\partial_y\hat{v}_+(\xi,0) - \tau\alpha_+\hat{v}_+(\xi,0) + it_+(\xi + i\tau s\gamma,0)\hat{v}_+(\xi,0)\big]\\ &\quad - a^-_{nn}(0)\big[\partial_y\hat{v}_-(\xi,0) - \tau\alpha_-\hat{v}_-(\xi,0) + it_-(\xi + i\tau s\gamma,0)\hat{v}_-(\xi,0)\big].\end{aligned} \tag{3.31}$$

For simplicity, we denote

$$V_\pm(\xi) := a^\pm_{nn}(0)\big[\partial_y\hat{v}_\pm(\xi,0) - \tau\alpha_\pm\hat{v}_\pm(\xi,0) + it_\pm(\xi + i\tau s\gamma,0)\hat{v}_\pm(\xi,0)\big], \tag{3.32}$$

so that

$$V_+(\xi) - V_-(\xi) = \eta_1(\xi). \tag{3.33}$$

Moreover, we define

$$m_\pm(\xi,y) := \sqrt{\frac{B_\pm(\xi,\xi,y)}{a^\pm_{nn}(y)}}.$$

From (3.9) we have

$$\lambda_1|\xi|^2 \leq B_\pm(\xi,\xi,y) \leq \lambda_1^{-1}|\xi|^2, \quad \forall\, y \in \mathbb{R},\, \forall\, \xi \in \mathbb{R}^{n-1}, \tag{3.34}$$

and, from (3.3),

$$|\partial_y B_\pm(\xi,\eta,y)| \leq M_1|\xi||\eta|, \quad \forall\, \xi,\, \eta \in \mathbb{R}^{n-1}, \tag{3.35}$$

where M_1 depends only on λ_0 and M_0. In a similar way, we list here some useful bounds, that can be easily obtained from (3.9) and (3.3).

$$\lambda_2|\xi| \leq m_\pm(\xi,y) \leq \lambda_2^{-1}|\xi|, \tag{3.36}$$

$$|\partial_y m_\pm(\xi, y)| \leq M_2|\xi|, \tag{3.37}$$
$$|t_\pm(\xi, y)| \leq \lambda_3^{-1}|\xi|, \tag{3.38}$$
$$|\partial_y t_\pm(\xi, y)| \leq M_3|\xi|, \tag{3.39}$$
$$|\zeta_\pm(\xi, y)| \leq (\lambda_0\lambda_1)^{-1}(|\xi|^2 + s^2\tau^2), \tag{3.40}$$
$$|\partial_y \zeta_\pm(\xi, y)| \leq M_4(|\xi|^2 + s^2\tau^2). \tag{3.41}$$

Here $\lambda_2 = \sqrt{\lambda_0\lambda_1}$, λ_3 depends only on λ_0, while M_2, M_3 and M_4 depends only on λ_0 and M_0.

3.2 Derivation of the Carleman estimate for the simple case

The derivation of the Carleman estimate (3.7) is a simple consequence of the auxiliary Proposition 3.1 stated below and proved in the following Section 3.3 via the inverse Fourier transform. We first set

$$L := \sup_{\xi \in \mathbb{R}^{n-1}\setminus\{0\}} \frac{m_+(\xi, 0)}{m_-(\xi, 0)}.$$

Note that, by (3.36), $\lambda_2^2 \leq L \leq \lambda_2^{-2}$. Now we introduce the fundamental assumption on the coefficients $\alpha_\pm$ in the weight function. As in [LL], we choose positive α_+ and α_-, such that

$$L < \frac{\alpha_+}{\alpha_-}. \tag{3.42}$$

This choice will only be conditioned by λ_0. These constants will be fixed. Denote the factor

$$\Lambda = (|\xi|^2 + \tau^2)^{1/2}.$$

We now state our main tool.

Proposition 3.1 *There exist τ_0, s_0, ρ, β and C, depending only on λ_0 and M_0, such that for $\tau \geq \tau_0$, $\operatorname{supp} \hat{v}_\pm(\xi, \cdot) \subset [-\rho, \rho]$, $s \leq s_0 < 1$, we have*

$$\begin{aligned}
&\frac{1}{\tau}\sum_\pm ||\partial_y^2 \hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^2}{\tau}\sum_\pm ||\partial_y \hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} \\
&\quad + \frac{\Lambda^4}{\tau}\sum_\pm ||\hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda \sum_\pm |V_\pm(\xi)|^2 + \Lambda^3 \sum_\pm |\hat{v}_\pm(\xi, 0)|^2 \\
&\quad \leq C\left(\sum_\pm ||P_\pm \hat{v}_\pm(\xi, \cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2\right).
\end{aligned} \tag{3.43}$$

Here $\mathbb{R}_\pm = \{y \in \mathbb{R} \,:\, y \gtrless 0\}$.

Proof of Theorem 3.1. Substituting (3.24) and the definitions of η_0, η_1 (see (3.30), (3.31)) into the right hand side of (3.43) implies

$$\begin{aligned}&\frac{1}{\tau}\sum_{\pm}||\partial_y^2\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}+\frac{\Lambda^2}{\tau}\sum_{\pm}||\partial_y\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}+\frac{\Lambda^4}{\tau}\sum_{\pm}||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}\\&+\Lambda\sum_{\pm}|V_\pm(\xi)|^2+\Lambda^3\sum_{\pm}|\hat{v}_\pm(\xi,0)|^2\\\le&C\left(\sum_{\pm}||f(\xi,\cdot)||^2_{L^2(\mathbb{R})}+\Lambda|\widehat{e^{\tau\phi(\cdot,0)}\theta_1}(\cdot)|^2+\Lambda^3|\widehat{e^{\tau\phi(\cdot,0)}\theta_0}(\cdot)|^2\right).\end{aligned}\tag{3.44}$$

Recalling (3.32), it is not hard to see that

$$\Lambda\sum_{\pm}|\partial_y\hat{v}_\pm(\xi,0)|^2\le C\left(\Lambda\sum_{\pm}|V_\pm(\xi)|^2+\Lambda^3\sum_{\pm}|\hat{v}_\pm(\xi,0)|^2\right).\tag{3.45}$$

Since $\Lambda^4\ge|\xi|^2\tau^2+|\xi|^4+\tau^4$, $|\xi|^3+|\xi|^2\tau+|\xi|\tau^2+\tau^3\le C\Lambda^3$, and $\Lambda^3\le C'(|\xi|^3+\tau^3)$, by integrating in ξ, we can deduce from (3.44) and (3.45) that

$$\begin{aligned}&\sum_{\pm}\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^kv_\pm|^2+\sum_{\pm}[\nabla_xv_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\sum_{\pm}[\partial_yv_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\&+\sum_{\pm}\tau^2[v_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\sum_{\pm}\tau\int_{\mathbb{R}^{n-1}}|\nabla_xv_\pm(x,0)|^2dx\\&+\sum_{\pm}\tau\int_{\mathbb{R}^{n-1}}|\partial_yv_\pm(x,0)|^2dx+\sum_{\pm}\tau^3\int_{\mathbb{R}^{n-1}}|v_\pm(x,0)|^2dx\\\le&C\Big(||f||^2_{L^2(\mathbb{R}^n)}+[e^{\tau\phi(\cdot,0)}\theta_1(\cdot)]^2_{1/2,\mathbb{R}^{n-1}}+[\nabla_x\big(e^{\tau\phi(\cdot,0)}\theta_0(\cdot)\big)]^2_{1/2,\mathbb{R}^{n-1}}\\&+\tau\int_{\mathbb{R}^{n-1}}e^{2\tau\phi(x,0)}|\theta_1|^2dx+\tau^3\int_{\mathbb{R}^{n-1}}e^{2\tau\phi(x,0)}|\theta_0|^2dx\Big).\end{aligned}\tag{3.46}$$

Replacing $v_\pm=e^{\tau\phi_\pm}w_\pm$ into (3.46) immediately leads to (3.7). □

3.3 Proof of Proposition 3.1

Let κ be the positive number

$$\kappa=\frac{1}{2}\left(1-L\frac{\alpha_-}{\alpha_+}\right)\tag{3.47}$$

depending only on λ_0 and M_0. The proof of Proposition 3.1 will be divided into three cases

$$\begin{cases} \tau \geq \dfrac{\lambda_2^2|\xi|}{2s_0}, \\ \dfrac{m_+(\xi,0)}{(1-\kappa)\,\alpha_+} \leq \tau \leq \dfrac{\lambda_2^2|\xi|}{2s_0}, \\ \tau \leq \dfrac{m_+(\xi,0)}{(1-\kappa)\,\alpha_+}. \end{cases}$$

Recall that $\lambda_2 = \sqrt{\lambda_0\lambda_1}$ (from (3.36)) depends only on λ_0. Of course, we first choose a small $s_0 < 1$, depending on λ_0 and M_0 only, such that

$$\frac{m_+(\xi,0)}{(1-\kappa)\,\alpha_+} \leq \frac{\lambda_2^2|\xi|}{2s_0}, \quad \forall\, \xi \in \mathbb{R}^n.$$

A smaller value s_0 will be chosen later in the proof.

We need to introduce here some further notations. First of all, let us denote by

$$P_\pm^0,\ E_\pm^0,\ \text{and}\ F_\pm^0$$

the operators defined by (3.25), (3.26) and (3.27), respectively, in the special case $s = 0$. We also give special names to these functions that will be used in the proof:

$$\omega_+(\xi,y) = a_{nn}^+(y)F_+\hat{v}_+(\xi,y), \quad \omega_-(\xi,y) = a_{nn}^-(y)E_-\hat{v}_-(\xi,y) \tag{3.48}$$

and, for the special case $s = 0$,

$$\omega_+^0(\xi,y) = a_{nn}^+(y)F_+^0\hat{v}_+(\xi,y), \quad \omega_-^0(\xi,y) = a_{nn}^-(y)E_-^0\hat{v}_-(\xi,y). \tag{3.49}$$

Case 1:

$$\tau \geq \frac{\lambda_2^2|\xi|}{2s_0} \tag{3.50}$$

Note that, in this case, we have $|\xi| \leq 2\lambda_2^{-2}s_0\tau$, which implies

$$\tau \leq \Lambda \leq \sqrt{5}\lambda_2^{-2}\tau. \tag{3.51}$$

We will need several lemmas. In the first one, we estimate the difference $P_\pm\hat{v}_\pm - P_\pm^0\hat{v}_\pm$.

Lemma 3.2 *Let $\tau \geq 1$ and assume (3.50), then we have*

$$|P_\pm\hat{v}_\pm(\xi,y) - P_\pm^0\hat{v}_\pm(\xi,y)| \leq Cs\tau\big[\tau(\alpha_\pm + 1 + \beta|y|)|\hat{v}_\pm(\xi,y)| + |\partial_y\hat{v}_\pm(\xi,y)|\big], \tag{3.52}$$

where C depends only on λ_0 and M_0.

Proof. It should be noted that

$$\zeta_{\pm}(\xi,y)|_{s=0} = \frac{B_{\pm}(\xi,\xi,y)}{a^{\pm}_{nn}(y)}.$$

By simple calculations, and dropping $\pm$ for the sake of shortness, we can write

$$P\hat{v}(\xi,y) - P^0\hat{v}(\xi,y) = I_1 + I_2 + I_3, \tag{3.53}$$

where

$$\begin{aligned} I_1 &= \big(it(\xi+i\tau s\gamma,y) - it(\xi,y)\big)a_{nn}(y)\big(\partial_y - \tau\varphi' + it(\xi+i\tau s\gamma,y)\big)\hat{v}, \\ I_2 &= \big(\partial_y - \tau\varphi' + it(\xi,y)\big)a_{nn}(y)\big(it(\xi+i\tau s\gamma,y) - it(\xi,y)\big)\hat{v}, \end{aligned}$$

and

$$I_3 = a^{\pm}_{nn}(y)\zeta_{\pm}(\xi,y) - B_{\pm}(\xi,\xi,y).$$

By linearity of t with respect to its first argument (see (3.23)) and by (3.38), we have

$$|t(\xi+i\tau s\gamma,y) - t(\xi,y)| = |t(i\tau s\gamma,y)| \le \lambda_3^{-1}s\tau,$$

which, together with (3.2) and (3.50), gives the estimate

$$\begin{aligned} |I_1| &\le \lambda_3^{-1}\lambda_0^{-1}s\tau\{|\partial_y\hat{v}| + \tau(\alpha_{\pm}+\beta|y|)|\hat{v}| + \lambda_3^{-1}(|\xi|+s\tau)|\hat{v}|\} \\ &\le Cs\tau\{|\partial_y\hat{v}| + [\tau(\alpha_{\pm}+\beta|y|) + s\tau]|\hat{v}|\}, \end{aligned} \tag{3.54}$$

where C depends on λ_0 only. On the other hand, by linearity of t and by (3.39), we obtain

$$|\partial_y\left(t(\xi+i\tau s\gamma,y) - t(\xi,y)\right)| = |\partial_y\left(t(i\tau s\gamma,y)\right)| \le M_3 s\tau,$$

which, together with (3.2), (3.3) and (3.50), gives the estimate

$$|I_2| \le Cs\tau\{|\partial_y\hat{v}| + [\tau(\alpha_{\pm}+\beta|y|) + s\tau]|\hat{v}|\}, \tag{3.55}$$

where C depends on λ_0 and M_0 only.

Finally, by (3.22), (3.34) and (3.50),

$$|I_3| = |2is\tau B_{\pm}(\xi,\gamma,y) - s^2\tau^2 B_{\pm}(\gamma,\gamma,y)| \le Cs\tau^2 \tag{3.56}$$

where C depends only on λ_0. Putting together (3.53), (3.55), (3.54), and (3.56) gives (3.52). □

Lemma 3.2 allows us to estimate $||P^0_{\pm}\hat{v}_{\pm}(\xi,\cdot)||_{L^2(\mathbb{R}_{\pm})}$ instead of $||P_{\pm}\hat{v}_{\pm}(\xi,\cdot)||_{L^2(\mathbb{R}_{\pm})}$. Let us now go further and note that, similarly to (3.28) and (3.29), we have

$$P^0_{+}\hat{v}_{+} = E^0_{+}a^{+}_{nn}(y)F^0_{+}\hat{v}_{+} - \hat{v}_{+}\partial_y\big(a^{+}_{nn}(y)m_{+}(\xi,y)\big),$$

$$P^0_{-}\hat{v}_{-} = F^0_{-}a^{+}_{nn}(y)E^0_{-}\hat{v}_{-} + \hat{v}_{-}\partial_y\big(a^{-}_{nn}(y)m_{-}(\xi,y)\big).$$

We can easily obtain, from (3.3) and (3.37), that

$$|P_+^0\hat{v}_+ - E_+^0 a_{nn}^+(y) F_+^0 \hat{v}_+| \le C|\xi||\hat{v}_+| \tag{3.57}$$

and

$$|P_-^0\hat{v}_- - F_-^0 a_{nn}^+(y) E_-^0 \hat{v}_-| \le C|\xi||\hat{v}_-|. \tag{3.58}$$

where C depends only on λ_0 and M_0.

Lemma 3.3 *Let* $\tau \ge 1$ *and assume* (3.50). *There exists a positive constant* C *depending only on* λ_0 *and* M_0 *such that, if* $s_0 \le 1/C$ *then we have*

$$\begin{aligned}&\Lambda|a_{nn}^+(0)F_+^0\hat{v}_+(\xi,0)|^2 + \Lambda^3|\hat{v}_+(\xi,0)|^2 + \Lambda^4||\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda^2||\partial_y\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\\ &\le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}&-\Lambda|a_{nn}^-(0)E_-^0\hat{v}_-(\xi,0)|^2 - \Lambda^3|\hat{v}_-(\xi,0)|^2 + \Lambda^4||\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} + \Lambda^2||\partial_y\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)}\\ &\le C||P_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)},\end{aligned} \tag{3.60}$$

where $\mathrm{supp}(\hat{v}_+(\xi,\cdot)) \subset [0,\frac{1}{\beta}]$ *and* $\mathrm{supp}(\hat{v}_-(\xi,\cdot)) \subset [-\frac{\alpha_-}{2\beta},0]$.

Proof. Since $\mathrm{supp}\,\hat{v}_+(x,y)$ is compact, $\hat{v}_+(\xi,y)\equiv 0$ when $|y|$ is large and the same holds for the function $\omega_+^0(\xi,y)$ defined in (3.49). We now compute

$$\begin{aligned}&||E_+^0\omega_+^0(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\\ &= \int_0^\infty |\partial_y\omega_+^0(\xi,y) + it_+(\xi,y)\omega_+^0(\xi,y)|^2dy + \int_0^\infty [\tau\alpha_+ + \tau\beta y + m_+(\xi,y)]^2|\omega_+^0(\xi,y)|^2dy\\ &-2\Re\int_0^\infty [\tau\alpha_+ + \tau\beta y + m_+(\xi,y)]\bar{\omega}_+^0(\xi,y)[\partial_y\omega_+^0(\xi,y) + it_+(\xi,y)\omega_+^0(\xi,y)]dy.\end{aligned} \tag{3.61}$$

Integrating by parts, we easily get

$$\begin{aligned}&-2\Re\int_0^\infty [\tau\alpha_+ + \tau\beta y + m_+(\xi,y)]\bar{\omega}_+^0(\xi,y)[\partial_y\omega_+^0(\xi,y) + it_+(\xi,y)\omega_+^0(\xi,y)]dy\\ &= [\tau\alpha_+ + m_+(\xi,0)]|\omega_+^0(\xi,0)|^2 + \int_0^\infty [\tau\beta + \partial_y m_+(\xi,y)]|\omega_+^0(\xi,y)|^2dy.\end{aligned} \tag{3.62}$$

By (3.50) and (3.37), we have that

$$\tau\beta + \partial_y m_+(\xi,y) \ge \tau\beta - M_2|\xi| \ge \tau\beta - 2\tau s_0\lambda_2^{-2}M_2 \ge \tau\beta/2 \ge 0 \tag{3.63}$$

provided $0 < s_0 \le \frac{\beta\lambda_2^2}{4M_2}$. Combining (3.51), (3.61), (3.62) and (3.63) yields

$$\begin{aligned}||E_+^0\omega_+^0(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} &\ge \int_0^\infty [\tau\alpha_+ + \tau\beta y + m_+(\xi,y)]^2|\omega_+^0(\xi,y)|^2dy\\ &\quad +[\tau\alpha_+ + m_+(\xi,0)]|\omega_+^0(\xi,0)|^2\\ &\ge C^{-1}\Lambda^2\int_0^\infty |\omega_+^0(\xi,y)|^2dy + C^{-1}\Lambda|\omega_+^0(\xi,0)|^2,\end{aligned} \tag{3.64}$$

where C depends only on λ_0.

Similarly, we have that

$$\lambda_0^{-2}||\omega_+^0(\xi,\cdot)||_{L^2(\mathbb{R}_+)}^2 \geq \int_0^\infty |\partial_y \hat{v}_+(\xi,y) + it_+(\xi,y)\hat{v}_+(\xi,y)|^2 dy$$
$$+\int_0^\infty [\tau\alpha_+ + \tau\beta y - m_+(\xi,y)]^2|\hat{v}_+(\xi,y)|^2 dy + [\tau\alpha_+ - m_+(\xi,0)]|\hat{v}_+(\xi,0)|^2$$
$$+\int_0^\infty [\tau\beta - \partial_y m_+(\xi,y)]|\hat{v}_+(\xi,y)|^2 dy. \tag{3.65}$$

The assumption (3.50) and (3.36) imply

$$\tau\alpha_+ + \tau\beta y - m_+(\xi,y) \geq \tau\alpha_+ - \lambda_2^{-1}|\xi| \geq \tau\alpha_+ - 2\lambda_2^{-3}\tau s_0 \geq \tau\alpha_+/2$$

provided $0 < s_0 \leq \frac{\alpha_+\lambda_2^3}{4}$. Thus, by choosing

$$0 < s_0 \leq \min\left\{1, \frac{\beta\lambda_2^2}{4M_2}, \frac{\alpha_+\lambda_2^3}{4}\right\},$$

we obtain from (3.63) and (3.65)

$$\begin{aligned} C||\omega_+^0(\xi,\cdot)||_{L^2(\mathbb{R}_+)}^2 \; &\geq \; \int_0^\infty |\partial_y \hat{v}_+(\xi,y) + it_+(\xi,y)\hat{v}_+(\xi,y)|^2 dy \\ &+ \; \Lambda^2 \int_0^\infty |\hat{v}_+(\xi,y)|^2 dy + \Lambda|\hat{v}_+(\xi,0)|^2. \end{aligned} \tag{3.66}$$

Additionally, we can see that

$$\begin{aligned} &\int_0^\infty |\partial_y \hat{v}_+(\xi,y) + it_+(\xi,y)\hat{v}_+(\xi,y)|^2 dy \\ &\geq \varepsilon \int_0^\infty \left(|\partial_y \hat{v}_+(\xi,y)|^2 - 2|\partial_y \hat{v}_+(\xi,y)||t_+(\xi,y)\hat{v}_+(\xi,y)| + |t_+(\xi,y)\hat{v}_+(\xi,y)|^2\right) dy \\ &\geq \varepsilon \int_0^\infty \left(\frac{1}{2}|\partial_y \hat{v}_+(\xi,y)|^2 - |t_+(\xi,y)|^2|\hat{v}_+(\xi,y)|^2\right) dy \\ &\geq \frac{\varepsilon}{2}\int_0^\infty |\partial_y \hat{v}_+(\xi,y)|^2 dy - \lambda_3^{-1}\varepsilon|\xi|^2 \int_0^\infty |\hat{v}_+(\xi,y)|^2 dy, \end{aligned} \tag{3.67}$$

for any $0 < \varepsilon < 1$. Choosing ε sufficiently small, we obtain, from (3.66) and (3.67),

$$C||\omega_+^0(\xi,\cdot)||_{L^2(\mathbb{R}_+)}^2 \geq \int_0^\infty |\partial_y \hat{v}_+(\xi,y)|^2 dy + \Lambda^2 \int_0^\infty |\hat{v}_+(\xi,y)|^2 dy + \Lambda|\hat{v}_+(\xi,0)|^2, \tag{3.68}$$

where C depends only on λ_0 and M_0.

Combining (3.64) and (3.68) yields

$$\begin{aligned} &\Lambda^2 \int_0^\infty |\partial_y \hat{v}_+(\xi,y)|^2 + \Lambda^4 \int_0^\infty |\hat{v}_+(\xi,y)|^2 + \Lambda^3|\hat{v}_+(\xi,0)|^2 + \Lambda|\omega_+^0(\xi,0)|^2 \\ &\leq C||E_+^0\omega_+^0(\xi,\cdot)||_{L^2(\mathbb{R}_+)}^2, \end{aligned} \tag{3.69}$$

where C depends only on λ_0 and M_0. From (3.52), since $\mathrm{supp}(\hat{v}_+(\xi,\cdot)) \subset [0,1/\beta]$ and (3.50) holds, we have

$$\begin{aligned} ||P_+^0\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} &\leq 2||P_+^0\hat{v}_+(\xi,\cdot)||_{L^2(\mathbb{R}_+)} \\ &\quad + Cs_0^2\left(\Lambda^2\int_0^\infty |\partial_y\hat{v}_+(\xi,y)|^2 + \Lambda^4\int_0^\infty |\hat{v}_+(\xi,y)|^2\right) \end{aligned} \tag{3.70}$$

Moreover, by (3.57) and (3.50),

$$||E_+^0\omega_+^0(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} \leq 2||P_+^0\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + Cs_0^2\Lambda^2\int_0^\infty |\hat{v}_+(\xi,y)|^2. \tag{3.71}$$

Finally, by (3.69), (3.70) and (3.71) we get (3.59), provided s_0 is small enough.

Now, we proceed to prove (3.60). Applying the same arguments leading to (3.62), we have that

$$\begin{aligned} &||F_-^0\omega_-^0(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} \\ \geq &\int_{-\infty}^0 [\tau\alpha_- + \tau\beta y - m_-(\xi,y)]^2|\omega_-^0(\xi,y)|^2dy - [\tau\alpha_- - m_-(\xi,0)]|\omega_-^0(\xi,0)|^2 \\ &+\int_{-\infty}^0 [\tau\beta - \partial_y m_-(\xi;y)]|\omega_-^0(\xi,y)|^2dy. \end{aligned} \tag{3.72}$$

By (3.36) and (3.50) and since $\mathrm{supp}(\hat{v}_-(\xi,\cdot)) \subset [-\frac{\alpha_-}{2\beta},0]$, we can see that

$$\tau\alpha_- + \tau\beta y - m_-(\xi,y) \geq \tau\alpha_-/2 - \lambda_2^{-1}|\xi| \geq \tau\alpha_-/2 - 2\lambda_2^{-3}\tau s_0 \geq \tau\alpha_-/4 \tag{3.73}$$

provided $0 < s_0 \leq \frac{\alpha_-\lambda_2^3}{8}$. From (3.72) and (3.73), it follows

$$\begin{aligned} ||F_-^0\omega_-^0(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} &\geq \frac{\alpha_-^2}{16}\tau^2\int_0^\infty |\omega_-^0(\xi,y)|^2dy - \tau\alpha_-|\omega_-^0(\xi,0)|^2 \\ &\geq C\Lambda^2\int_0^\infty |\omega_-^0(\xi,y)|^2dy - C\Lambda|\omega_-^0(\xi,0)|^2. \end{aligned} \tag{3.74}$$

Arguing as before and recalling (3.51) we obtain (3.60). □

We now take into account the transmission conditions.

Lemma 3.4 *Let* $\tau \geq 1$ *and assume* (3.50). *There exists a positive constant* C *depending only on* λ_0 *and* M_0 *such that if* $s_0 \leq 1/C$ *then*

$$\begin{aligned} &\Lambda\sum_{\pm}|V_\pm(\xi)|^2 + \Lambda^3\sum_{\pm}|\hat{v}_\pm(\xi,0)|^2 + \Lambda^4\sum_{\pm}||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda^2\sum_{\pm}||\partial_y\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \\ &\qquad \leq C\sum_{\pm}||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + C\Lambda|\eta_1(\xi)|^2 + C\Lambda^3|\eta_0(\xi)|^2, \end{aligned} \tag{3.75}$$

where $\mathrm{supp}(\hat{v}_\pm(\xi,\cdot)) \subset [-\frac{c_0}{2\beta},\frac{c_0}{\beta}]$ *with* $c_0 = \min(\alpha_-,1)$.

Proof. It follows from (3.59) and (3.49) that, for some C depending only on λ_0 and M_0,

$$\Lambda|\omega_0^+(\xi,0)|^2+\Lambda^3|\hat{v}_+(\xi,0)|^2\le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}. \tag{3.76}$$

By (3.32), (3.49), (3.36) and (3.38) we easily get

$$V_+(\xi)=\omega_0^+(\xi,0)-a_{nn}^+(0)(\tau st_+(\gamma,0)+m_+(\xi,0))\hat{v}_+(\xi,0),$$

and hence

$$\Lambda|V_+(\xi)|^2\le 2\Lambda|\omega_0^+(\xi,0)|^2+C\Lambda^3|\hat{v}_+(\xi,0)|^2\le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}, \tag{3.77}$$

where C depends only on λ_0 and M_0.

By (3.30) and (3.59), we have that

$$\Lambda^3|\hat{v}_-(\xi,0)|^2\le 2\Lambda^3|\hat{v}_+(\xi,0)|^2+2\Lambda^3|\eta_0(\xi)|^2\le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}+2\Lambda^3|\eta_0(\xi)|^2. \tag{3.78}$$

Using the definition of η_1 (see (3.31)) and (3.77), we also deduce that

$$\Lambda|V_-(\xi)|^2\le 2\Lambda|V_+(\xi)|^2+2\Lambda|\eta_1(\xi)|^2\le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}+2\Lambda|\eta_1(\xi)|^2. \tag{3.79}$$

Putting together (3.76), (3.77), (3.78) and (3.79) implies

$$\Lambda^3\sum_\pm|\hat{v}_\pm(\xi,0)|^2+\Lambda\sum_\pm|V_\pm(\xi)|^2\le C||P_+\hat{v}_+(\xi,0)||^2_{L^2(\mathbb{R}_+)}+2\Lambda^3|\eta_0(\xi)|^2+2\Lambda|\eta_1(\xi)|^2. \tag{3.80}$$

We now use (3.59) and (3.60) and get

$$\begin{aligned}&\Lambda^4\sum_\pm||\hat{v}_\pm(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}+\Lambda^2\sum_\pm||\partial_y\hat{v}_\pm(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}\\ \le&C\sum_\pm||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}+\Lambda|\omega_0^-(\xi,0)|^2+\Lambda^3|\hat{v}_-(\xi,0)|^2\end{aligned}$$

Arguing similarly as we did for (3.77) and using (3.79) and (3.80), we get

$$\begin{aligned}&\Lambda^4\sum_\pm||\hat{v}_\pm(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}+\Lambda^2\sum_\pm||\partial_y\hat{v}_\pm(\xi,\cdot)||_{L^2(\mathbb{R}_\pm)}\\ \le&C\sum_\pm||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}+2\Lambda|V_-(\xi)|^2+C\Lambda^3|\hat{v}_-(\xi,0)|^2\\ \le&C\left(\sum_\pm||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}+\Lambda|\eta_1(\xi)|^2+\Lambda^3|\eta_0(\xi)|^2\right),\end{aligned} \tag{3.81}$$

where C depends on λ_0 and M_0 only. The proof is complete by combining (3.80) and (3.81). □

Since $\tau \geq 1$, it is easily seen that (3.75) implies

$$\Lambda \sum_{\pm} |V_{\pm}(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_{\pm}(\xi,0)|^2 + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_{\pm})} + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_{\pm})}$$
$$\leq C \left(\sum_{\pm} ||P_{\pm}\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_{\pm})} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2 \right), \tag{3.82}$$

where C depends on λ_0 and M_0 only.

Case 2:

$$\frac{m_+(\xi,0)}{(1-\kappa)\,\alpha_+} \leq \tau \leq \frac{\lambda_2^2|\xi|}{2s_0}. \tag{3.83}$$

In this case, (3.36) implies

$$\frac{\lambda_2|\xi|}{\alpha_+} \leq \tau \leq \frac{\lambda_2^2|\xi|}{2s_0}. \tag{3.84}$$

In addition, in view of the definition of $\zeta_{\pm}$, (3.34), (3.83), and recalling that $\lambda_2 = \sqrt{\lambda_0\lambda_1}$ and $s \leq s_0$, we have that

$$\Re\zeta_{\pm} \geq \frac{3}{4}\lambda_2^2|\xi|^2. \tag{3.85}$$

It is not hard to see from (3.40), (3.41), (3.84), (3.85) that

$$|\partial_y \sqrt{\zeta_{\pm}}| \leq M_5|\xi|, \tag{3.86}$$

where M_5 depends only on λ_0 and M_0. Moreover, if we set $R_{\pm} = \Re\sqrt{\zeta_{\pm}} \geq 0$ and $J_{\pm} = \Im\sqrt{\zeta_{\pm}}$, then (3.86) gives

$$|\partial_y R_{\pm}| + |\partial_y J_{\pm}| \leq M_5|\xi|. \tag{3.87}$$

Using (3.86), we can easily obtain from (3.28), (3.29) that

$$|P_+\hat{v}_+(\xi,y) - E_+a^+_{nn}(y)F_+\hat{v}_+(\xi,y)| \leq C|\xi||\hat{v}_+(\xi,y)| \tag{3.88}$$

and

$$|P_-\hat{v}_-(\xi,y) - F_-a^-_{nn}(y)E_-\hat{v}_-(\xi,y)| \leq C|\xi||\hat{v}_-(\xi,y)|, \tag{3.89}$$

where C depends only on λ_0 and M_0.

We now prove the following lemma.

Lemma 3.5 *Assume* (3.83). *There exists a positive constant C depending only on λ_0 and M_0 such that, if $0 < s_0 \le C^{-1}$, $\beta \ge C$ and $\tau \ge C$, then we have*

$$\begin{aligned}&\Lambda|V_+(\xi) + a^+_{nn}(0)\sqrt{\zeta_+(\xi,0)}\hat{v}_+(\xi,0)|^2 + \Lambda^2||a^+_{nn}(y)F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\\ \le& C||E_+a^+_{nn}(y)F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\end{aligned} \tag{3.90}$$

and

$$\begin{aligned}&\Lambda|V_+(\xi) + a^+_{nn}(0)\sqrt{\zeta_+(\xi,0)}\hat{v}_+(\xi,0)|^2 + \Lambda^3|\hat{v}_+(\xi,0)|^2\\ &+ \Lambda^4\int_0^\infty |\hat{v}_+(\xi,y)|^2dy + \Lambda^2\int_0^\infty |\partial_y\hat{v}_+(\xi,y)|^2dy \le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\end{aligned} \tag{3.91}$$

provided $\mathrm{supp}(\hat{v}_+(\xi,\cdot)) \subset [0,\frac{1}{\beta}]$.

Proof. We write

$$E_+\omega_+(\xi,y) = [\partial_y + it_+(\xi + i\tau s\gamma, y) - \tau\varphi'_+ - \sqrt{\zeta_+}]\omega_+(\xi,y) := I_3 - I_4,$$

where $I_3 = \partial_y\omega_+ + it_+(\xi + i\tau s\gamma, y)\omega_+ - iJ_+\omega_+$ and $I_4 = \tau\alpha_+\omega_+ + \tau\beta y\omega_+ + R_+\omega_+$. Our task now is to estimate

$$||E_+\omega_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} = \int_0^\infty |I_3|^2dy + \int_0^\infty [\tau\alpha_+ + \tau\beta y + R_+]^2|\omega_+|^2dy - 2\Re\int_0^\infty I_3\bar{I}_4dy. \tag{3.92}$$

Observe that

$$\begin{aligned}-2\Re\int_0^\infty I_3\bar{I}_4 =& -\int_0^\infty [\tau\alpha_+ + \tau\beta y + R_+(\xi,y)]\partial_y(|\omega_+(\xi,y)|^2)dy\\ &+ 2\int_0^\infty [\tau\alpha_+ + \tau\beta y + R_+(\xi,y)]t_+(\tau s\gamma,y)|\omega_+(\xi,y)|^2dy\\ =& \int_0^\infty [\tau\beta + \partial_yR_+(\xi,y)]|\omega_+(\xi,y)|^2dy + [\tau\alpha_+ + R_+(\xi,0)]|\omega_+(\xi,0)|^2\\ &+ 2\int_0^\infty [\tau\alpha_+ + \tau\beta y + R_+(\xi,y)]t_+(\tau s\gamma,y)|\omega_+(\xi,y)|^2dy\\ \ge& \int_0^\infty [\tau\beta + \partial_yR_+(\xi,y) - \lambda_3^{-1}s\tau(\tau\alpha_+ + \tau\beta y + R_+)]|\omega_+(\xi,y)|^2dy\\ &+ [\tau\alpha_+ + R_+(\xi,0)]|\omega_+(\xi,0)|^2,\end{aligned} \tag{3.93}$$

where in the last inequality we have used the fact that $R_+ \ge 0$. Combining (3.92) and (3.93) yields

$$\begin{aligned}&||E_+\omega_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\\ \ge& \int_0^\infty [(\tau\alpha_+ + \tau\beta y + R_+)^2 + \tau\beta + \partial_yR_+(\xi,y) - \lambda_3^{-1}s\tau(\tau\alpha_+ + \tau\beta y + R_+)]|\omega_+(\xi,y)|^2dy\\ &+ [\tau\alpha_+ + R_+(\xi,0)]|\omega_+(\xi,0)|^2\\ \ge& \frac{\Lambda^2}{C}\int_0^\infty |\omega_+(\xi,y)|^2dy + \frac{\Lambda}{C}|\omega_+(\xi,0)|^2\end{aligned} \tag{3.94}$$

provided s_0 is small enough. Formulas (3.32) and (3.27) give

$$\omega_+(\xi,0) = V_+(\xi) + a^+_{nn}(0)\sqrt{\zeta_+(\xi,0)}\hat{v}_+(\xi,0), \tag{3.95}$$

which leads to (3.90) by (3.94).

We now want to derive (3.91). Let us write

$$F_+\hat{v}_+ = [\partial_y + it_+(\xi + i\tau s\gamma; y) - \tau\varphi'_+ + \sqrt{\zeta_+}]\hat{v}_+ := I_5 - I_6,$$

where $I_5 = \partial_y\hat{v}_+ + it_+(\xi + i\tau s\gamma; y)\hat{v}_+ + iJ_+\hat{v}_+$ and $I_6 = \tau\alpha_+\hat{v}_+ + \tau\beta y\hat{v}_+ - R_+\hat{v}_+$. Thus, we have

$$\begin{aligned} &||F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} \\ =&\int_0^\infty |I_5|^2dy + \int_0^\infty [\tau\alpha_+ + \tau\beta y - R_+]^2|\hat{v}_+(\xi,y)|^2dy - 2\Re\int_0^\infty I_5\bar{I}_6dy. \end{aligned} \tag{3.96}$$

Repeating the computations of (3.93) and (3.94) yields

$$\begin{aligned} &||F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} \\ \geq&\int_0^\infty |I_5|^2dy + \int_0^\infty [\tau\alpha_+ + \tau\beta y - R_+]^2|\hat{v}_+(\xi,y)|^2dy + \int_0^\infty (\tau\beta - \partial_yR_+)|\hat{v}_+(\xi,y)|^2dy \\ &- Cs\tau\int_0^\infty |\tau\alpha_+ + \tau\beta y - R_+||\hat{v}_+(\xi,y)|^2dy + [\tau\alpha_+ - R_+(\xi,0)]|\hat{v}_+(\xi,0)|^2. \end{aligned} \tag{3.97}$$

We observe that

$$R_+^2 = \frac{\Re\zeta_+ + |\zeta_+|}{2}$$

and, by simple calculations,

$$|\zeta_\pm| \leq -\Re\zeta_\pm + 2\frac{B_\pm(\xi,\xi,y)}{a^\pm_{nn}(y)}, \tag{3.98}$$

which gives the estimate

$$R_+(\xi,y) \leq \sqrt{\frac{B_+(\xi,\xi;y)}{a^+_{nn}(y)}} = m_+(\xi,y). \tag{3.99}$$

From (3.83) and (3.99), we deduce that

$$\tau\alpha_+ - R_+(\xi,0) \geq \tau\alpha_+ - m_+(\xi,0) \geq \tau\alpha_+ - (1-\kappa)\tau\alpha_+ = \kappa\tau\alpha_+. \tag{3.100}$$

On the other hand, using (3.100), (3.87) and (3.84), we can obtain that for $y \geq 0$

$$\begin{aligned} \tau\alpha_+ + \tau\beta y - R_+(\xi,y) =&\tau\alpha_+ - R_+(\xi,0) + \tau\beta y - R_+(\xi,y) + R_+(\xi,0) \\ \geq&\kappa\tau\alpha_+ + y(\tau\beta - C\tau) \geq \kappa\tau\alpha_+ \end{aligned}$$

provided β is large enough. Furthermore, if $0 \leq y \leq 1/\beta$, then

$$[\tau\alpha_+ + \tau\beta y - R_+]^2 + (\tau\beta - \partial_y R_+) - Cs\tau|\tau\alpha_+ + \tau\beta y - R_+| \geq (\kappa\tau\alpha_+)^2/4 \quad (3.101)$$

provided s_0 is small enough and τ is large enough. Now it follows from (3.97), (3.100), and (3.101) and arguing as in (3.67), that

$$\begin{aligned} &C||F_+\hat{v}_+(\xi,y)||^2_{L^2(\mathbb{R}_+)} \\ \geq &\int_0^\infty |\partial_y \hat{v}_+(\xi,y)|^2 dy + \Lambda^2 \int_0^\infty |\hat{v}_+(\xi,y)|^2 dy + \Lambda|\hat{v}_+(\xi,0)|^2. \end{aligned} \quad (3.102)$$

Finally, by (3.88), (3.90), and (3.102), we can easily derive (3.91) provided $\beta \geq C$, $\tau \geq C$ and $s_0 \leq 1/C$ for some C depending on λ_0 and M_0. □

Similarly, we can prove that

Lemma 3.6 *Assume* (3.83). *There exists a positive constant C depending only on λ_0 and M_0 such that, if $0 < s_0 \leq C^{-1}$ and $\tau \geq C$ then we have*

$$\begin{aligned} &-\Lambda|V_-(\xi) - a^-_{nn}(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0)|^2 + \Lambda||a^-_{nn}(y)E_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} \\ \leq &C||F_-a^-_{nn}(y)E_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} \end{aligned} \quad (3.103)$$

and

$$\begin{aligned} &-\Lambda|V_-(\xi) - a^-_{nn}(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0)|^2 - \Lambda^3|\hat{v}_-(\xi,0)|^2 + \Lambda^3\int_{-\infty}^0 |\hat{v}_-(\xi,y)|^2 dy \\ &+ \Lambda\int_{-\infty}^0 |\partial_y\hat{v}_-(\xi,y)|^2 dy \leq C||P_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)}, \end{aligned} \quad (3.104)$$

provided $\operatorname{supp}(\hat{v}_-(\xi,\cdot)) \subset [-\frac{\alpha_-}{2\beta}, 0]$.

Proof. Let $\omega_-(\xi,y) = a^-_{nn}(y)E_-\hat{v}_-(\xi,y) = a^-_{nn}(y)[\partial_y + it_-(\xi + i\tau s\gamma, y) - \tau\varphi'_- - \sqrt{\zeta_-}]\hat{v}_-(\xi,y)$. If we write

$$F_-\omega_-(\xi,y) = I_7 - I_8,$$

where

$$\begin{aligned} I_7 &= \partial_y\omega_- + it_-(\xi,y)\omega_- + iJ_-\omega_- \\ I_8 &= \tau\alpha_-\omega_- + \tau\beta y\omega_- + t_-(\tau s\gamma, y)\omega_- - R_-\omega_-, \end{aligned}$$

we have

$$
\begin{aligned}
&||F_-\omega_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} \geq -2\Re\int_{-\infty}^0 I_7\bar{I}_8 dy\\
&= -\int_{-\infty}^0 [\tau\alpha_- + \tau\beta y + t_-(\tau s\gamma, y) - R_-(\xi,y)]\partial_y(|\omega_-(\xi,y)|^2)dy\\
&= \int_{-\infty}^0 [\tau\beta + \partial_y t_-(\tau s\gamma,y) - \partial_y R_-(\xi,y)]|\omega_-(\xi,y)|^2 dy\\
&\quad - [t_-(\tau s\gamma,0) + \tau\alpha_- - R_-(\xi,0)]|\omega_-(\xi,0)|^2\\
&\geq \int_{-\infty}^0 \tau[\beta - M_3 s - 2M_5 s_0\lambda_2^{-2}]|\omega_-(\xi,y)|^2 dy - (\lambda_3 s + \alpha_+)\tau|\omega_-(\xi,0)|^2,
\end{aligned}
$$

hence, by (3.84),

$$
||F_-\omega_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} \geq C\Lambda\int_{-\infty}^0 |\omega_-(\xi,y)|^2 dy - C\Lambda|\omega_-(\xi,0)|^2, \tag{3.105}
$$

provided s_0 is small enough. Since, by (3.32) and (3.26),

$$
\omega_-(\xi,0) = V_-(\xi) - a^-_{nn}(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0),
$$

we get (3.103).

To derive (3.104), we denote

$$
E_-\hat{v}_-(\xi,y) = I_9 - I_{10},
$$

where

$$
\begin{aligned}
I_9 &= \partial_y\hat{v}_- + it_-(\xi,y)\hat{v}_- - iJ_-\hat{v}_-,\\
I_{10} &= \tau\alpha_-\hat{v}_- + \tau\beta y\hat{v}_- + t_-(\tau s\gamma,y)\hat{v}_- + R_-\hat{v}_-.
\end{aligned}
$$

Observe that if $-\frac{\alpha_-}{2\beta} \leq y \leq 0$ then

$$
\tau\alpha_- + \tau\beta y + t_-(\tau s\gamma,y) + R_- \geq \tau\alpha_-/2 - \lambda_3^{-1}s\tau \geq \tau\alpha_-/4 \tag{3.106}
$$

provided s_0 is small. Furthermore, by choosing again s_0 small, we can make

$$
\tau\beta + \partial_y R_- + \partial_y t_-(\tau s\gamma,y) \geq \tau\left(\beta - 2M_5 s_0\lambda_2^{-2} - M_3 s_0\right) \geq 0. \tag{3.107}
$$

With the help of (3.106) and (3.107), and arguing as in (3.67) we get

$$
\begin{aligned}
&C||E_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)}\\
&\geq \int_{-\infty}^0 |\partial_y\hat{v}_-(\xi,y)|^2 dy + \Lambda^2\int_{-\infty}^0 |\hat{v}_-(\xi,y)|^2 dy - \Lambda|\hat{v}_-(\xi,0)|^2.
\end{aligned}
\tag{3.108}
$$

Using (3.103), (3.108) and (3.89), we obtain (3.104) provided τ is large. □

Lemma 3.7 *Assume* (3.83)*. There exists a positive constant* C*, depending only on* λ_0 *and* M_0*, such that if* $s_0 \le C^{-1}$*,* $\beta \ge C$ *and* $\tau \ge C$ *then we have*

$$\Lambda \sum_{\pm} |V_\pm(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi,0)|^2 + \Lambda^3 \sum_{\pm} ||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda \sum_{\pm} ||\partial_y \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}$$
$$\le C \left(\sum_{\pm} ||P_\pm \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3 |\eta_0(\xi)|^2 \right), \tag{3.109}$$

provided $\mathrm{supp}(\hat{v}_\pm(\xi,\cdot)) \subset [-\frac{c_0}{2\beta}, \frac{c_0}{\beta}]$ *with* $c_0 = \min(\alpha_-, 1)$*.*

Proof. We obtain from (3.91) that

$$\Lambda|\omega_+(\xi,0)|^2 + \Lambda^3|\hat{v}_+(\xi,0)|^2 \le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}. \tag{3.110}$$

On the other hand,

$$\Lambda|V_+(\xi)|^2 \le 2\Lambda|\omega_+(\xi,0)|^2 + C\Lambda^3|\hat{v}_+(\xi,0)|^2 \le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}. \tag{3.111}$$

Using the definition of η_0 and (3.110), we see that

$$\Lambda^3|\hat{v}_-(\xi,0)|^2 \le 2\Lambda^3|\hat{v}_+(\xi,0)|^2 + 2\Lambda^3|\eta_0(\xi)|^2 \le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3|\eta_0(\xi)|^2. \tag{3.112}$$

Summing up (3.110) and (3.112) yields

$$\Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi,0)|^2 \le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3|\eta_0(\xi)|^2. \tag{3.113}$$

Likewise, the definition of η_1 and (3.111) lead to

$$\Lambda|V_-(\xi)|^2 \le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda|\eta_1(\xi)|^2. \tag{3.114}$$

Putting together (3.111), (3.113), and (3.114), we deduce that

$$\Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi,0)|^2 + \Lambda \sum_{\pm} |V_\pm(\xi)|^2 \le C||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + 2\Lambda^3|\eta_0(\xi)|^2 + 2\Lambda|\eta_1(\xi)|^2. \tag{3.115}$$

Finally, we first use (3.91), recall that $\Lambda \ge \tau \ge 1$, (3.104), and then (3.114), (3.115) to derive

$$\Lambda^3 \sum_{\pm} ||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda \sum_{\pm} ||\partial_y \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}$$
$$\le C \sum_{\pm} ||P_\pm \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda|\hat{V}_-(\xi) - a^-_{nn}(0)R_-(\xi,0)\hat{v}_-(\xi,0)|^2 + \Lambda^3|\hat{v}_-(\xi,0)|^2$$
$$\le C \left(\sum_{\pm} ||P_\pm \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2 \right). \tag{3.116}$$

The proof is complete by combining (3.115) and (3.116). ☐

We conclude Case 2 by observing that (3.109) can be written in the form

$$\Lambda\sum_{\pm}|V_{\pm}(\xi)|^2+\Lambda^3\sum_{\pm}|\hat{v}_{\pm}(\xi,0)|^2+\frac{1}{\tau}\big(\Lambda^4\sum_{\pm}||\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_{\pm})}+\Lambda^2\sum_{\pm}||\partial_y\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_{\pm})}\big)$$
$$\leq C\left(\sum_{\pm}||P_{\pm}\hat{v}_{\pm}(\xi,\cdot)||^2_{L^2(\mathbb{R}_{\pm})}+\Lambda|\eta_1(\xi)|^2+\Lambda^3|\eta_0(\xi)|^2\right), \tag{3.117}$$

where C depends only on λ_0 and M_0.

Case 3:

$$\tau\leq\frac{m_+(\xi,0)}{(1-\kappa)\,\alpha_+}. \tag{3.118}$$

In this case, we have

$$\tau\leq\frac{2\lambda_2^{-1}|\xi|}{\alpha_++L\alpha_-}\quad\text{(from (3.36), (3.47))}.$$

From the definition of $\zeta_{\pm}$ (see (3.22)) and the inequality

$$B_{\pm}(\xi,\xi;y)-s^2\tau^2B_{\pm}(\gamma,\gamma;y)\geq\lambda_1|\xi|^2-\lambda_1^{-1}s^2\tau^2\geq\frac{\lambda_1}{4}|\xi|^2,$$

that holds for s_0 is sufficiently small, we can derive the estimates

$$\begin{cases}\Re\zeta_{\pm}\geq\frac{\lambda_2^2}{4}|\xi|^2,\\ R_{\pm}\geq\frac{\lambda_2}{2}|\xi|,\\ |J_{\pm}|\leq 4\lambda_2^{-3}s\tau,\\ |\partial_y\zeta_{\pm}|\leq M_4\left(1+\frac{4s_0^2\lambda_2^{-2}}{(\alpha_++L\alpha_-)^2}\right)|\xi|^2:=M_6|\xi|^2,\\ |\partial_y\sqrt{\zeta_{\pm}}|\leq\frac{M_6}{\lambda_2}|\xi|:=M_7|\xi|.\end{cases} \tag{3.119}$$

Lemma 3.8 *Assume* (3.118). *There exist a positive constant C such that, if $s_0\leq C^{-1}$ and $\tau\geq C$, then we have*

$$\Lambda|\omega_+(\xi,0)|^2+\Lambda^2\int_0^{\infty}|\omega_+(\xi,y)|^2dy+\int_0^{\infty}|\partial_y\omega_+(\xi,y)|^2dy\leq C||E_+\omega_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}. \tag{3.120}$$

Furthermore, if $\operatorname{supp}(\hat{v}_-(\xi,\cdot))\subset[-\frac{\alpha_-}{2\beta},0]$, *then*

$$\Lambda^2\int_{-\infty}^{0}|\hat{v}_-(\xi,y)|^2dy+\int_{-\infty}^{0}|\partial_y\hat{v}_-(\xi,y)|^2dy\leq C||E_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)}+C\Lambda|\hat{v}_-(\xi,0)|^2. \tag{3.121}$$

Proof. We write

$$E_+\omega_+ = I_{11} - I_{12},$$

where

$$\begin{aligned} I_{11} &= \partial_y\omega_+ + it_+(\xi,y)\omega_+ - iJ_+\omega_+, \\ I_{12} &= \tau\alpha_+\omega_+ + \tau\beta y\omega_+ + R_+\omega_+ + t_+(\tau s\gamma,y)\omega_+, \end{aligned}$$

and thus

$$\begin{aligned} &||E_+\omega_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} \\ =&\int_0^\infty |I_{11}|^2dy + \int_0^\infty [\tau\alpha_+ + \tau\beta y + R_+ + t_+(\tau s\gamma,y)]^2|\omega_+(\xi,y)|^2dy - 2\Re\int_0^\infty I_{11}\bar{I}_{12}dy. \end{aligned} \tag{3.122}$$

We first estimate

$$\begin{aligned} &-2\Re\int_0^\infty I_{11}\bar{I}_{12} \\ &= \int_0^\infty [\tau\beta + \partial_y R_+(\xi,y) + \partial_y t_+(\tau s\gamma,y)]|\omega_+(\xi,y)|^2dy \\ &\quad + [\tau\alpha_+ + R_+(\xi,0) + t_+(\tau s\gamma,0)]|\omega_+(\xi,0)|^2 \qquad (3.123) \\ &\geq -(M_7|\xi| - M_3 s\tau)\int_0^\infty |\omega_+(\xi,y)|^2dy + \left(\tau\alpha_+ + \frac{\lambda_2}{\sqrt{2}} - \lambda_3^{-1}s\tau\right)|\omega_+(\xi,0)|^2 \\ &\geq -C\Lambda\int_0^\infty |\omega_+(\xi,y)|^2dy + C\Lambda|\omega_+(\xi,0)|^2, \end{aligned}$$

provided s_0 is small enough. Combining (3.122) and arguing as in (3.67), we get (3.120). Likewise, we obtain (3.121). □

Lemma 3.9 *Assume* (3.118). *There exists a positive constants C, depending on λ_0, M_0, such that if $s_0 \leq C^{-1}$, $\tau \geq C$, and $\beta \geq C$, then, for* $\mathrm{supp}(\hat{v}_+(\xi,\cdot)) \subset [0, \frac{1}{\beta}]$, *we have that*

$$\frac{\Lambda^2}{\tau}\int_0^\infty |\hat{v}_+(\xi,y)|^2dy + \frac{1}{\tau}\int_0^\infty |\partial_y\hat{v}_+(\xi,y)|^2dy \leq C\left(||F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda|\hat{v}_+(\xi,0)|^2\right). \tag{3.124}$$

Proof. Expressing

$$F_+\hat{v}_+ = I_{13} - I_{14},$$

where

$$\begin{aligned} I_{13} &= \partial_y\hat{v}_+ + it_+(\xi,y)v_+ + iJ_+\hat{v}_+ \\ I_{14} &= \tau\alpha_+\hat{v}_+ + \tau\beta y\hat{v}_+ - R_+\hat{v}_+ + t_+(\tau s\gamma,y)\hat{v}_+, \end{aligned}$$

we can compute

$$\begin{aligned}
&||F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\\
=&\int_0^\infty |I_{13}|^2dy+\int_0^\infty p|\hat{v}_+(\xi,y)|^2dy+[\tau\alpha_+-R_+(\xi,0)+t_+(\tau s\gamma,0)]|\hat{v}_+(\xi,0)|^2.
\end{aligned} \tag{3.125}$$

where $p=[-\tau\alpha_+-\tau\beta y+R_+-t_+(\tau s\gamma,y)]^2+(\tau\beta-\partial_yR_++\partial_yt_+(\tau s\gamma,y))$.

We want to claim that

$$p\geq C\frac{\Lambda^2}{\tau}. \tag{3.126}$$

It follows from (3.119) and (3.118) that for $0\leq y\leq 1/\beta$

$$\begin{aligned}
&R_+-\tau\alpha_+-\tau\beta y-t_+(\tau s\gamma,y)\\
\geq&\frac{\lambda_2}{2}|\xi|-\tau(\alpha_++1+\lambda_3^{-1}s_0)\geq\frac{\lambda_2}{4}|\xi|
\end{aligned} \tag{3.127}$$

provided $|\xi|\geq C_2\tau=4\lambda_2^{-1}(\alpha_++1+\lambda_3^{-1}s_0)\tau$. By (3.127), we can easily obtain (3.126) in the case of $|\xi|\geq C_2\tau$ with τ large. On the other hand, when $|\xi|\leq C_2\tau$, we can estimate

$$p\geq\tau\beta-\partial_yR_++\partial_yt_+(\tau s\gamma,y)\geq\tau\beta-M_7C_2\tau-M_3s\tau\geq\frac{\beta}{2}\tau\geq\frac{\beta}{2}\frac{\Lambda^2}{\tau} \tag{3.128}$$

provided β is big enough. The estimate (3.124) is an easy consequence of (3.125) and (3.126). □

Lemma 3.10 *Assume* (3.118). *There exist positive constants C and ρ_1, depending only λ_0 and M_0 such that if* $\mathrm{supp}(\hat{v}_-(\xi,\cdot))\subset[-\rho_1,0]$ *then*

$$\Lambda|\omega_-(\xi,0)|^2+\Lambda^2||\omega_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)}\leq C||F_-\omega_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)}. \tag{3.129}$$

Proof. From (3.48), we see that

$$\mathrm{supp}(\omega_-(\xi,\cdot))\subseteq\mathrm{supp}(\hat{v}_-(\xi,\cdot)).$$

We first compute

$$\begin{aligned}
&\Re\int_{-\infty}^0|\xi|(F_-\omega_-)\bar{\omega}_-dy\\
=&\Re\int_{-\infty}^0|\xi|\partial_y\omega_-\bar{\omega}_-dy-\int_{-\infty}^0|\xi|[\tau\alpha_-+\tau\beta y+t_-(\tau s\gamma,y)-R_-(\xi,y)]|\omega_-|^2dy\\
=&\frac{1}{2}|\xi||\omega_-(\xi,0)|^2+\int_{-\infty}^0|\xi|[R_-(\xi;y)-\tau\alpha_--\tau\beta y-t_-(\tau s\gamma,y)]|\omega_-|^2dy.
\end{aligned} \tag{3.130}$$

We hope to show that

$$C_*|\xi| \leq R_-(\xi,y) - \tau\alpha_- - \tau\beta y - t_-(\tau s\gamma, y). \tag{3.131}$$

Assume that (3.131) is true. From (3.130) and (3.131), it follows that

$$\begin{aligned}
&\frac{1}{2}|\xi||\omega_-(\xi,0)|^2 + \int_{-\infty}^0 C_*|\xi|^2|\omega_-(\xi,y)|^2 dy \\
\leq &\Re \int_{-\infty}^0 |\xi|(F_-\omega_-)\bar{\omega}_- \\
\leq &\frac{C_*}{2}\int_{-\infty}^0 |\xi|^2|\omega_-(\xi,y)|^2 dy + C\int_{-\infty}^0 |F_-\omega_-(\xi,y)|^2 dy,
\end{aligned} \tag{3.132}$$

which implies (3.129).

To establish (3.131), we first note that, by simple calculations,

$$|m_-(\xi,0) - R_-(\xi,0)| \leq Cs|\xi|,$$

which can be used to derive for $y \leq 0$

$$\begin{aligned}
&R_-(\xi,y) - \tau\alpha_- - \tau\beta y - t_-(\tau s\gamma, y) \\
\geq &m_-(\xi,0) - |R_-(\xi,0) - m_-(\xi,0)| - |R_-(\xi,y) - R_-(\xi,0)| - \tau\alpha_- - \lambda_3^{-1}\tau s \\
\geq &m_-(\xi,0) - \tau\alpha_- - C(s+|y|)|\xi|.
\end{aligned} \tag{3.133}$$

On the other hand, by the definition of L, (3.36) and (3.118), we can estimate

$$m_-(\xi,0) - \tau\alpha_- \geq \frac{m_+(\xi,0)}{L}[1 - \frac{L\alpha_-}{(1-\kappa)\alpha_+}] \geq m_+(\xi,0)\frac{\kappa}{L(1-\kappa)} \geq \frac{\lambda_2\kappa}{(1-\kappa)L}|\xi|. \tag{3.134}$$

Combining (3.133) and (3.134) yields (3.131) provided s and $|y|$ are small. □

.

Lemma 3.11 *Assume* (3.118). *There exists C, depending only on λ_0 and M_0, such that if $s_0 \leq C^{-1}$, $\tau \geq C$, $\beta \geq C$, then for* $\text{supp}(\hat{v}_+(\xi,\cdot)) \subset [0, \frac{1}{\beta}]$ *we have*

$$\begin{aligned}
&\Lambda|V_+(\xi) + a_{nn}^+(0)\sqrt{\zeta_+(\xi,0)}v_+(\xi,0)|^2 + \Lambda^2||F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} \\
\leq &C\left(||P_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda^2||\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)}\right).
\end{aligned} \tag{3.135}$$

Furthermore, if $\text{supp}(\hat{v}_-(\xi,\cdot)) \subset [-\rho_1, 0]$, *for ρ_1 as in Lemma 3.10, then*

$$\begin{aligned}
&\Lambda|V_-(\xi) - a_{nn}^-(0)\sqrt{\zeta_-(\xi,0)}\hat{v}_-(\xi,0)|^2 + \Lambda^2||E_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} \\
\leq &C\left(||P_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} + \Lambda^2||\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)}\right).
\end{aligned} \tag{3.136}$$

Proof. Inequality (3.135) follows from (3.120) and (3.88). Similarly, (3.136) follows from (3.129) and (3.89). □

Lemma 3.12 *There exist C, ρ_2, depending only on λ_0 and M_0, such that if $s_0 \le C^{-1}$, $\tau \ge C$, $\beta \ge C$ then for* $\mathrm{supp}(\hat{v}_\pm(\xi,\cdot)) \subset [-\rho_2, \rho_2]$ *we have that*

$$\begin{aligned}&\Lambda \sum_{\pm} |V_\pm(\xi)|^2 + \Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi,0)|^2 + \frac{\Lambda^4}{\tau} \sum_{\pm} ||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^2}{\tau} \sum_{\pm} ||\partial_y \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \\ \le& C\left(\sum_{\pm} ||P_\pm \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2 \right).\end{aligned} \tag{3.137}$$

Proof. In view of (3.119),

$$\begin{aligned}&\Lambda|a^+_{nn}(0)\sqrt{\zeta_+(\xi,0)}\hat{v}_+(\xi,0) + a^-_{nn}(0)\sqrt{\zeta_-(\xi,0)}\hat{v}_+(\xi,0)|^2 \\ \ge& \Lambda|a^+_{nn}(0)R_+(\xi,0)\hat{v}_+(\xi,0) + a^-_{nn}(0)R_-(\xi,0)\hat{v}_+(\xi,0)|^2 \\ \ge& \frac{1}{C}\Lambda^3|\hat{v}_+(\xi,0)|^2,\end{aligned}$$

hence, by (3.30) and (3.33), we have

$$\begin{aligned}&\frac{1}{C}\Lambda^3|\hat{v}_+(\xi,0)|^2 \\ \le& \Lambda|a^+_{nn}(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0) + a^-_{nn}(0)\sqrt{\zeta_-}(\hat{v}_-(\xi,0)+\eta_0)|^2 \\ =& \Lambda|V_+ + a^+_{nn}(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0) + a^-_{nn}(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0) - V_- - \eta_1 - a^-_{nn}\sqrt{\zeta_-}\eta_0|^2 \\ \le& 4\left(\Lambda|V_+ + a^+_{nn}(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + \Lambda|V_- - a^-_{nn}(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0)|^2 + \Lambda|\eta_1|^2 + \Lambda^3|\eta_0|^2\right).\end{aligned} \tag{3.138}$$

From (3.135), (3.136) and (3.30), we can estimate

$$\begin{aligned}&\Lambda^3 \sum_{\pm} |\hat{v}_\pm(\xi,0)|^2 \\ \le& C\left(\sum_{\pm} ||P_\pm \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda^2||\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2 \right).\end{aligned} \tag{3.139}$$

Again from (3.135) and (3.136), we have the following estimate

$$
\begin{aligned}
&\Lambda|V_+|^2\\
\le&2\Lambda|V_+ + a_{nn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + 2\Lambda|a_{nn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2\\
\le&2\Lambda|V_+ + a_{nn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + C\Lambda^3|\hat{v}_+(\xi,0)|^2\\
\le&2\Lambda|V_+ + a_{nn}^+(0)\sqrt{\zeta_+}\hat{v}_+(\xi,0)|^2 + C\left(\Lambda|V_- - a_{nn}^-(0)\sqrt{\zeta_-}\hat{v}_-(\xi,0)|^2 + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2\right)\\
\le&C\left(\sum_{\pm}||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda^2||\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2\right).
\end{aligned}
\tag{3.140}
$$

We then obtain from (3.33) that

$$
\Lambda\sum_{\pm}|V_\pm(\xi)|^2 \le C\left(\sum_{\pm}||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda^2||\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2\right).
\tag{3.141}
$$

Combining (3.121), (3.124), (3.135) (3.136) and (3.139), we deduce that

$$
\begin{aligned}
&\frac{\Lambda^4}{\tau}\sum_{\pm}||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^2}{\tau}\sum_{\pm}||\partial_y\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}\\
\le&C\left(\Lambda^2||F_+\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda^2||E_-\hat{v}_-(\xi,\cdot)||^2_{L^2(\mathbb{R}_-)} + \Lambda^3\sum_{\pm}|\hat{v}_\pm(\xi,0)|^2\right)\\
\le&C\left(\sum_{\pm}||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda^2||\hat{v}_+(\xi,\cdot)||^2_{L^2(\mathbb{R}_+)} + \Lambda|\eta_1(\xi)|^2 + \Lambda^3|\eta_0(\xi)|^2\right)
\end{aligned}
\tag{3.142}
$$

Finally, putting together (3.139), (3.141) and (3.142) yields

$$
\begin{aligned}
&\Lambda\sum_{\pm}|V_\pm|^2 + \Lambda^3\sum_{\pm}|\hat{v}_\pm(\xi,0)|^2 + \frac{\Lambda^2}{\tau}\sum_{\pm}||\partial_y\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau}\sum_{\pm}||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}\\
\le&C\left(\sum_{\pm}||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda|\eta_1|^2 + \Lambda^3|\eta_0|^2 + \Lambda^2\sum_{\pm}||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}\right)
\end{aligned}
$$

that gives (3.137) if we take τ large enough to absorb the term $C\Lambda^2\sum_\pm||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}$. □

Now are ready to finish the proof of Theorem 3.1. Combining all cases (3.82), (3.117), (3.137), we conclude that

$$
\begin{aligned}
&\Lambda\sum_{\pm}|V_\pm|^2 + \Lambda^3\sum_{\pm}|\hat{v}_\pm(\xi,0)|^2 + \frac{\Lambda^2}{\tau}\sum_{\pm}||\partial_y\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau}\sum_{\pm}||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}\\
\le&C\left(\sum_{\pm}||P_\pm\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \Lambda|\eta_1|^2 + \Lambda^3|\eta_0|^2\right).
\end{aligned}
\tag{3.143}
$$

Recall that

$$P_\pm \hat{v}_\pm = \big(\partial_y - \tau\varphi'_\pm + it_\pm(\xi + i\tau s\gamma, y)\big) a^\pm_{nn}(y) \big(\partial_y - \tau\varphi'_\pm + it_\pm(\xi + i\tau s\gamma, y)\big)\hat{v}_\pm - a^\pm_{nn}(y)\zeta_\pm(\xi, y)\hat{v}_\pm,$$

which implies

$$|\partial_y^2 \hat{v}_\pm| \leq C\left(|P_\pm \hat{v}_\pm| + \Lambda|\partial_y \hat{v}_\pm| + \Lambda^2|\hat{v}_\pm|\right),$$

where C depends only on λ_0 and M_0. Therefore, we can derive

$$\frac{1}{\tau}\sum_\pm ||\partial_y^2 \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} \leq C\left(\sum_\pm ||P_\pm \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^4}{\tau}\sum_\pm ||\hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)} + \frac{\Lambda^2}{\tau}\sum_\pm ||\partial_y \hat{v}_\pm(\xi,\cdot)||^2_{L^2(\mathbb{R}_\pm)}\right). \tag{3.144}$$

The estimate (3.43) follows directly from (3.143) and (3.144). □

4 Step 2 - The Carleman estimate for general coefficients

Having at disposal the Carleman estimate when $A_\pm = A_\pm(y)$, we want to derive it for $A_\pm(x, y)$. The main idea is to "approximate" $A_\pm(x, y)$ with coefficients depending on y only. For this purpose we will make use of a special kind of partition of unity introduced in the next section.

4.1 Partition of unity and auxiliary results

In this section we collect some results on a partition of unity that will be crucial in our proof. In particular we will carefully describe how this partition of unity behaves with respect to the function spaces that we use.

Let $\vartheta_0 \in C_0^\infty(\mathbb{R})$ and $0 \leq \vartheta \leq 1$ such that

$$\vartheta_0(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \geq 3/2. \end{cases}$$

Also let $\vartheta(x) = \vartheta_0(x_1)\cdots\vartheta_0(x_{n-1})$, then we have

$$\vartheta(x) = \begin{cases} 1, & x \in Q_1(0), \\ 0, & x \in \mathbb{R}^{n-1} \setminus Q_{3/2}(0). \end{cases}$$

Given $\mu \geq 1$ and $g \in \mathbb{Z}^{n-1}$, we define

$$x_g = g/\mu$$

and

$$\vartheta_{g,\mu}(x) = \vartheta(\mu(x - x_g)).$$

It is not hard to see that

$$\operatorname{supp} \vartheta_{g,\mu} \subset Q_{3/2\mu}(x_g) \subset Q_{2/\mu}(x_g)$$

and

$$|D^k \vartheta_{g,\mu}| \leq C_1 \mu^k (\chi_{Q_{3/2\mu}(x_g)} - \chi_{Q_{1/\mu}(x_g)}), \quad k = 0, 1, 2, \tag{4.1}$$

where $C_1 \geq 1$ depends only on n.

For $g \in \mathbb{Z}^{n-1}$, let $A_g = \{g' \in \mathbb{Z}^{n-1} \,|\, \operatorname{supp} \vartheta_{g',\mu} \cap \operatorname{supp} \vartheta_{g,\mu} \neq \emptyset\}$, then

$$card(A_g) \text{ depends only on } n. \tag{4.2}$$

Thus, we can define

$$\bar{\vartheta}_\mu(x) := \sum_{g \in \mathbb{Z}^N} \vartheta_{g,\mu} \geq 1, \quad x \in \mathbb{R}^{n-1}. \tag{4.3}$$

It is clear that (4.1) implies

$$|D^k \bar{\vartheta}_\mu| \leq C_2 \mu^k, \tag{4.4}$$

where $C_2 \geq 1$ depends on n. If we set

$$\eta_{g,\mu}(x) = \vartheta_{g,\mu}(x)/\bar{\vartheta}_\mu(x), \quad x \in \mathbb{R}^{n-1},$$

then we have that

$$\begin{cases} \sum_{g \in \mathbb{Z}^{n-1}} \eta_{g,\mu} = 1, \quad x \in \mathbb{R}^{n-1}, \\ \operatorname{supp} \eta_{g,\mu} \subset Q_{3/2\mu}(x_g) \subset Q_{2/\mu}(x_g), \\ |D^k \eta_{g,\mu}| \leq C_3 \mu^k \chi_{Q_{3/2\mu}(x_g)}, \quad k = 0, 1, 2, \end{cases} \tag{4.5}$$

where $C_3 \geq 1$ depends on n.

In Section 2 we have recalled the definition of $H^{1/2}(\mathbb{R}^{n-1})$ and its seminorm $[\cdot]_{1/2,\mathbb{R}^{n-1}}$, in what follows we will also need the seminorm

$$[f]_{1/2,Q_r} = \left[\int_{Q_r} \int_{Q_r} \frac{|f(x) - f(y)|^2}{|x - y|^n} dx dy \right]^{1/2}, \tag{4.6}$$

where $Q_r = Q_r(0) = \{x \in \mathbb{R}^{n-1} : |x_j| \leq r, \, j = 1, 2, \cdots, n-1\}$.

Lemma 4.1 *Let $f \in C^\infty(\mathbb{R}^{n-1})$ and $\operatorname{supp} f \subset Q_{3r/4}$ for some $r \leq 1$. There exists a positive constant C, depending only on n, such that*

$$[f]^2_{1/2,Q_r} + \frac{C^{-1}}{r} \int_{Q_r} |f(x)|^2 dx \leq ||f||^2_{H^{1/2}(\mathbb{R}^{n-1})} \leq [f]^2_{1/2,Q_r} + \frac{C}{r} \int_{Q_r} |f(x)|^2 dx. \tag{4.7}$$

Proof. It follows easily from (2.9) and (4.6), that

$$[f]^2_{1/2,\mathbb{R}^{n-1}} = I + [f]^2_{1/2,Q_r}, \tag{4.8}$$

where

$$I = 2\int_{\mathbb{R}^{n-1}\setminus Q_r}\int_{Q_r}\frac{|f(x)-f(y)|^2}{|x-y|^n}dydx = 2\int_{\mathbb{R}^{n-1}\setminus Q_r}\int_{Q_{3r/4}}\frac{|f(y)|^2}{|x-y|^n}dydx.$$

Note that there is a positive constant $C_n < 1$, depending only on n, such that, for $x \in \mathbb{R}^{n-1}\setminus Q_r$ and $y \in Q_{3r/4}$, we have

$$C_n^{-1}|x| \le |x-y| \le C_n|x|,$$

hence, by using Fubini theorem, there is a constant C depending only on n, such that

$$\frac{C^{-1}}{r}\int_{Q_r}|f(y)|^2dy \le I \le \frac{C}{r}\int_{Q_r}|f(y)|^2dy,$$

that, together with (4.8), gives (4.7). □

Proposition 4.1 *Let* $\{\varsigma_g\}_{g\in\mathbb{Z}^{n-1}}$ *be a family of smooth functions such that* $\operatorname{supp}\varsigma_g \subset Q_{\frac{3}{2\mu}}(x_g)$*, then*

$$[\sum_{g\in\mathbb{Z}^{n-1}}\varsigma_g]^2_{1/2,\mathbb{R}^{n-1}} \le C\big(\sum_{g\in\mathbb{Z}^{n-1}}[\varsigma_g]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} + \sum_{g\in\mathbb{Z}^{n-1}}\mu\int_{Q_{\frac{2}{\mu}}(x_g)}|\varsigma_g|^2\big), \tag{4.9}$$

where C depends only on n.

Proof. Let $x' = \mu x$ and $y' = \mu y$, then

$$\begin{aligned}[\sum_g\varsigma_g]^2_{1/2,\mathbb{R}^{n-1}} &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|\sum_g\varsigma_g(x)-\sum_g\varsigma_g(y)|^2}{|x-y|^n}dydx\\ &=\mu^{2-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|\sum_g\varsigma_g(x'/\mu)-\sum_g\varsigma_g(y'/\mu)|^2}{|x'-y'|^n}dy'dx'.\end{aligned}$$

It is enough to consider $n=2$ and denote $\varsigma_g(x'/\mu) = \varsigma_j(x')$. Note that $\operatorname{supp}\varsigma_j \subset Q_{\frac{3}{2}}(j) = \{x\in\mathbb{R}||x-j|\le 3/2\}$. We write

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\sum_{j\in\mathbb{Z}}\varsigma_j(x)-\sum_{j\in\mathbb{Z}}\varsigma_j(y)|^2}{|x-y|^2}\,dxdy = I_1 + I_2, \tag{4.10}$$

where

$$I_1 := \int_{\mathbb{R}^2}\frac{|\sum_{j\in\mathbb{Z}}\varsigma_j(x)-\sum_{j\in\mathbb{Z}}\varsigma_j(y)|^2}{|x-y|^2}\chi_{\{|x-y|<1\}}\,dxdy$$

$$I_2 := \int_{\mathbb{R}^2}\frac{|\sum_{j\in\mathbb{Z}}\varsigma_j(x)-\sum_{j\in\mathbb{Z}}\varsigma_j(y)|^2}{|x-y|^2}\chi_{\{|x-y|\ge1\}}\,dxdy.$$

Let us first estimate I_2. It is not hard to check that

$$I_2 \leq \int_{\mathbb{R}^2} \frac{2|\sum_{j\in\mathbb{Z}} \varsigma_j(x)|^2 + 2|\sum_{j\in\mathbb{Z}} \varsigma_j(y)|^2}{|x-y|^2} \chi_{\{|x-y|\geq 1\}} dxdy$$
$$=4\int_{\mathbb{R}^2} \frac{|\sum_{j\in\mathbb{Z}} \varsigma_j(x)|^2}{|x-y|^2} \chi_{\{|x-y|\geq 1\}} dxdy \leq 8\int_{\mathbb{R}} |\sum_{j\in\mathbb{Z}} \varsigma_j(x)|^2 dx.$$

Since the cardinality of A_g only depends on n, we have for $n=2$ that

$$|\sum_{j\in\mathbb{Z}} \varsigma_j(x)|^2 \leq 7\sum_{j\in\mathbb{Z}} |\varsigma_j(x)|^2$$

and hence

$$I_2 \leq 56\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\varsigma_j(x)|^2 dx = 56\sum_{j\in\mathbb{Z}} \int_{Q_2(j)} |\varsigma_j(x)|^2 dx. \tag{4.11}$$

Next, for I_1, we can see that

$$I_1 = \int_{\mathbb{R}^2} \frac{|\sum_{j\in\mathbb{Z}} \varsigma_j(x) - \sum_{j\in\mathbb{Z}} \varsigma_j(y)|^2}{|x-y|^2} \chi_{\{|x-y|<1\}} dxdy$$
$$\leq \sum_{i\in\mathbb{Z}} \int_{Q_2(i)} \int_{\mathbb{R}} \frac{|\sum_{j\in\mathbb{Z}} \varsigma_j(x) - \sum_{j\in\mathbb{Z}} \varsigma_j(y)|^2}{|x-y|^2} \chi_{\{|x-y|<1\}} dydx.$$

We note for each $x \in Q_2(i)$ that

$$\sum_{j\in\mathbb{Z}} \varsigma_j(x) = \sum_{|j-i|\leq 3} \varsigma_j(x)$$

and

$$\operatorname{dist}(Q_2(i), Q_2(l)) \geq 1, \quad |l-i| \geq 5.$$

Therefore, we have

$$I_1 \leq \sum_{i\in\mathbb{Z}} \int_{Q_2(i)} \sum_{|l-i|\leq 4} \int_{Q_2(l)} \frac{|\sum_{|j-i|\leq 3} \varsigma_j(x) - \sum_{k\in\mathbb{Z}} \varsigma_k(y)|^2}{|x-y|^2} dydx$$
$$=\sum_{i\in\mathbb{Z}} \sum_{|l-i|\leq 4} \int_{Q_2(i)} \int_{Q_2(l)} \frac{|\sum_{|j-i|\leq 3} \varsigma_j(x) - \sum_{|k-l|\leq 3} \varsigma_k(y)|^2}{|x-y|^2} dydx.$$

For $|l-i| \leq 4$, we note for $y \in Q_2(l)$ and $x \in Q_2(i)$ that

$$\sum_{|k-l|\leq 3} \varsigma_k(y) = \sum_{|k-i|\leq 7} \varsigma_k(y)$$

and

$$\sum_{|j-i|\leq 3} \varsigma_j(x) = \sum_{|j-i|\leq 7} \varsigma_j(x).$$

Thus, we can derive

$$\begin{aligned}
I_1 \leq & \sum_{i\in\mathbb{Z}} \int_{Q_2(i)} \sum_{|l-i|\leq 4} \int_{Q_2(l)} \frac{|\sum_{|k-i|\leq 7}(\varsigma_k(x)-\varsigma_k(y))|^2}{|x-y|^2} dydx \\
\leq & 7 \sum_{i\in\mathbb{Z}} \sum_{|l-i|\leq 4} \sum_{|k-i|\leq 7} \int_{Q_2(i)} \int_{Q_2(l)} \frac{|(\varsigma_k(x)-\varsigma_k(y))|^2}{|x-y|^2} dydx.
\end{aligned}$$

Since $Q_2(l) \subset Q_6(i)$ when $|l-i| \leq 4$, we obtain that

$$\begin{aligned}
I_1 \leq & 7 \sum_{i\in\mathbb{Z}} \sum_{|l-i|\leq 4} \sum_{|k-i|\leq 7} \int_{Q_2(i)} \int_{Q_6(i)} \frac{|(\varsigma_k(x)-\varsigma_k(y))|^2}{|x-y|^2} dydx \\
\leq & 10^3 \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|(\varsigma_k(x)-\varsigma_k(y))|^2}{|x-y|^2} dydx \qquad (4.12) \\
\leq & 10^3 \sum_{k\in\mathbb{Z}} \{[\varsigma_k]^2_{1/2,Q_2(k)} + \int_{Q_2(k)} |\varsigma_k(x)|^2 dx\},
\end{aligned}$$

where we used (4.7) in the last inequality. Combining (4.11) and (4.12), the proof is complete. □

Proposition 4.2 *Let $F \in C^\infty(\mathbb{R}^{n-1}) \cap H^{1/2}(\mathbb{R}^{n-1})$ with $\operatorname{supp} F \subset Q_{3/2\mu}(x_g)$, and let a be a function satisfying*

$$|a(z)| \leq E_a, \quad |a(x) - a(x')| \leq K_a |x - x'|, \tag{4.13}$$

for $z, x, x' \in \operatorname{supp} \eta_{g,\mu}$ and E_a, K_a positive constants. Then, there is a constant C depending only on n such that,

$$[aF]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} \leq C \left(E_a^2 [F]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} + K_a^2 \mu^{-1} \int_{Q_{\frac{2}{\mu}}(x_g)} |F(y)|^2 dy \right). \tag{4.14}$$

Proof. In view of (4.6), we have

$$\begin{aligned}
[aF]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} = & \int_{Q_{\frac{2}{\mu}}(x_g)} \int_{Q_{\frac{2}{\mu}}(x_g)} \frac{|a(x)F(x) - a(y)F(y)|^2}{|x-y|^n} dxdy \\
\leq 2 & \int_{Q_{\frac{2}{\mu}}(x_g)} \int_{Q_{\frac{2}{\mu}}(x_g)} \left(\frac{|a(x)|^2 \, |F(x) - F(y)|^2}{|x-y|^n} + \frac{|F(y)|^2 \, |a(x) - a(y)|^2}{|x-y|^n} \right) dxdy, \\
\leq C & \left(E_a^2 [F]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} + K_a^2 \mu^{-1} \int_{Q_{\frac{2}{\mu}}(x_g)} |F(y)|^2 dy \right).
\end{aligned}$$

□

Proposition 4.3 *Let $f \in C^\infty(\mathbb{R}^{n-1}) \cap H^{1/2}(\mathbb{R}^{n-1})$. Then*

$$\sum_{g\in\mathbb{Z}^{n-1}} [f\eta_{g,\mu}]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} \leq C\left([f]^2_{1/2,\mathbb{R}^{n-1}} + \mu\int_{\mathbb{R}^{n-1}} |f(y)|^2 dy\right). \tag{4.15}$$

Proof. It follows from (4.6) that

$$[f\eta_{g,\mu}]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} \leq I + 2\int_{Q_{\frac{2}{\mu}}(x_g)}\int_{Q_{\frac{2}{\mu}}(x_g)} \frac{|\eta_{g,\mu}(x)|^2\,|f(x)-f(y)|^2}{|x-y|^n}\,dxdy, \tag{4.16}$$

where

$$I = 2\int_{Q_{\frac{2}{\mu}}(x_g)}\int_{Q_{\frac{2}{\mu}}(x_g)} |f(y)|^2\,|\eta_{g,\mu}(x)-\eta_{g,\mu}(y)|^2\,|x-y|^{-n}dxdy.$$

Using (4.5), we can estimate

$$I \leq 2C_3^2\mu^2\int_{Q_{\frac{2}{\mu}}(x_g)}\int_{Q_{\frac{2}{\mu}}(x_g)} \frac{|f(y)|^2}{|x-y|^n}\,dxdy \leq C\mu\int_{Q_{\frac{2}{\mu}}(x_g)} |f(y)|^2 dy. \tag{4.17}$$

Using the fact that for any $g \in \mathbb{Z}^{n-1}$, the cardinality of $\{g' \in \mathbb{Z}^{n-1} | Q_{2/\mu}(x_g) \cap Q_{2/\mu}(x_{g'}) \neq \emptyset\}$ is finite and only depends on n and adding up with respect to $g \in \mathbb{Z}^{n-1}$, we get (4.15). □

Proposition 4.4 *Let $f \in C^\infty(\mathbb{R}^{n-1}) \cap H^{1/2}(\mathbb{R}^{n-1})$. Then*

$$\sum_{g\in\mathbb{Z}^{n-1}} [f\,\nabla_x\eta_{g,\mu}]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} \leq C\left(\mu^2[f]^2_{1/2,\mathbb{R}^{n-1}} + \mu^3\int_{\mathbb{R}^{n-1}} |f(y)|^2 dy\right). \tag{4.18}$$

We omit the proof that proceeds in the same way as that of Proposition 4.3.

4.2 Estimate of the left hand side of the Carleman estimate, I

We are ready to derive the Carleman estimate for general coefficients. In order to make clear the procedure that we follow let us introduce and recall some notations and definitions. Let $0 < \delta \leq 1$ and define

$$A^\delta_\pm(x,y) := A_\pm(\delta x, \delta y), \tag{4.19}$$

$$\mathcal{L}_\delta(x,y,\partial)w := \sum_\pm H_\pm \mathrm{div}_{x,y}(A^\delta_\pm(x,y)\nabla_{x,y}w_\pm), \tag{4.20}$$

and the transmission conditions

$$\begin{cases}\theta_0(x) = w_+(x,0) - w_-(x,0),\\ \theta_1(x) = A_+(x,0)\nabla_{x,y}w_+(x,0)\cdot\nu - A_-(x,0)\nabla_{x,y}w_-(x,0)\cdot\nu.\end{cases}$$

Next, with $x_g = g/\mu\ g \in \mathbb{Z}^{n-1}$, we define

$$\begin{cases} A_{\pm}^{\delta,g}(y) := A_{\pm}^{\delta}(x_g, y) = A_{\pm}(\delta x_g, \delta y), \\ \mathcal{L}_{\delta,g}(y,\partial)w := \sum_{\pm} H_{\pm}\mathrm{div}_{x,y}(A_{\pm}^{\delta,g}(y)\nabla_{x,y}w_{\pm}). \end{cases}$$

It is readily seen that

$$\lambda_0|z|^2 \le A_{\pm}^{\delta,g}(y)z\cdot z \le \lambda_0^{-1}|z|^2,\ \forall y \in \mathbb{R},\ \forall\, z \in \mathbb{R}^n$$

and

$$|A_{\pm}^{\delta,g}(y') - A_{\pm}^{\delta,g}(y)| \le M_0\delta|y'-y|.$$

Concerning the weight functions, let us introduce the following notations

$$\begin{cases} h_\varepsilon(x) := -\varepsilon|x|^2/2, \\ H_\varepsilon(x,x_g) := \varepsilon|x-x_g|^2/2, \\ \psi_\varepsilon(x,y) := \varphi(y) + h_\varepsilon(x), \\ \psi_{\varepsilon,g}(x,y) := \varphi(y) + \nabla_x h_\varepsilon(x_g)\cdot(x-x_g) + h_\varepsilon(x_g), \end{cases}$$

where $\varphi(y)$ is defined in (2.7). Moreover assume that $\alpha_+, \alpha_-, \beta$ are fixed positive numbers such that $\beta \ge \beta_0$ and $\lambda_2^{-1} < \frac{\alpha_+}{\alpha_-}$, in such a way that condition (3.42) is satisfied by the operator $\mathcal{L}_{\delta,g}(y,\partial)$ and Theorem 3.1 holds true for such an operator.

Note that

$$\psi_{\varepsilon,g}(x,y) - \psi_\varepsilon(x,y) = H_\varepsilon(x,x_g). \tag{4.21}$$

We define

$$\begin{aligned} \Xi(w) :=& \sum_{\pm}\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^k w_\pm|^2 e^{2\tau\psi_\varepsilon}dxdy \\ &+ \sum_{\pm}\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^k w_\pm(x,0)|^2 e^{2\tau\psi_\varepsilon(x,0)}dx \\ &+ \sum_{\pm}\tau^2[e^{\tau\psi_\varepsilon(\cdot,0)}w_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\ &+ \sum_{\pm}[\partial_y(e^{\tau\psi_{\varepsilon,\pm}}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_{\pm}[\nabla_x(e^{\tau\psi_\varepsilon}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}, \end{aligned} \tag{4.22}$$

where we note that $\Xi(w)$ corresponds to the left hand side of (2.10).

In the present subsection we prove that if $\mathrm{supp}\, w \subset \mathfrak{U} := B_{1/2}\times[-r_0,r_0]$ and if we choose

$$\tau \ge 1/\varepsilon \quad \text{and} \quad \mu = (\varepsilon\tau)^{1/2}, \tag{4.23}$$

then

$$\Xi(w) \le C\sum_{g\in\mathbb{Z}^{n-1}}\Xi(w\eta_{g,\mu}) + CR_1, \tag{4.24}$$

where

$$R_1 := (\varepsilon\tau)^{1/2}\sum_{\pm}\int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon(x,0)}(|\partial_y w_\pm(x,0)|^2 + |\nabla_x w_\pm(x,0)|^2 + \tau^2|w_\pm(x,0)|^2)dx$$

and C depends only on λ_0, M_0.

In order to obtain (4.24), we estimate from above each term in (4.22), by (4.5),

$$w_\pm = \sum_{g\in\mathbb{Z}^{n-1}} w_\pm \eta_{g,\mu}(x). \tag{4.25}$$

From (4.2), (4.21) and (4.25), we can see that

$$\begin{aligned}&\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^k w_\pm|^2 e^{2\tau\psi_\varepsilon}\,dxdy\\ &\le C\sum_{g\in\mathbb{Z}^{n-1}}\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^k(w_\pm\eta_{g,\mu})|^2 e^{2\tau\psi_{\varepsilon,g,\pm}}dxdy\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}&\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^k w_\pm(x,0)|^2 e^{2\tau\psi_\varepsilon(x,0)}\,dx\\ &\le C\sum_{g\in\mathbb{Z}^{n-1}}\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^k(w_\pm\eta_{g,\mu})(x,0)|^2 e^{2\tau\psi_{\varepsilon,g}(x,0)}\,dx,\end{aligned} \tag{4.27}$$

where C depends only on n.

Using (4.9), we obtain

$$\begin{aligned}&[\nabla_x(e^{\tau\psi_\varepsilon}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} = [\nabla_x(e^{\tau\psi_\varepsilon}\sum_g w_\pm\eta_{g,\mu})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\ &\le C\sum_{g\in\mathbb{Z}^{n-1}}\left([\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)} + \mu\int_{Q_{\frac{2}{\mu}}(x_g)}|\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(x,0)|^2dx\right).\end{aligned} \tag{4.28}$$

Since

$$\begin{aligned}&\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(x,0)\\ =&e^{\tau\psi_\varepsilon(x,0)}\eta_{g,\mu}\nabla_x w_\pm(x,0) + e^{\tau\psi_\varepsilon(x,0)}w_\pm\nabla_x\eta_{g,\mu}(x,0) - (\varepsilon\tau x)e^{\tau\psi_\varepsilon(x,0)}\eta_{g,\mu}w_\pm(x,0),\end{aligned}$$

by (4.9) and (4.2), we have that

$$\begin{aligned}&\sum_{g\in\mathbb{Z}^{n-1}}\mu\int_{Q_{\frac{2}{\mu}}(x_g)}|\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(x,0)|^2dx\\ \le&C\left(\mu\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\nabla_x w_\pm(x,0)|^2dx + \mu^5\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|w_\pm(x,0)|^2dx\right).\end{aligned} \tag{4.29}$$

In order to estimate $\sum_{g\in\mathbb{Z}^{n-1}}[\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(\cdot,0)]^2_{1/2,Q_{2/\mu}(x_g)}$ we need to observe the following easy consequence of Lemma 4.1, Proposition 4.2 and of (4.21):

Lemma 4.2 *If* $\operatorname{supp} f\subset Q_{3/2\mu}(x_g)$, *then we have that*

$$[fe^{\tau\psi_\varepsilon(\cdot,0)}]^2_{1/2,\mathbb{R}^{n-1}}\le C[fe^{\tau\psi_{\varepsilon,g}(\cdot,0)}]^2_{1/2,\mathbb{R}^{n-1}}+\mu\int_{Q_{2/\mu}(x_g)}|f(x)|^2e^{2\tau\psi_\varepsilon(x,0)}dx, \tag{4.30}$$

and

$$[fe^{\tau\psi_{\varepsilon,g}(\cdot,0)}]^2_{1/2,\mathbb{R}^{n-1}}\le C[fe^{\tau\psi_\varepsilon(\cdot,0)}]^2_{1/2,Q_{2/\mu}(x_g)}+\mu\int_{Q_{2/\mu}(x_g)}|f(x)|^2e^{2\tau\psi_\varepsilon(x,0)}dx, \tag{4.31}$$

where C depends only on n.

Similarly, we can show that

Lemma 4.3

$$\begin{aligned}[xe^{\tau\psi_\varepsilon(\cdot,0)}\eta_{g,\mu}w_\pm]^2_{1/2,Q_{2/\mu}(x_g)}\le C\Bigg([e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\eta_{g,\mu}w_\pm]^2_{1/2,Q_{2/\mu}(x_g)}&\\ +\frac{1}{\mu}\int_{Q_{2/\mu}(x_g)}|\eta_{g,\mu}w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx\Bigg).&\end{aligned} \tag{4.32}$$

It is time to estimate $\sum_\pm\sum_{g\in\mathbb{Z}^{n-1}}[\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}$. Since

$$\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(x,0)=e^{\tau\psi_\varepsilon}\nabla_x(\eta_{g,\mu}w_\pm)(x,0)-(\varepsilon\tau x)e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm(x,0),$$

we can deduce from (4.30) and (4.32) that

$$\begin{aligned}&\sum_{g\in\mathbb{Z}^{n-1}}[\nabla_x(e^{\tau\psi_\varepsilon}\eta_{g,\mu}w_\pm)(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}\le\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_\varepsilon(\cdot,0)}\nabla_x(\eta_{g,\mu}w_\pm)(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}\\&+(\varepsilon\tau)^2\sum_{g\in\mathbb{Z}^{n-1}}[xe^{\tau\psi_\varepsilon(\cdot,0)}\eta_{g,\mu}w_\pm)(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}\\ \le&C\Bigg(\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\nabla_x(\eta_{g,\mu}w_\pm)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}+\mu\sum_{g\in\mathbb{Z}^{n-1}}\int_{Q_{2/\mu}(x_g)}|\nabla_x(\eta_{g,\mu}w_\pm)|^2e^{2\tau\psi_\varepsilon(x,0)}dx\\&+(\varepsilon\tau)^2\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\eta_{g,\mu}w_\pm)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}+\mu^{-1}(\varepsilon\tau)^2\sum_{g\in\mathbb{Z}^{n-1}}\int_{Q_{2/\mu}(x_g)}|\eta_{g,\mu}w_\pm|^2e^{2\tau\psi_\varepsilon(x,0)}dx\Bigg)\\ \le&C\Bigg(\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\nabla_x(\eta_{g,\mu}w_\pm)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}+(\varepsilon\tau)^2\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\eta_{g,\mu}w_\pm)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}\\&+\sum_\pm\mu\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\nabla_xw_\pm(x,0)|^2dx+\mu^3\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|w_\pm(x,0)|^2dx\Bigg).\end{aligned} \tag{4.33}$$

Combining (4.28), (4.29) and (4.33) yields

$$[\nabla_x(e^{\tau\psi_\varepsilon}\sum_g w_\pm\eta_{g,\mu})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \le C\left(\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\nabla_x(\eta_{g,\mu}w_\pm)(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}\right.$$
$$+\mu^4\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\eta_{g,\mu}w_\pm(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}+\mu\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\nabla_x w_\pm(x,0)|^2dx$$
$$\left.+\mu^5\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|w_\pm(x,0)|^2dx\right). \tag{4.34}$$

In a similar way, we can estimate the terms $[\partial_y(e^{\tau\psi_{\varepsilon\pm}}\sum_g w_\pm\eta_{g,\mu})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}$ and $\tau^2[e^{\tau\psi_\varepsilon(\cdot,0)}\sum_g w_\pm\eta_{g,\mu}]^2_{1/2,\mathbb{R}^{n-1}}$ and finally get (4.24). Notice that in deriving (4.24) we make use of $\mu^4=(\varepsilon\tau)^2\le\tau^2$.

4.3 Estimate of the left hand side of the Carleman estimate, II

In this section, we will continue to estimate the upper bound of $\Xi(w)$ using (4.24). The task now is to connect the estimate to the operator $\mathcal{L}(x,y,\partial)$ given in (2.1). To this aim we apply Theorem 3.1 to the function $w\eta_{g,\mu}$ with the weight function $\psi_{\varepsilon,g}=\varphi(y)-\varepsilon x_g\cdot x+\varepsilon|x_g|^2/2$. In order to do this we note that if $\operatorname{supp} w\subset\mathfrak{U}:=B_{1/2}\times[-r_0,r_0]$ and $\mu\ge 4$, then either $|x_g|\le 1$ or $\operatorname{supp}\eta_{g,\mu}\cap B_{1/2}=\emptyset$. Thus, in both the cases, we can apply Theorem 3.1.

By applying (3.7) and by adding up with respect to $g\in\mathbb{Z}^{n-1}$, we obtain that

$$\sum_{g\in\mathbb{Z}^{n-1}}\Xi(w\eta_{g,\mu})\le C\sum_{g\in\mathbb{Z}^{n-1}}(d^{(1)}_{g,\mu}+d^{(2)}_{g,\mu}+d^{(3)}_{g,\mu}), \tag{4.35}$$

where

$$d^{(1)}_{g,\mu}=\int_{\mathbb{R}^n}|\mathcal{L}_{\delta,g}(y,\partial)(w\eta_{g,\mu})|^2e^{2\tau\psi_{\varepsilon,g}}dxdy,$$
$$d^{(2)}_{g,\mu}=\tau^3\int_{\mathbb{R}^{n-1}}|e^{\tau\psi_{\varepsilon,g}(x,0)}\theta_{0;g,\mu}(x)|^2dx+[\nabla_x(e^{\tau\psi_{\varepsilon,g}}\theta_{0;g,\mu})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}},$$
$$d^{(3)}_{g,\mu}=\tau\int_{\mathbb{R}^{n-1}}|e^{\tau\psi_{\varepsilon,g}(x,0)}\theta_{1;g,\mu}(x)|^2dx+[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\theta_{1;g,\mu}(\cdot)]^2_{1/2,\mathbb{R}^{n-1}},$$

where we set

$$\theta_{0;g,\mu}(x):=w_+(x,0)\eta_{g,\mu}(x)-w_-(x,0)\eta_{g,\mu}(x)=\theta_0(x)\eta_{g,\mu}, \tag{4.36}$$

$$\theta_{1;g,\mu}(x):=A^{\delta,g}_+(0)\nabla_{x,y}(w_+\eta_{g,\mu})\cdot\nu-A^{\delta,g}_-(0)\nabla_{x,y}(w_-\eta_{g,\mu})\cdot\nu. \tag{4.37}$$

We will estimate the three terms of (4.35) separately.

Estimate of $\sum_{g\in\mathbb{Z}^{n-1}} d^{(1)}_{g,\mu}$.

By straightforward computations, we obtain that

$$\begin{aligned}&\mathcal{L}_{\delta,g}(y,\partial)(w_\pm\eta_{g,\mu})\\ \le&|\mathcal{L}_\delta(x,y,\partial)(w_\pm\eta_{g,\mu})|+|\mathcal{L}_\delta(x,y,\partial)(w_\pm\eta_{g,\mu})-\mathcal{L}_{\delta,g}(y,\partial)(w_\pm\eta_{g,\mu})|\\ \le&\eta_{g,\mu}|\mathcal{L}_\delta(x,y,\partial)(w_\pm)|+C\left(\delta\mu^{-1}|D^2w_\pm|\,\chi_{Q_{\frac{2}{\mu}}(x_g)}+\mu|Dw_\pm|\,\chi_{Q_{\frac{2}{\mu}}(x_g)}+\mu^2|w_\pm|\,\chi_{Q_{\frac{2}{\mu}}(x_g)}\right),\end{aligned}$$

which implies

$$\sum_{g\in\mathbb{Z}^{n-1}} d^{(1)}_{g,\mu}\le C\sum_\pm\int_{\mathbb{R}^n_\pm}|\mathcal{L}_\delta(x,y,\partial)(w_\pm)|^2\,e^{2\tau\psi_\varepsilon}dxdy+CR_2, \tag{4.38}$$

where

$$\begin{aligned}R_2=&\delta^2\mu^{-2}\sum_\pm\int_{\mathbb{R}^n_\pm}|D^2w_\pm|^2\,e^{2\tau\psi_{\varepsilon,\pm}}dxdy+\mu^2\sum_\pm\int_{\mathbb{R}^n_\pm}|Dw_\pm|^2\,e^{2\tau\psi_{\varepsilon,\pm}}dxdy\\ &+\mu^4\sum_\pm\int_{\mathbb{R}^n_\pm}|w_\pm|^2\,e^{2\tau\psi_{\varepsilon,\pm}}dxdy.\end{aligned}$$

Estimate of $\sum_{g\in\mathbb{Z}^{n-1}} d^{(2)}_{g,\mu}$.

It is obvious that

$$\sum_{g\in\mathbb{Z}^{n-1}}\tau^3\int_{\mathbb{R}^{n-1}}|e^{\tau\psi_{\varepsilon,g}(x,0)}\theta_{0;g,\mu}(x)|^2dxC\tau^3\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\theta_0(x)|^2dx, \tag{4.39}$$

where C depends only on n. Next, we note that $\nabla_x(e^{\tau\psi_{\varepsilon,g}}\theta_{0;g,\mu})=e^{\tau\psi_{\varepsilon,g}}\nabla_x\theta_{0;g,\mu}-\tau\varepsilon x_g e^{\tau\psi_{\varepsilon,g}}\theta_{0;g,\mu}$. From (4.31), (4.15), and (4.36), it follows that

$$\begin{aligned}&\sum_{g\in\mathbb{Z}^{n-1}}[\nabla_x(e^{\tau\psi_{\varepsilon,g}}\theta_{0;g,\mu})(\cdot,0)]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}\\ \le&C\left(\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_\varepsilon(\cdot,0)}\nabla_x\theta_{0;g,\mu}]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}+(\tau\varepsilon)^2[e^{\tau\psi_\varepsilon(\cdot,0)}\omega_0]^2_{1/2,\mathbb{R}^{n-1}}\right.\\ &\left.+\mu\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\nabla_x\theta_0|^2dx+\mu^5\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\theta_0|^2dx\right).\end{aligned} \tag{4.40}$$

On the other hand, by (4.15), (4.18) and (4.32), we have that

$$\begin{aligned}&\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_\varepsilon(\cdot,0)}\nabla_x\theta_{0;g,\mu}]^2_{1/2,Q_{\frac{2}{\mu}}(x_g)}\\ \lesssim&[\nabla_x(e^{\tau\psi_\varepsilon(\cdot,0)}\theta_0)]^2_{1/2,\mathbb{R}^{n-1}}+\mu^4[e^{\tau\psi_\varepsilon(\cdot,0)}\theta_0]^2_{1/2,\mathbb{R}^{n-1}}\\ &+\mu\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\nabla_x\theta_0|^2dx+\mu^5\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\theta_0|^2dx.\end{aligned} \tag{4.41}$$

Finally, combining (4.39) and (4.41) yields

$$\sum_{g\in\mathbb{Z}^{n-1}} d^{(2)}_{g,\mu} \le C\left([\nabla_x(e^{\tau\psi_\varepsilon}\theta_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \tau^3\int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon(x,0)}|\theta_0|^2dx + R_3\right), \quad (4.42)$$

where

$$R_3 = \mu^4[e^{\tau\psi_\varepsilon(\cdot,0)}\theta_0]^2_{1/2,\mathbb{R}^{n-1}} + \mu\int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon(x,0)}|\nabla_x\theta_0|^2dx + \mu^5\int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon(x,0)}|\theta_0|^2dx.$$

Estimate of $\sum_{g\in\mathbb{Z}^{n-1}} d^{(3)}_{g,\mu}$.

Straightforward computations show that we can write $\theta_{1;g,\mu}$ as

$$\theta_{1;g,\mu} = \theta_1\eta_{g,\mu} + J^{(1)}_{g,\mu} + J^{(2)}_{g,\mu} + J^{(3)}_{g,\mu}, \quad (4.43)$$

where

$$\begin{aligned} J^{(1)}_{g,\mu} =& w_+A_+(\delta x,0)\nabla_{x,y}\eta_{g,\mu}\cdot\nu - w_-A_-(\delta x,0)\nabla_{x,y}\eta_{g,\mu}\cdot\nu,\\ J^{(2)}_{g,\mu} =& \eta_{g,\mu}(A_+(\delta x_g,0) - A_+(\delta x,0))\nabla_{x,y}w_+\cdot\nu\\ &- \eta_{g,\mu}(A_-(\delta x_g,0) - A_-(\delta x,0))\nabla_{x,y}w_-\cdot\nu,\\ J^{(3)}_{g,\mu} =& w_+(A_+(\delta x_g,0) - A_+(\delta x,0))\nabla_{x,y}\eta_{g,\mu}\cdot\nu\\ &- w_-(A_-(\delta x_g,0) - A_-(\delta x,0))\nabla_{x,y}\eta_{g,\mu}\cdot\nu. \end{aligned}$$

It is easy to compute that

$$\begin{aligned} |J^{(1)}_{g,\mu}| &\le C\mu\sum_\pm |w_\pm(x,0)|\,\chi_{Q_{\frac{2}{\mu}}(x_g)},\\ |J^{(2)}_{g,\mu}| &\le C\delta\mu^{-1}\sum_\pm |\nabla_{x,y}w_\pm(x,0)|\eta_{g,\mu},\\ |J^{(3)}_{g,\mu}| &\le C\delta\mu^{-1}\sum_\pm |\nabla_{x,y}\eta_{g,\mu}||w_\pm(x,0)|, \end{aligned} \quad (4.44)$$

where C depends on λ_0, M_0 and n. Putting together (4.43) and (4.44) implies

$$\begin{aligned} \sum_{g\in\mathbb{Z}^{n-1}} \tau\int_{\mathbb{R}^{n-1}} |e^{\tau\psi_{\varepsilon,g}(x,0)}\theta_{1;g,\mu}(x)|^2dx \le C\Bigg(\tau\int_{\mathbb{R}^{n-1}} |\omega_1|^2e^{2\tau\psi_\varepsilon(x,0)}dx\\ + \delta^2\varepsilon^{-1}\sum_\pm\int_{\mathbb{R}^{n-1}} |\nabla_{x,y}w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx\\ +(\delta^2\tau+\tau^2\varepsilon)\sum_\pm\int_{\mathbb{R}^{n-1}} |w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx\Bigg). \end{aligned} \quad (4.45)$$

We turn to the second term of $d^{(3)}_{g,\mu}$. We first derive from (4.31) and (4.15) that

$$\begin{aligned} &\sum_{g\in\mathbb{Z}^{n-1}} [e^{\tau\psi_{\varepsilon,g}(\cdot,0)}\theta_1\eta_{g,\mu}]^2_{1/2,\mathbb{R}^{n-1}}\\ \le& C[e^{\tau\psi_\varepsilon(\cdot,0)}\theta_1]^2_{1/2,\mathbb{R}^{n-1}} + C\mu\int_{\mathbb{R}^{n-1}} |\theta_1|^2e^{2\tau\psi_\varepsilon(x,0)}dx. \end{aligned} \quad (4.46)$$

Again by (4.31), (4.14) and (4.18) we get

$$\begin{aligned}&\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}J^{(1)}_{g,\mu}]^2_{1/2,\mathbb{R}^{n-1}}\\ \le& C\left(\mu^3\sum_{\pm}\int_{\mathbb{R}^{n-1}}|w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx+\mu^2\sum_{\pm}[e^{\tau\psi_\varepsilon(\cdot,0)}w_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\right).\end{aligned}\tag{4.47}$$

We now go to the next term $\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}J^{(2)}_{g,\mu}]^2_{1/2,\mathbb{R}^{n-1}}$. By (4.31), (4.14) and (4.32), we have that

$$\begin{aligned}&\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}J^{(2)}_{g,\mu}]^2_{1/2,\mathbb{R}^{n-1}}\\ \le& C\sum_{\pm}\Bigg(\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_\varepsilon(\cdot,0)}\eta_{g,\mu}(A_\pm(\delta x_g,0)-A_\pm(\delta x,0))\nabla_{x,y}w_\pm\cdot\nu]^2_{1/2,Q_{2/\mu}(x_g)}\\ &+\mu\sum_{g\in\mathbb{Z}^{n-1}}\int_{Q_{2/\mu}(x_g)}|A_\pm(\delta x_g,0)-A_\pm(\delta x,0)|^2|\nabla_{x,y}w_\pm|^2e^{2\tau\psi_\varepsilon(x,0)}dx\Bigg)\\ \le& C\sum_{\pm}\Bigg(\delta^2\mu^{-2}\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_\varepsilon(\cdot,0)}\eta_{g,\mu}\nabla_{x,y}w_\pm(\cdot,0)]^2_{1/2,Q_{2/\mu}(x_g)}\\ &+\delta^2\mu^{-1}\int_{\mathbb{R}^{n-1}}|\nabla_{x,y}(w_\pm e^{\tau\psi_\varepsilon})(x,0)|^2dx+\delta^2\mu^{-1}\tau^2\int_{\mathbb{R}^{n-1}}|w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx\Bigg)\\ \le& C\sum_{\pm}\Bigg(\delta^2\mu^{-2}[\nabla_{x,y}(w_\pm e^{\tau\psi_\varepsilon(\cdot,0)})]^2_{1/2,\mathbb{R}^{n-1}}+\delta^2\mu^{-2}\tau^2[e^{\tau\psi_\varepsilon(\cdot,0)}w_\pm]^2_{1/2,\mathbb{R}^{n-1}}\\ &+\delta^2\mu^{-1}\int_{\mathbb{R}^{n-1}}|\nabla_{x,y}w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx+\delta^2\mu^{-1}\tau^2\int_{\mathbb{R}^{n-1}}|w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx\Bigg).\end{aligned}\tag{4.48}$$

Now we estimate $\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}J^{(3)}_{g,\mu}]^2_{1/2,\mathbb{R}^{n-1}}$. In view of (4.31), (4.14), and (4.18), we can obtain that

$$\begin{aligned}&\sum_{g\in\mathbb{Z}^{n-1}}[e^{\tau\psi_{\varepsilon,g}(\cdot,0)}J^{(3)}_{g,\mu}]^2_{1/2,\mathbb{R}^{n-1}}\\ \le& C\left(\delta^2\sum_{\pm}[e^{\tau\psi_\varepsilon(\cdot,0)}w_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\delta^2\mu\sum_{\pm}\int_{\mathbb{R}^{n-1}}|w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx\right),\end{aligned}\tag{4.49}$$

where C depends on λ_0, M_0 and n. Finally, combining (4.45), (4.46), (4.47), (4.48), and (4.49) implies

$$\sum_{g\in\mathbb{Z}^{n-1}}d^{(3)}_{g,\mu}\le C\left(\tau\int_{\mathbb{R}^{n-1}}|\theta_1|^2e^{2\tau\psi_\varepsilon(x,0)}dx+[e^{\tau\psi_\varepsilon(\cdot,0)}\theta_1]^2_{1/2,\mathbb{R}^{n-1}}+R_4\right),\tag{4.50}$$

where

$$R_4 =\delta^2\mu^{-2}\sum_{\pm}[\nabla_{x,y}(w_\pm e^{\tau\psi_\varepsilon})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + (\mu^2+\delta^2\mu^{-2}\tau^2)\sum_{\pm}[e^{\tau\psi_\varepsilon(\cdot,0)}w_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}$$
$$+ \delta^2\varepsilon^{-1}\sum_{\pm}\int_{\mathbb{R}^{n-1}}|\nabla_{x,y}w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx$$
$$+ (\varepsilon\tau^2+\delta^2\tau+\delta^2\mu^{-1}\tau^2)\sum_{\pm}\int_{\mathbb{R}^{n-1}}|w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx.$$

Consequently, we have from (4.24), (4.35), (4.38), (4.42) and (4.50) that

$$\begin{aligned}\Xi(w) \le C\Bigg(&\sum_{\pm}\int_{\mathbb{R}^n_\pm}|L_\delta(x,y,\partial)(w_\pm)|^2\, e^{2\tau\psi_{\varepsilon,\pm}}dxdy + [e^{\tau\psi_\varepsilon(\cdot,0)}\theta_1]^2_{1/2,\mathbb{R}^{n-1}}\\ &+ [\nabla_x(e^{\tau\psi_\varepsilon}\theta_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \tau^3\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\theta_0(x)|^2dx\\ &+\tau\int_{\mathbb{R}^{n-1}}e^{2\tau\psi_\varepsilon(x,0)}|\theta_1(x)|^2dx + R_5\Bigg),\end{aligned} \tag{4.51}$$

where

$$R_5 =\delta^2\mu^{-2}\sum_{\pm}\int_{\mathbb{R}^n_\pm}|D^2w_\pm|^2\, e^{2\tau\psi_\varepsilon}dxdy + \mu^2\sum_{\pm}\int_{\mathbb{R}^n_\pm}|Dw_\pm|^2\, e^{2\tau\psi_\varepsilon}dxdy$$
$$+ \mu^4\sum_{\pm}\int_{\mathbb{R}^n_\pm}|w_\pm|^2\, e^{2\tau\psi_{\varepsilon,\pm}}dxdy + (\mu+\delta^2\varepsilon^{-1})\sum_{\pm}\int_{\mathbb{R}^{n-1}}|Dw_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx$$
$$+ \mu\tau^2\sum_{\pm}\int_{\mathbb{R}^{n-1}}|w_\pm(x,0)|^2e^{2\tau\psi_\varepsilon(x,0)}dx + (\mu^4+\delta^2\mu^{-2}\tau^2)\sum_{\pm}[e^{\tau\psi_\varepsilon(\cdot,0)}w_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}$$
$$+ \delta^2\mu^{-2}\sum_{\pm}[D(w_\pm e^{\tau\psi_\varepsilon})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}.$$

We now set $\delta=\varepsilon$ and choose a sufficiently small δ_0 and a sufficiently large τ_0, both depending on λ_0, M_0, n such that if $\varepsilon\le\delta_0$ and $\tau\ge\tau_0$, then R_5 on the right hand side of (4.51) can be absorbed by $\Xi(w)$ (defined in (4.22)). In other words, we have proved that

$$\begin{aligned}&\sum_{\pm}\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_\pm}|D^kw_\pm|^2e^{2\tau\psi_\varepsilon}dxdy + \sum_{\pm}\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^kw_\pm(x,0)|^2e^{2\psi_\varepsilon(x,0)}dx\\ &+\sum_{\pm}\tau^2[e^{\tau\psi_\varepsilon}w_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_{\pm}[\partial_y(e^{\tau\psi_{\varepsilon,\pm}}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_{\pm}[\nabla_x(e^{\tau\psi_\varepsilon}w_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\ \le& C\Bigg(\sum_{\pm}\int_{\mathbb{R}^n_\pm}|\mathcal{L}_\delta(x,y,\partial)(w_\pm)|^2\, e^{2\tau\psi_\varepsilon}dxdy + [e^{\tau\psi_\varepsilon(\cdot,0)}\theta_1]^2_{1/2,\mathbb{R}^{n-1}} + [\nabla_x(e^{\tau\psi_\varepsilon}\theta_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\ &+\tau^3\int_{\mathbb{R}^{n-1}}|\theta_0|^2e^{2\tau\psi_\varepsilon(x,0)}dx + \tau\int_{\mathbb{R}^{n-1}}|\theta_1|^2e^{2\tau\psi_\varepsilon(x,0)}dx\Bigg).\end{aligned} \tag{4.52}$$

Now, applying (4.52) to the function $w(x,y) = u(\delta x, \delta y)$ and by a standard change of variable, we have (2.10), where $\phi_{\delta,\pm}$ is given by (2.8).

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