Uniqueness for the inverse boundary value problem with singular potentials in 2D

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Abstract

In this paper we consider the inverse boundary value problem for the Schrödinger equation with potential in L^p class, $p > 4/3$. We show that the potential is uniquely determined by the boundary measurements.

1 Introduction

In this work we study the inverse boundary value problem for the Schrödinger equation with singular potentials in the plane. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary $\partial\Omega$. Let $q \in L^p(\Omega)$ with $p > 1$ and assume that 0 is not a Dirichlet eigenvalue of the Schrödinger operator $\Delta + q$ in Ω. Then the Dirichlet-Neumann map $\Lambda_q : u|_{\partial\Omega} \mapsto \partial_\nu u|_{\partial\Omega}$, where u satisfies $\Delta u + qu = 0$ in Ω, is well-defined (see [3, Lemma 5.1.3] for the precise statement). Here we are concerned with the unique determination of q from the knowledge of Λ_q, namely, whether

$$\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2. \tag{1}$$

We will prove that (1) is indeed true for $L^p(\Omega)$ potentials with $p > 4/3$.

Our method follows from the strategies introduced by Bukhgeim in [5] where he showed that (1) holds true for C^1 potentials. The key ingredient in Bukhgeim's method is the invention of special complex geometrical optics solutions with non-degenerate singular phases, namely, $\Phi(z) = (z - z_0)^2$, $z, z_0 \in \mathbb{C}$. Using this type of complex geometrical optics solutions, Bukhgeim was able to prove the global uniqueness by the method of stationary phase. Bukhgeim's result was later improved to $L^p(\Omega)$ with $p > 2$ in [8] and [4].

In this paper, we push the uniqueness result even further to $p > 4/3$. To do so, we need to prove the existence of complex geometrical optics solutions with phase Φ for such potentials. In fact, we show that such complex geometrical optics solutions exist for all potentials in $L^p(\Omega)$ with $p > 1$. The improvement relies on a new estimate for the conjugated Cauchy operator (see Lemma 4.3). Having constructed the complex geometrical optics solutions, we then perform

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the usual step — substituting such special solutions into Alessandrini's identity. In order to obtain the dominating term containing the difference of potentials in the method of stationary phase, we need to derive more refined estimates of terms of various orders in Alessandrini's identity. In this step, we need to use the fact that the knowledge the DN map improves the integrability of the potential. In other words, $\Lambda_{q_1} = \Lambda_{q_2}$ for $q_1, q_2 \in L^p(\Omega)$ with $3/4 < p < 2$ implies $q_1 - q_2 \in L^2(\Omega)$ (see [12]).

We would also like to mention another related paper. It was shown in [10, Thm 2.3] that if $p > 1$ then for every $z_0 \in \Omega$ there exists a *generic set* of potentials in L^p for which its value in a neighbourhood of z_0 is recoverable. It was also remarked in [10] that the neighbourhood of z_0 also depends on the chosen potential in the generic family which is determined by the choice of $z_0 \in \Omega$. Though the assumption on the L^p space of the potential is more general, the dependence of the generic set on the choice of the point $z_0 \in \Omega$ and the dependence of the neighbourhood on the potential makes it unclear how a global identifiability result would follow from [10, Thm 2.3].

Intuitively, the uniqueness of the inverse boundary value problem is strongly related to the unique continuation property. For higher dimensions ($n \geq 3$), it is known that the unique continuation holds for any solution $u \in H^{2,\frac{2n}{n+2}}_{loc}(\Omega)$ when $q \in L^{n/2}_{loc}$ (scale-invariant potentials) [9], where $H^{2,s}_{loc}(\Omega) = \{u \in L^1_{loc}(\Omega) : \Delta u \in L^s_{loc}(\Omega)\}$. In this situation, the global uniqueness of the inverse boundary value problem with $q \in L^{n/2}$ was established in [6] (also see related result in [7] for $n = 3, q \in W^{-1,3}$). When $n = 2$, the unique continuation holds relative to $u \in H^{2,s}_{loc}$ for $q \in L^p_{loc}$ with any $p > 1$, where $s = \max\{1, \frac{2p}{p+2}\}$ [1]. Prior to [1], a weaker result which stated that the unique continuation property holds relative to $u \in H^{2,2}_{loc}$ for $q \in L^p_{loc}$ with $p > 4/3$ was proved in [11]. Our uniqueness theorem of the inverse boundary value problem is consistent with the unique continuation result relative to $u \in H^{2,2}_{loc}$. It remains an interesting problem to close the gap of the uniqueness theorem for the inverse boundary value problem for $q \in L^p(\Omega)$ with $1 < p \leq 4/3$.

2 Main results

Theorem 2.1. *Let $q_1, q_2 \in L^p(\Omega)$ with $4/3 < p < 2$. Assume 0 is not a Dirichlet-eigenvalue of either potential and that their Dirichlet-Neumann maps are identical $\Lambda_{q_1} = \Lambda_{q_2}$. Then $q_1 = q_2$.*

Let us fix some notational convention before stating our second theorem. Throughout this text we shall always assume the following, which we call *the usual assumptions*: Let $\Omega, X \subset \mathbb{R}^2$ be bounded domains and $\overline{\Omega} \subset\subset X$. Fix a cut-off function $\chi \in C^\infty_0(X)$ such that $\chi \equiv 1$ on Ω. Also, whenever we let a function belong to $L^p(\Omega)$ for any p we automatically extend it by zero to $L^p(X)$. Finally, let $\tau > 1$ and $z_0 \in \mathbb{C} \equiv \mathbb{R}^2$ and set $\Phi(z) = (z - z_0)^2$. Moreover should φ_j or S_j be mentioned, then they refer to definitions 3.5 and 3.3. We now state the existence of complex geometrical optics solutions.

Theorem 2.2. *Let the usual assumptions hold and $q_j \in L^p(\Omega)$ with $1 < p < 2$. For $j \in \{1, 2\}$ let $\beta_j = \beta_j(z_0)$ be uniformly bounded over a parameter $z_0 \in \mathbb{C}$. Then for any z_0 we may define the function φ_j from Definition 3.5 and the* *series*

$$f_j(z) = \sum_{m=0}^{\infty} F_{j,m}(z) = e^{-i\tau(\Phi+\overline{\Phi})} + \varphi_j(z) + S_j\varphi_j(z) + \dots$$

from Definition 3.6. The latter converges uniformly in the variable $z \in X$ for τ large enough and

$$\|F_{j,m}\|_\infty \leq (C\tau^{-\alpha})^m$$

for some $C = C(p, \chi, q_j, \sup_{z_0} |\beta_j|)$, where $\alpha > 0$ is a constant obtained in Proposition 4.4. Moreover we have $f_j \in W^{1,2}(X)$. Lastly, $(\Delta + q_1)(e^{i\tau\Phi} f_1) = 0$ and $(\Delta + q_2)(e^{i\tau\overline{\Phi}} f_2) = 0$ in Ω.

3 Notation and CGO solution buildup

In this section we shall start by defining the operators and notation used in the rest of the paper. At the same time we reduce the construction of Bukhgeim-type [5] complex geometrical optics solutions to an integral equation.

We shall use the complex geometrical optics solutions for $(\Delta + q_j)u_j = 0$ of the form

$$u_1 = e^{i\tau\Phi} f_1, \qquad u_2 = e^{i\tau\overline{\Phi}} f_2.$$

The special form of f_1, f_2 which we are going to use was first defined in [8] and used in [4] for proving uniqueness for the boundary value inverse problem when the potentials are in L^p, $p > 2$.

Definition 3.1. With the usual assumptions, define the differential operators

$$D_1 f = -4e^{-i\tau(\Phi+\overline{\Phi})}\partial(e^{i\tau(\Phi+\overline{\Phi})}\overline{\partial} f),$$
$$D_2 f = -4e^{-i\tau(\Phi+\overline{\Phi})}\overline{\partial}(e^{i\tau(\Phi+\overline{\Phi})}\partial f).$$

Lemma 3.2. *Let the usual assumptions hold. For $q_j \in L^1(\Omega)$ and $f_j \in L^\infty(\Omega)$ we have*

$$(\Delta + q_1)(e^{i\tau\Phi} f_1) = 0 \qquad \Leftrightarrow \qquad D_1 f_1 = q_1 f_1,$$
$$(\Delta + q_2)(e^{i\tau\overline{\Phi}} f_2) = 0 \qquad \Leftrightarrow \qquad D_2 f_2 = q_2 f_2,$$

all in Ω.

Proof. Use $\Delta = 4\partial\overline{\partial} = 4\overline{\partial}\partial$ for distributions on Ω. □

For inverting the operators D_1 and D_2 we will have to use conjugated versions of the *Cauchy operators* $\overline{\partial}^{-1}$ and ∂^{-1}. We have included a short reminder of their properties in Section 6.

Definition 3.3. Let the usual assumptions hold. The we define the operators S_1, S_2 acting on $f \in L^\infty(X)$ by

$$S_1 f = -\frac{1}{4}\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}q_1 f)\big),$$
$$S_2 f = -\frac{1}{4}\partial^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi\overline{\partial}^{-1}(e^{i\tau(\Phi+\overline{\Phi})}q_2 f)\big).$$

Remark 3.4. We have $D_j S_j f = q_j f$ in Ω (but not in $X \setminus \Omega$).

Definition 3.5. Let the usual assumptions hold and $q_j \in L^1(\Omega)$. Then for any given $z_0 \in \mathbb{C}$ and $\beta_j = \beta_j(z_0)$, we define functions of $z \in X$ by

$$\varphi_1 = \frac{1}{4}\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi(\beta_1(z_0) - \partial^{-1}q_1)\big),$$
$$\varphi_2 = \frac{1}{4}\partial^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi(\beta_2(z_0) - \overline{\partial}^{-1}q_2)\big).$$

Note that $\partial^{-1}q_1$ and $\overline{\partial}^{-1}q_2$ inside the parenthesis do not depend on z_0. For example

$$\varphi_1(z) = \frac{1}{4\pi}\int \frac{e^{-i\tau((z'-z_0)^2+(\overline{z}'-\overline{z_0})^2)}\chi(z')\left(\beta_1(z_0) - \partial^{-1}q_1(z')\right)}{z - z'} dm(z').$$

Definition 3.6. Let the usual assumptions hold. For $j \in \{1, 2\}$ define

$$f_j = e^{-i\tau(\Phi+\overline{\Phi})} + \sum_{m=0}^{\infty} S_j^m \varphi_j$$

where S_j is as in Definition 3.3 and φ_j as in Definition 3.5. For convenience we write

$$F_{j,0} = e^{-i\tau(\Phi+\overline{\Phi})}, \qquad F_{j,m} = S_j^{m-1}\varphi_j$$

when $m \in \mathbb{N}$, $m \geq 1$. Hence $f_j = \sum_{m=0}^{\infty} F_{j,m}$.

We have made enough definitions now to show the structure of the complex geometrical optics solutions. Given $z_0 \in \mathbb{C}$ and $\beta_j = \beta_j(z_0)$ we can show formally that if

$$u_1 = e^{i\tau\Phi} f_1, \qquad u_2 = e^{i\tau\overline{\Phi}} f_2,$$

then $(\Delta + q_j)u_j = 0$ in Ω. This follows from writing

$$f_j = e^{-i\tau(\Phi+\overline{\Phi})} + \varphi_j + S_j(f_j - e^{-i\tau(\Phi+\overline{\Phi})}),$$

applying D_j and noting that $D_j(e^{-i\tau(\Phi+\overline{\Phi})}) = 0$, $D_j S_j$ is the multiplication operator by q_j in Ω, and $D_j\varphi_j = q_j e^{-i\tau(\Phi+\overline{\Phi})}$ in Ω. Then $D_j f_j = q_j f_j$ and Lemma 3.2 gives the rest. For proving actual existence and estimates, see the proof at the end of the next section.

4 Estimates for conjugated operators and CGO existence

In this section we will start by showing that a fundamental operator of the form $a \mapsto \overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}a)$ has decay properties as $\tau \to \infty$. Section 7 contains the technical cut-off function estimates. Once estimates for this fundamental operator have been shown we can prove the required estimates for S_j from Definition 3.3. At the end of this section all the details for proving the existence of complex geometrical optics solutions for L^p-potentials with $1<p<2$, i.e. Theorem 2.2, will be given. From now on, we define $p*$ satisfying the relation

$$\frac{1}{p}=\frac{1}{2}+\frac{1}{p*}.$$

Thus, we have $p*>2$.

Definition 4.1. Let the usual assumptions hold. Define the operator $\mathcal{T}$ by

$$\mathcal{T}a=\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}a).$$

Lemma 4.2. *Let the usual assumptions hold and $\mathcal{T}$ as in Definition 4.1. Then, for $p*>2$ we can extend $\mathcal{T}$ to a mapping $W_0^{1,p*}(X)\to L^\infty(X)$ with norm estimate*

$$\|\mathcal{T}a\|_{L^\infty(X)}\le C\tau^{-1/p*}\|a\|_{W^{1,p*}(X)},$$

where $W_0^{1,p*}(X)$ is the completion of $C_0^\infty(X)$ under the $W^{1,p*}$-norm.

Proof. Let $\psi\in C_0^\infty(\mathbb{R}^2)$ be a test function supported in $B(\bar{0},2)$ with $0\le\psi\le 1$ and $\psi\equiv 1$ in $B(\bar{0},1)$. Write $\psi_\tau(z)=\psi(\tau^{1/2}(z-z_0))$. Let $h(z)=(1-\psi_\tau(z))/(\overline{z}-\overline{z_0})$. By integration by parts (Lemma 6.3) we have

$$\begin{aligned}\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}a)&=\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\psi_\tau a\big)\\&\quad-\frac{1}{2i\tau}\big(e^{-i\tau(\Phi+\overline{\Phi})}ha-\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}\overline{\partial}ha)-\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}h\overline{\partial}a)\big).\end{aligned}$$

Then recall that by Lemma 6.4 we have $\overline{\partial}^{-1}\colon L^{p*}(X)\to W^{1,p*}(X)$, the latter of which is embedded into $L^\infty(X)$ since $p*>2$. Hence, by taking the $L^\infty(X)$-norm we have

$$\|\mathcal{T}a\|_\infty\le C\big(\|\psi_\tau a\|_{p*}+\tau^{-1}(\|ha\|_\infty+\|\overline{\partial}ha\|_{p*}+\|h\overline{\partial}a\|_{p*})\big).$$

The claim follows from Hölder's inequality and lemmas 7.1 and 7.3 after estimating

$$\|\mathcal{T}a\|_\infty\le C\big(\|\psi_\tau\|_{p*}+\tau^{-1}(\|h\|_\infty+\|\overline{\partial}h\|_{p*})\big)(\|a\|_\infty+\|\overline{\partial}a\|_{p*})$$

and noting that $\tau^{-1/2}\le\tau^{-1/p*}$ since $\tau>1$. $\square$

Lemma 4.3. *Let the usual assumptions hold and $\mathcal{T}$ be as in Definition 4.1. Assume that $2<p*<\infty$ and $1/2+1/p*\ge 1/q>1/2$. Then we can extend $\mathcal{T}$ to a mapping $W_0^{1,q}(X)\to L^{p*}(X)$ with norm*

$$\|\mathcal{T}a\|_{L^{p*}(X)}\le C\tau^{1/q-1-1/p*}\|a\|_{W^{1,q}(X)}$$

where $C=C(p,q,X)$.

Proof. Let $\psi\in C_0^\infty(\mathbb{R}^2)$ be a test function supported in $B(0,2)$ with $0\le\psi\le 1$ and $\psi\equiv 1$ in $B(0,1)$. For $\tau>0$ and $z_0\in\mathbb{R}^2$ write $\psi_\tau(z)=\psi(\tau^{1/2}(z-z_0))$. Let $h(z)=(1-\psi_\tau(z))/(\overline{z}-\overline{z_0})$. Integration by parts gives

$$\begin{aligned}\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}a)&=\overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\psi_\tau a\big)\\&\quad-\frac{1}{2i\tau}\big(e^{-i\tau(\Phi+\overline{\Phi})}ha-\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}\overline{\partial}ha)-\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})}h\overline{\partial}a)\big)\end{aligned}$$

by Lemma 6.3.

Sobolev embedding and Lemma 6.4 imply that $\overline{\partial}^{-1} : L^p(X) \to L^{p*}(X)$. Taking the $L^{p*}(X)$-norm gives

$$\|\mathcal{T}a\|_{p*} \leq C\big(\|\psi_\tau a\|_p + \tau^{-1}(\|ha\|_{p*} + \|\overline{\partial}ha\|_p + \|h\overline{\partial}a\|_p)\big).$$

Again, recall that $W^{1,q}(X) \hookrightarrow L^{q*}(X)$ where $1/q* = 1/q - 1/2$. Hölder's inquality gives

$$\begin{aligned}
\|\psi_\tau a\|_p &\leq \|\psi_\tau\|_{r_1} \|a\|_{q*}, & \frac{1}{r_1} &= \frac{1}{p} - \frac{1}{q*} = 1 + \frac{1}{p*} - \frac{1}{q}, \\
\|ha\|_{p*} &\leq \|h\|_{r_2} \|a\|_{q*}, & \frac{1}{r_2} &= \frac{1}{2} + \frac{1}{p*} - \frac{1}{q}, \\
\|\overline{\partial}ha\|_p &\leq \|\overline{\partial}h\|_{r_1} \|a\|_{q*}, \\
\|h\overline{\partial}a\|_p &\leq \|h\|_{r_2} \|\overline{\partial}a\|_q .
\end{aligned}$$

Lemmas 7.1 and 7.3 then give

$$\begin{aligned}
\|\psi_\tau a\|_p &\leq C\tau^{-1-1/p*+1/q} \|a\|_{q*}, \\
\|ha\|_{p*} &\leq C\tau^{-1/p*+1/q} \|a\|_{q*}, \\
\|\overline{\partial}ha\|_p &\leq C\tau^{-1/p*+1/q} \|a\|_{q*} \\
\|h\overline{\partial}a\|_p &\leq C\tau^{-1/p*+1/q} \|\overline{\partial}a\|_q,
\end{aligned}$$

which implies the claim. □

Proposition 4.4. *Let the usual assumptions hold and let $q_1, q_2 \in L^p(\Omega)$ where $1 < p < 2$. Then we can extend S_1 and S_2 from Definition 3.3 to the following maps with corresponding norm estimates*

$$\begin{aligned}
S_j &: L^\infty(X) \to L^{p*}(X), & \|S_j f\|_{p*} &\leq C\tau^{-1/2} \|f\|_\infty, \\
S_j &: L^\infty(X) \to L^\infty(X), & \|S_j f\|_\infty &\leq C\tau^{-\alpha} \|f\|_\infty,
\end{aligned}$$

where $C = C(p,\chi) \|q_j\|_p$ and $0 < \alpha < 1/p$ with $\alpha = \alpha(p)$. If in addition $p > 4/3$ then we have the extension

$$S_j : L^{p*}(X) \to L^{p*/2}(X), \qquad \|S_j f\|_{p*/2} \leq C\tau^{-1/2} \|f\|_{p*}.$$

Proof. We shall prove the claim for $j = 1$. The other case follows similarly. Using the notation from Lemma 4.3 we can write $-4S_1 f = \mathcal{T}(\chi\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}q_1 f))$. The lemma combined with Lemma 6.4 gives us

$$\|S_1 f\|_{q*} \leq C\tau^{-1/2} \left\|\chi\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}q_1 f)\right\|_{W^{1,q}} \leq C\tau^{-1/2} \|q_1 f\|_q$$

whenever $2 < q* < \infty$ and $1/q = 1/2 + 1/q*$. For the first estimate choose $q = p$, $q* = p*$, and for the third one $1/q = 1/p + 1/p*$, $q* = p*/2$. Hölder's inequality implies the rest.

The second claim follows by interpolation. Let $2 < Q < \infty$ and $1 < q < p$. If $q_1 \in L^Q(X)$ then by Lemma 4.2

$$\|S_1 f\|_\infty \leq C\tau^{-1/Q} \left\|\chi\partial^{-1}(e^{i\tau(\Phi+\overline{\Phi})}q_1 f)\right\|_{W^{1,Q}}$$

and Lemma 6.4 gives the bound $C\tau^{-1/Q}\left\|q_1\right\|_Q\left\|f\right\|_\infty$. The latter lemma and Sobolev embedding imply that $\partial^{-1} : L^q \to L^{q*}$ and $\overline{\partial}^{-1} : L^{q*} \to L^\infty$. Hence $\left\|S_1 f\right\|_\infty \le C\left\|q_1\right\|_q\left\|f\right\|_\infty$. Since $q < p < Q$ and $1/Q > 0$ interpolation gives us the second estimate with some $\alpha > 0$. □

Lemma 4.5. *Let the usual assumptions hold and $q_j \in L^p(\Omega)$ with $1 < p < 2$. Then φ_j, the function of $z \in X$ given by Definition 3.5, is in $L^\infty(X)$ with norm*

$$\left\|\varphi_j\right\|_\infty \le C\tau^{-\alpha}$$

for $\alpha > 0$ as in Proposition 4.4, and is in $L^{p*}(X)$ satisfying

$$\left\|\varphi_j\right\|_{p*} \le C\tau^{-1/2},$$

where $C = C(p,\chi)(\left\|q_j\right\|_p + |\beta_j(z_0)|)$.

Proof. Note that $\varphi_1 = \frac{1}{4}\beta_1(z_0)\overline{\partial}^{-1}\left(e^{-i\tau(\Phi+\overline{\Phi})}\chi\right) + S_1(e^{-i\tau(\Phi+\overline{\Phi})})$ and use Proposition 4.4 and lemmas 4.2 and 4.3. □

Proof of Theorem 2.2. By Proposition 4.4 and Lemma 4.5 we have

$$\left\|S_j f\right\|_\infty \le C(p,\chi)\left\|q_j\right\|_p\tau^{-\alpha}\left\|f\right\|_\infty, \quad \left\|\varphi_j\right\|_\infty \le C(p,\chi)(\left\|q_j\right\|_p + |\beta_j(z_0)|)\tau^{-\alpha}$$

where these norms are over the variable $z \in X$. Hence we get $\left\|F_{j,m}\right\|_\infty \le C^m\tau^{-m\alpha}$ for some $C = C(p,\chi,q_j,\sup_{z_0}|\beta_j(z_0)|)$ when $m \ge 0$. If $\tau > C^{1/\alpha}$ then the series for f_j converges in $L^\infty(X)$.

The following observations, which are each easy to check in $\mathscr{D}'(X)$, imply that $D_j f_j = q_j f_j$ in Ω. Note that β_j are functions of the parameter z_0 but constant in the variable z. Recall the definitions 3.1, 3.3 and 3.5 of D_j, S_j and φ_j. Then

- $f_j = e^{-i\tau(\Phi+\overline{\Phi})} + \varphi_j + S_j(f_j - e^{-i\tau(\Phi+\overline{\Phi})})$,
- $D_j(e^{-i\tau(\Phi+\overline{\Phi})}) = 0$,
- $D_j S_j f = q_j f$ in Ω,
- $D_j\varphi_j = q_j e^{-i\tau(\Phi+\overline{\Phi})}$ in Ω.

Lemma 3.2 shows that we indeed get solutions to $(\Delta + q_j)u_j = 0$.

Recall that $1/p = 1/2 + 1/p*$. Then by Lemma 6.4 we have $\overline{\partial}^{-1}, \partial^{-1} : L^{p*} \to W^{1,p*}$ which embeds to $W^{1,2}$ locally since $p* > 2$. Moreover by Sovolev embedding we have $\overline{\partial}^{-1}, \partial^{-1} : L^p \to L^{p*}$. Hence by the first item above $f_j \in W^{1,2}(X)$. □

5 Proof of the main result

We will prove Theorem 2.1 in this section. The proof will be split into several lemmas. By Alessandrini's identity

$$\int_{\Omega} (q_1 - q_2) u_1 u_2 dx = 0$$

for solutions u_j to $(\Delta + q_j)u_j = 0$ in Ω. For any parameter $z_0 \in \mathbb{C}$ let u_j be the complex geometrical optics solution given by Theorem 2.2. Recall that they are defined in $X \supset \Omega$ but are solutions only in Ω. Then the product $u_1 u_2$ will be a series of terms, and these will have to be estimated carefully. Lemmas 5.1 – 5.6 deal with this. The main proof follows.

Lemma 5.1. *Let the usual assumptions hold and* $1 < p < 2$ *with* $q_1, q_2 \in L^p(\Omega)$*. Let* $f_1, f_2 \in L^\infty(X)$ *be as in Theorem 2.2 and set* $u_1 = e^{i\tau\Phi} f_1$ *and* $u_2 = e^{i\tau\overline{\Phi}} f_2$*. Then*

$$\frac{2\tau}{\pi} \int (q_1 - q_2) u_1 u_2 dx = \sum_{k+l=0}^{\infty} \frac{2\tau}{\pi} \int (q_1 - q_2) e^{i\tau(\Phi + \overline{\Phi})} F_{1,k} F_{2,l} dx$$

where $k, l \geq 0$ *and the sum converges in the* $L^\infty(X)$*-norm with respect to* z_0*.*

Proof. We can estimate

$$\left| \int \sum_{k+l=N}^{\infty} (q_1 - q_2) e^{i\tau(\Phi + \overline{\Phi})} F_{1,k} F_{2,l} dx \right| \leq \int \sum_{k+l=N}^{\infty} |q_1 - q_2| \, |F_{1,k}| \, |F_{2,l}| \, dx$$

and use the z_0-independent estimates for $F_{j,m}$ from Theorem 2.2 to see that the remainder tends to zero as $N \to \infty$. Hence the sum can be taken out of the integral and the claim follows. ☐

Lemma 5.2. *Let the usual assumptions hold and* $q_1, q_2 \in L^p(\Omega)$ *with* $4/3 < p < 2$*. For* $j \in \{1, 2\}$ *and* $m \in \mathbb{N}$ *take* $F_{j,m}$ *as in Definition 3.6.*

Then if $q_1 - q_2 \in L^2(\Omega)$ *we have*

$$\left| \tau \int (q_1 - q_2) e^{i\tau(\Phi + \overline{\Phi})} F_{1,k} F_{2,l} dx \right| \leq C^{k+l} \tau^{-(k+l-2)\alpha}.$$

when $k+l \geq 3$*. Here* $C = C(p, \chi, \|q_1\|_p, \|q_2\|_p, \|q_1 - q_2\|_2)(1 + |\beta_1(z_0)| + |\beta_2(z_0)|)$ *and* $\alpha > 0$ *is as in Proposition 4.4.*

Proof. We may assume that $k \geq l$. By Hölder's inequality the integral can be estimated with

$$C\tau \|q_1 - q_2\|_2 \|F_{1,k}\|_{p*/2} \|F_{2,l}\|_\infty$$

because $p > 4/3$ imples $p* > 4$ for $1/p = 1/2 + 1/p*$ and then $1/2 + 2/p* \leq 1$. Proposition 4.4 and Lemma 4.5 imply the following estimates

$$\begin{aligned}
&\|F_{j,0}\|_\infty = 1, \quad \|F_{j,1}\|_\infty \leq C\tau^{-\alpha}, \quad \|F_{j,1}\|_{p*} \leq C\tau^{-1/2}, \\
&\|F_{j,m+1}\|_\infty \leq C\tau^{-\alpha} \|F_{j,m}\|_\infty, \\
&\|F_{j,m+1}\|_{p*} \leq C\tau^{-1/2} \|F_{j,m}\|_\infty, \\
&\|F_{j,m+1}\|_{p*/2} \leq C\tau^{-1/2} \|F_{j,m}\|_{p*}
\end{aligned}$$

for $j \in \{1,2\}$ and $m = 1, 2, \ldots$. These imply

$$\|F_{1,k}\|_{p*/2} \leq C^k \tau^{-1-(k-2)\alpha}, \qquad \|F_{2,l}\|_\infty \leq C^l \tau^{-l\alpha}$$

for $k \geq 2$, $l \geq 0$. The claim is direct consequence. $\square$

From Lemma 5.2, we can see that the higher order terms decay in τ whenever $k+l \geq 3$. A more refined estimate shows that the term of $k+l=2$ also decays.

Lemma 5.3. *Let the usual assumption hold and $q_1, q_2 \in L^p(\Omega)$ with $4/3 < p < 2$. For $j \in \{1,2\}$ and $m \in \mathbb{N}$ let $F_{j,m}$ be as in Definition 3.6. Assume that $q_1 - q_2 \in L^2(\Omega)$. For $k+l=2$, we have*

$$\left| \tau \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx \right| \leq C\tau^{1/p-3/4},$$

where the constant C is of the form $C(p, \chi, \|q_1\|_p, \|q_2\|_p, \|q_1 - q_2\|_2)(1+|\beta_1(z_0)|+|\beta_2(z_0)|)$.

Proof. We can assume that $k \geq l$. There are two cases: $k = 2, l = 0$ and $k = l = 1$. Start with the first one. The integral with $F_{1,2}F_{2,0}$ is

$$-\frac{1}{4}\tau \int (q_1 - q_2) \overline{\partial}^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} \chi \partial^{-1} (e^{i\tau(\Phi+\overline{\Phi})} q_1 \varphi_1)\big) dx$$

by Definition 3.3. We have $q_1 - q_2 \in L^2(\Omega)$ and hence we should take the $L^{r*}(X)$-norm of the remaining factor for any $r* \geq 2$.

We note that $\varphi_1 \in L^{p*}(X)$ and $q_1 \in L^p(X)$. Hence their product is in $L^q(X)$ with $1/q = 1/p + 1/p* = 2/p - 1/2$. Choose $1/r* = 1/p - 1/4$. Then $2 < r* < \infty$ and $1/2 + 1/r* \geq 1/q > 1/2$ since $4/3 < p < 2$. Hence by Lemma 4.3

$$\left\| \overline{\partial}^{-1} \big(e^{-i\tau(\Phi+\overline{\Phi})} \chi \partial^{-1} (e^{i\tau(\Phi+\overline{\Phi})} q_1 \varphi_1)\big) \right\|_{r*} \leq C\tau^{1/q-1-1/r*} \|q_1\|_p \|\varphi_1\|_{p*}$$

and the exponential is $1/q - 1 - 1/r* = 1/p - 5/4$. Recall that $\|\varphi_1\|_{p*} \leq C\tau^{-1/2}$ by Lemma 4.5. The claim for $k=2$, $l=0$ follows.

In the case $k = l = 1$ note that $\beta_1(z_0) - \partial^{-1} q_1 \in W^{1,p}(X)$ by Lemma 6.4. Note that $4/3 < p < 2$ implies $1/2 + 1/4 \geq 1/p > 1/2$. Then by Lemma 4.3

$$\|\varphi_1\|_4 \leq C\tau^{1/p-1-1/4} \left\|\beta_1(z_0) - \partial^{-1} q_1\right\|_{W^{1,p}} \leq C\tau^{1/p-5/4}(|\beta_1(z_0)| + \|q_1\|_p).$$

When $k = l = 1$, the absolute value of the integral in the lemma statement becomes

$$\left| \tau \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} \varphi_1 \varphi_2 dx \right| \leq \|q_1 - q_2\|_2 \tau \|\varphi_1\|_4 \|\varphi_2\|_4$$

by Hölder's inequality. The claim follows since $\tau^{2/p-6/4} < \tau^{1/p-3/4}$ when $\tau > 1$ and $p > 4/3$. $\square$

We recall the method of stationary phase and its convergence in the L^2-sense before proceeding to deal with terms of order one and zero in the Alessandrini identity.

Lemma 5.4. *For $z_0 \in \mathbb{C}$, $\Phi(z) = (z - z_0)^2$ and $\tau \in \mathbb{R}$ define the operator*

$$Ef(z_0) = \frac{2\tau}{\pi} \int e^{-i\tau(\Phi+\overline{\Phi})} f(z) dm(z)$$

for $f \in C_0^\infty(\mathbb{C})$. Here $dm(z)$ is the two-dimensional Lebesgue measure in $\mathbb{C}$.

Then E can be extended to a unitary operator on $L^2(\mathbb{C})$ such that

$$\lim_{\tau \to \pm\infty} \|Ef - f\|_2 = 0.$$

Proof. Consider the function $z \mapsto 2\tau \exp(-i(z^2 + \overline{z}^2))/\pi$ defined on $\mathbb{C} \equiv \mathbb{R}^2$. Its Fourier transform is $\exp(i(\xi^2 + \overline{\xi}^2)/(16\tau))$ by for example [4]. We have $Ef = \frac{2\tau}{\pi} e^{-i(z^2+\overline{z}^2)} * f$ and hence $\mathscr{F}\{Ef\}(\xi) = e^{i\frac{\xi^2+\overline{\xi}^2}{16\tau}} \hat{f}(\xi)$. Parseval's theorem implies the unitary extension to $L^2(\mathbb{C})$. When $\tau \to \pm\infty$ the exponential tends to 1 pointwise. Dominated convergence and Parseval's theorem imply the second claim. □

The following way of dealing with the first order terms comes from [8, 4].

Lemma 5.5. *Let the usual assumptions hold and $q_1, q_2 \in L^p(\Omega)$ with $4/3 < p < 2$. For $j \in \{1, 2\}$ and $m \in \mathbb{N}$, let $F_{j,m}$ be as in Definition 3.6. Moreover let $\beta_j \in L^\infty(X)$ with respect to the z_0-variable. Then*

$$\lim_{\tau \to \infty} \left\| \frac{2\tau}{\pi} \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx \right\|_2 \leq C \left\| \beta_2 - \overline{\partial}^{-1} q_2 \right\|_{p*} \|q_1 - q_2\|_p$$

for $k = 1$, $l = 0$, where the $L^2(X)$-norm is taken over the variable z_0 and $C = C(p, \chi)$. A similar bound holds for $k = 0$ and $l = 1$.

Proof. Recall that $\varphi_2 = \frac{1}{4}\partial^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi(\beta_2(z_0) - \overline{\partial}^{-1} q_2)\big)$ and hence the integral becomes

$$\frac{\tau}{2\pi} \int (q_1 - q_2) \partial^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})}\chi(\beta_2(z_0) - \overline{\partial}^{-1} q_2)\big) dx$$

when $k = 1$, $l = 0$. By Fubini's theorem this is equal to

$$-\frac{\tau}{2\pi} \int e^{-i\tau(\Phi+\overline{\Phi})}\chi(\beta_2(z_0) - \overline{\partial}^{-1} q_2) \partial^{-1}(q_1 - q_2) dx,$$

and using the stationary phase operator of Lemma 5.4 it is equal to

$$\frac{1}{4} E\big(\chi \overline{\partial}^{-1} q_2 \partial^{-1}(q_1 - q_2)\big)(z_0) - \frac{1}{4}\beta_2(z_0) E\big(\chi \partial^{-1}(q_1 - q_2)\big)(z_0).$$

We have $\partial^{-1}(q_1 - q_2) \in L^{p*}(X)$ where $p* > 4$ since $p > 4/3$ (e.g. Lemma 6.4 and Sobolev embedding). Similarly $\chi \overline{\partial}^{-1} q_2 \in L^{p*}(\mathbb{C})$. Their product is in $L^2(\mathbb{C})$ since χ has compact support. Hence the operator E is being applied to $L^2(\mathbb{C})$-functions above. Since $z_0 \mapsto \beta_2(z_0)$ is uniformly bounded, the above converges to

$$\frac{1}{4}\chi(\overline{\partial}^{-1} q_2 - \beta_2) \partial^{-1}(q_1 - q_2)$$

in the $L^2(\mathbb{C})$-norm with respect to z_0 as $\tau \to \infty$ by Lemma 5.4. The claim follows from the norm estimates at the beginning of this paragraph. □

Lemma 5.6. *Let the usual assumptions hold and $q_1 - q_2 \in L^2(\Omega)$. Then*

$$\lim_{\tau\to\infty} \left\| \frac{2\tau}{\pi} \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,0} F_{2,0} dx - (q_1 - q_2) \right\|_2 = 0,$$

where $F_{j,0}$ are as in Definition 3.6 and the norm is taken with respect to the variable $z_0 \in \mathbb{C}$.

Proof. This follows directly from $F_{j,0} = \exp(-i\tau(\Phi + \overline{\Phi}))$ and the stationary phase Lemma 5.4. □

We are ready to prove uniqueness for the inverse problem with potential in L^p, $4/3 < p < 2$.

Proof of Theorem 2.1. In view of Green's identity and the symmetry of the DN map, we can see that the condition of identical Dirichlet-Neumann maps imply that

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0$$

for any solution $u_j \in W^{1,2}(\Omega)$ to $(\Delta + q_j)u_j = 0$ in Ω. We also note that Theorem 3 in [12] implies that $q_1 - q_2 \in H^t(\Omega)$ for any $t < 3 - 4/p$. Hence $q_1 - q_2 \in L^2(\Omega)$.

Let $\varepsilon > 0$ and take $\beta_j \in C_0^\infty(X)$ such that

$$\|\beta_1 - \partial^{-1} q_1\|_{L^{p*}(X)} < \varepsilon, \quad \|\beta_2 - \overline{\partial}^{-1} q_2\|_{L^{p*}(X)} < \varepsilon \tag{2}$$

which is possible since $\partial^{-1} q_1, \overline{\partial}^{-1} q_2 \in L^{p*}$ by Sobolev embedding and Lemma 6.4. Let $z_0 \in \mathbb{C}$ and from now β_1 and β_2 shall be evaluated at z_0 if not mentioned otherwise, and note that they are uniformly bounded. Then, given $\tau > 1$ large enough let $u_1 = e^{i\tau\Phi} f_1$ and $u_2 = e^{i\tau\overline{\Phi}} f_2$ be the solutions in the variable z with parameter z_0, given by Theorem 2.2. They are in $W^{1,2}(X)$ and satisfy $(\Delta + q_j)u_j = 0$ in Ω. By Lemma 5.1 we have

$$\frac{2\tau}{\pi} \int (q_1 - q_2) u_1 u_2 dx = \sum_{k+l=0}^{\infty} \frac{2\tau}{\pi} \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx.$$

In view of lemmas 5.2 and 5.3

$$\begin{aligned}\sum_{k+l=2}^{\infty} \frac{2\tau}{\pi} \left| \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx \right| &\le C\tau^{1/p-3/4} + \sum_{k+l=3}^{\infty} C^{k+l} \tau^{-(k+l-2)\alpha} \\ &= C \left(\tau^{1/p-3/4} + \sum_{N=3}^{\infty} (N+1)(C\tau)^{-(N-2)\alpha} \right)\end{aligned}$$

for any $z_0 \in \mathbb{R}^2$, and where $C = C_{p,q_1,q_2,\Omega}(1 + \|\beta_1\|_\infty + \|\beta_2\|_\infty)$. Recall that $p > 4/3$ so the first exponent is negative. Note that for τ sufficiently large, $(C\tau)^{-\alpha} < 1$ so the sum can be rewritten as

$$\begin{aligned}\sum_{N=3}^{\infty} (N+1)((C\tau)^{-\alpha})^{N-2} &= \sum_{N=1}^{\infty} (N+1)((C\tau)^{-\alpha})^N + 2\sum_{N=1}^{\infty} ((C\tau)^{-\alpha})^N \\ &= \frac{1}{(1 - (C\tau)^{-\alpha})^2} - 1 + \frac{2(C\tau)^{-\alpha}}{1 - (C\tau)^{-\alpha}},\end{aligned}$$

which tends to zero as $\tau \to \infty$. Hence the sum of the terms with $k + l \geq 2$ in the original sum tends to zero when β_j are fixed.

For the terms with $k + l \in \{0, 1\}$ we will use lemmas 5.5 and 5.6. By them

$$\lim_{\tau\to\infty} \left\| \sum_{k+l=0}^{1} \frac{2\tau}{\pi} \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx - (q_1 - q_2) \right\|_{L^2(X)}$$

$$\leq C(\|\beta_1 - \partial^{-1} q_1\|_{L^{p*}(X)} + \|\beta_2 - \overline{\partial}^{-1} q_2\|_{L^{p*}(X)}) \leq 2C\varepsilon$$

where the $L^2(X)$-norm is taken with respect to z_0 and this time C does not depend on β_1 or β_2. We can redo this whole argument for any $\varepsilon > 0$, and thus by Alessandrini's identity

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq \lim_{\tau\to\infty} \left\| q_1 - q_2 - \sum_{k+l=0}^{\infty} \frac{2\tau}{\pi} \int (q_1 - q_2) e^{i\tau(\Phi+\overline{\Phi})} F_{1,k} F_{2,l} dx \right\|_{L^2(\Omega)}$$

the latter of which can be made as small as we please by choosing β_1, β_2. The claim follows. $\square$

6 Appendix 1: Cauchy operator and integration by parts

We define the two fundamental tools for solving the two-dimensional inverse problem of the Schrödinger operator in this section: the Cauchy operators and an integration by parts formula for the Cauchy operator conjugated by an exponential. These were used by Bukhgeim [5] for solving the problem.

Definition 6.1. Let $u \in \mathscr{E}'(\mathbb{R}^2)$ be a compactly supported distribution. Then we define the *Cauchy operators* by

$$\overline{\partial}^{-1} u = \frac{1}{\pi z} * u, \qquad \partial^{-1} u = \frac{1}{\pi \overline{z}} * u.$$

Remark 6.2. The notations $\overline{\partial}^{-1}$ and ∂^{-1} cause no problems because $1/(\pi z)$ and $1/(\pi \overline{z})$ are the fundamental solutions to the operators $\overline{\partial} = (\partial_1 + i\partial_2)/2$ and $\partial = (\partial_1 - i\partial_2)/2$.

Lemma 6.3. *Let $\tau > 0$, $z_0 \in \mathbb{C}$ and $\Phi(z) = (z - z_0)^2$. Let $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi \equiv 1$ in a neighbourhood of 0, and write*

$$\psi_\tau(z) = \psi(\tau^{1/2}(z - z_0)), \qquad h(z) = \frac{1 - \psi_\tau(z)}{\overline{z} - \overline{z_0}}.$$

Then for $a \in C_0^\infty(\mathbb{R}^2)$ we have the integration by parts formula

$$\overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})} a) = \overline{\partial}^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})} \psi_\tau a\big)$$
$$- \frac{1}{2i\tau}\big(e^{-i\tau(\Phi+\overline{\Phi})} ha - \overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})} \overline{\partial} ha) - \overline{\partial}^{-1}(e^{-i\tau(\Phi+\overline{\Phi})} h \overline{\partial} a)\big).$$

If we had set $h(z) = (1 - \psi_\tau(z))/(z - z_0)$ instead then

$$\partial^{-1}(e^{-i\tau(\Phi+\overline{\Phi})} a) = \partial^{-1}\big(e^{-i\tau(\Phi+\overline{\Phi})} \psi a\big)$$
$$- \frac{1}{2i\tau}\big(e^{-i\tau(\Phi+\overline{\Phi})} ha - \partial^{-1}(e^{-i\tau(\Phi+\overline{\Phi})} \partial ha) - \partial^{-1}(e^{-i\tau(\Phi+\overline{\Phi})} h \partial a)\big).$$

Proof. The proof follows by differentiating $e^{-i\tau(\Phi+\overline{\Phi})}ha$ and noting that by Remark 6.2 the operators $\overline{\partial}^{-1}\overline{\partial}$ and $\partial^{-1}\partial$ are the identity on compactly supported distributions. $\square$

Lemma 6.4. *Let $X \subset \mathbb{R}^2$ be a bounded domain and $1 < p < \infty$. Then the Cauchy operators $\overline{\partial}^{-1}$ and ∂^{-1} are bounded $L^p(X) \to W^{1,p}(X)$.*

Proof. If $f \in L^p(X)$ we extend it by zero to $\mathbb{R}^2 \setminus X$ to create a compactly supported distribution and thus $\overline{\partial}^{-1} f$ is well defined by Definition 6.1. The convolution kernel $1/(\pi z)$ is locally integrable, so by Young's inequality

$$\left\| \overline{\partial}^{-1} f \right\|_{L^p(X)} \leq C \left\| f \right\|_{L^p(X)},$$

because in essence $\overline{\partial}^{-1} f$ has the same values in X as the convolution of f with the kernel $\chi_{X-X}(z)/(\pi z)$, where $X - X = \{z \in \mathbb{R}^2 \mid z = z_1 - z_2, z_j \in \mathbb{R}^2\}$.

For the derivatives note that by Remark 6.2 we have $\overline{\partial}\overline{\partial}^{-1} f = f$. On the other hand $\partial\overline{\partial}^{-1} f = \Pi f$ which is the Beurling transform, and hence bounded $L^p(X) \to L^p(X)$. For reference see for example Section 4.5.2 in [2] or [13] for a more classical approach. $\square$

7 Appendix 2: Cut-off function estimates

This section contains all the technical cut-off function construction and norm estimates used in the paper.

Lemma 7.1. *Let $\psi \in C_0^\infty(\mathbb{R}^2)$. For $z_0 \in \mathbb{R}^2$ and $\tau > 0$ write $\psi_\tau(z) = \psi(\tau^{1/2}(z - z_0))$. Then, given any vector $v \in \mathbb{C}^2$, we have*

$$\left\| \psi_\tau \right\|_{L^p(\mathbb{R}^2)} = \left\| \psi \right\|_{L^p(\mathbb{R}^2)} \tau^{-1/p}, \qquad \left\| v \cdot \nabla\psi_\tau \right\|_{L^p(\mathbb{R}^2)} = \left\| v \cdot \nabla\psi \right\|_{L^p(\mathbb{R}^2)} \tau^{1/2-1/p}$$

for $1 \leq p \leq \infty$.

Proof. This follows directly from the scaling properties and translation invariance of L^p-norms in $\mathbb{R}^2$. $\square$

Lemma 7.2. *Let $\tau > 0$ and set $\mathbb{R}^2_\tau = \mathbb{R}^2 \setminus B(0, \tau^{-1/2})$. Then*

$$\left\| z^{-a} \right\|_{L^p(\mathbb{R}^2_\tau)} = \left(\frac{2\pi}{ap-2} \right)^{1/p} \tau^{a/2-1/p}$$

for $a > 0$ and $2/a < p \leq \infty$.

Proof. This is a direct computation using the polar coordinates integral transform $\int_{\mathbb{R}^2_\tau} \ldots dz = \int_{\tau^{-1/2}}^{\infty} \int_{\mathbb{S}^1} \ldots d\sigma(\theta) r dr$, with $z = r\theta$. $\square$

Lemma 7.3. *Let $\psi \in C_0^\infty(\mathbb{R}^2)$ be a test function supported in $B(0,2)$ with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $B(0,1)$. For $\tau > 0$ and $z_0 \in \mathbb{R}^2$ write $\psi_\tau(z) = \psi(\tau^{1/2}(z - z_0))$. Let $h(z) = (1 - \psi_\tau(z))/(\overline{z} - \overline{z_0})$. Then*

$$\left\| h \right\|_{L^p(\mathbb{R}^2)} \leq C_p \tau^{1/2-1/p}$$

for $C_p < \infty$ when $2 < p \leq \infty$ and for any complex vector $v \in \mathbb{C}^2$ we have

$$\|v \cdot \nabla h\|_{L^p(\mathbb{R}^2)} \leq C_{\psi,p,v} \tau^{1-1/p}$$

for $C_{\psi,p,v} < \infty$ when $1 \leq p \leq \infty$. The same conclusions hold if we had defined h by dividing $1 - \psi_\tau$ by $z - z_0$ instead of its complex conjugate.

Proof. For the first claim note that $|h(z)| \leq |z - z_0|^{-1}$ and $\operatorname{supp} h \subset \mathbb{R}^2_\tau + z_0 = \mathbb{R}^2 \setminus B(z_0, \tau^{-1/2})$. Hence $\|h\|_{L^p(\mathbb{R}^2)} \leq \left\|z^{-1}\right\|_{L^p(\mathbb{R}^2_\tau)}$ and Lemma 7.2 takes care of the first estimate.

For the second estimate

$$v \cdot \nabla h(z) = \frac{v \cdot \nabla \psi_\tau(z)}{\overline{z} - \overline{z_0}} - \frac{1 - \psi_\tau(z)}{(\overline{z} - \overline{z_0})^2}.$$

The L^p-norm of the first term is bounded by $\|v \cdot \nabla \psi_\tau\|_{L^p} \left\|z^{-1}\right\|_{L^\infty(\mathbb{R}^2_\tau)}$ which is at most $C_{\psi,p,v}\tau^{1-1/p}$ according to lemmas 7.1 and 7.2. The second term is supported in $\mathbb{R}^2 \setminus B(z_0, \tau^{-1/2})$ and bounded pointwise by $|z - z_0|^{-2}$. Hence, as in the first paragraph, it has the required bound. □

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References

[1] W.O. Amrein, A.M. Berthier, and V. Georgescu. L^p-inequalities for the laplacian and unique continuation. *Annales de l'institut Fourier*, 31(3):153–168, 1981.

[2] Kari Astala, Tadeusz Iwaniec, and Gaven Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2009.

[3] Eemeli Blåsten. *On the Gel'fand-Calderón inverse problem in two dimensions*. Doctoral thesis, University of Helsinki, Finland, 2013.

[4] Eemeli Blåsten, Oleg Yu. Imanuvilov, and Masahiro Yamamoto. Stability and uniqueness for a two-dimensional inverse boundary value problem for less regular potentials. *Inverse Problems and Imaging*, 9(3):709–723, 2015.

[5] Alexander L. Bukhgeim. Recovering a potential from Cauchy data in the two-dimensional case. *J. Inverse Ill-Posed Probl.*, 16(1):19–33, 2008.

[6] David Dos Santos Ferreira, Carlos E. Kenig, and Mikko Salo. Determining an unbounded potential from cauchy data in admissible geometries. *Communications in Partial Differential Equations*, 38(1):50–68, 2013.

[7] Boaz Haberman. Unique determination of a magnetic Schrödinger operator with unbounded magnetic potential from boundary data. *Int. Math. Res. Notices*, 2016.

[8] Oleg Y. Imanuvilov and Masahiro Yamamoto. Inverse boundary value problem for linear Schrödinger equation in two dimensions. *ArXiv e-prints*, August 2012.

[9] David Jerison and Carlos E. Kenig. Unique continuation and absence of positive eigenvalues for Schrödinger operators. *Annals of Mathematics*, 121(3):463–488, 1985.

[10] Evgeny Lakshtanov and Boris Vainberg. Recovery of L^p-potential in the plane. *J. Inverse Ill-Posed Probl.*, 2017.

[11] J.C. Saut and B. Scheurer. Un théorème de prolongement unique pour des opérators elliptiques dont les coefficients ne sont pas localement bornés. *C.R.A.S., Paris*, 290A:595–598, 1980.

[12] Valery S. Serov and Lassi Päivärinta. New estimates of the Green-Faddeev function and recovering of singularities in the two-dimensional Schrödinger operator with fixed energy. *Inverse Problems*, 21(4):1291–1301, 2005.

[13] Ilia N. Vekua. *Generalized analytic functions.* Pergamon Press, London, 1962. Translation from the 1959 Russian edition.