Size estimates for the inverse boundary value problems of isotropic elasticity and complex conductivity in 3D

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Abstract

In the inverse boundary value problems of isotropic elasticity and complex conductivity, we derive estimates for the volume fraction of an inclusion whose physical parameters satisfy suitable gap conditions. In the elasticity case we require one measurement for the lower bound and another for the upper one. In the complex conductivity case we need three measurements for the lower bound and three for the upper. We accomplish this with the help of the "translation method" which consists of perturbing the minimum principle associated with the equation by either a null-Lagrangian or a quasi-convex quadratic form.

1 Introduction

An inverse boundary value problem is the question of identifying the interior physical properties of an object using measurements taken on the boundary of the object. One interesting sub-problem, in the case when the object consists of two homogeneous phases, would be to estimate the volume fraction of one of those phases. This is known as the size estimate problem. In this paper we consider the size estimate problem for the isotropic elasticity equation and for the isotropic complex conductivity equation. In the case of elasticity, we seek to estimate the size of the inclusion using boundary measurements

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of displacements and tractions. In the case of the complex conductivity equation, our method will use boundary measurements of potentials and current densities.

Unlike more general reconstruction procedures, which require a large number of measurements, size estimate methods usually rely on only a few measurements. One known approach to deriving size estimates is a PDE method using quantitative uniqueness estimates, the other, which we use here, is a variational method which is related to the estimates of effective properties of the composites. In the case of the conductivity equation estimates for the size of the inclusion have been derived in the work of Alessandrini-Rosset [2], Alessandrini-Rosset-Seo [1], Ikehata [5], Kang-Seo-Sheen [10], Milton [14], and others. The translation method which was introduced by Murat-Tartar [17], [18], [16] and Lurie-Cherkaev [11], [12], was used by Kang-Kim-Milton [6], Kang-Milton [8] to obtain lower and upper bounds for the relative volume of an inclusion. The same approach was also used to bound the volume fraction of two-phase materials in the 2D elasticity case [15] and in the shallow shell case [9].

One practical example where size estimates in the complex conductivity case may be useful is the estimation of the size of tumors or other anomalous tissue in biological samples. Biological tissue, when probed using alternating currents, may be modeled using complex conductivities. This is due to the fact that cell membranes act as capacitors, thus introducing an imaginary component to the conductivity (see [13]). There are a number of results known in this case such as the one of Beretta-Francini-Vessella [3], Kang-Kim-Lee-Li-Milton [7], and Milton-Thaler [19]. We want to point out that [3] is based on the PDE method, while both [7] and [16] use the variational approach. To put the current work in perspective, we remark that [7] uses the translation method for the 2D problem and [16] considers 2D and 3D problems using the splitting method.

1.1 Main results – elasticity

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let $\mathbb{C}$ be a two-valued elastic tensor defined on this domain. We will assume that $\mathbb{C}$ is isotropic, i.e., that its components are

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are known as the Lamé coefficients. The ellipticity condition for $\mathbb{C}$ in terms of the Lamé coefficients is

$$\lambda(x) + 2\mu(x) > 0, \quad \mu(x) > 0, \quad \forall x \in \Omega.$$

We will make a stronger assumption, namely that

$$3\lambda(x) + 2\mu(x) > 0, \quad \mu(x) > 0, \quad \forall x \in \Omega.$$

This is known as the strong convexity condition (see [20, Section 2.9]).

We denote by Ω_1 and Ω_2 the domains corresponding to the two phases and will use corresponding lower indices to identify the values the Lamé coefficients and any other quantity derived from them take on these two domains. We also define the volume fractions $f_1 := |\Omega_1|/|\Omega|$ and $f_2 = |\Omega_2|/|\Omega|$. In order to obtain our results it will be necessary to assume that the values of the Lamé coefficients in the two phases satisfy the gap conditions

$$\mu_1 > \mu_2, \quad 3\lambda_1 + 2\mu_1 > 3\lambda_2 + 2\mu_2. \tag{1}$$

A function $u : \Omega \to \mathbb{R}^3$ is a solution of the linear elasticity equation with elastic tensor $\mathbb{C}$ provided

$$\partial_j(C_{ijkl}\partial_k u_l) = 0, \quad i = 1, 2, 3.$$

We will denote by $\phi_i = u_i|_{\partial\Omega}$, $i = 1, 2, 3$, the Dirichlet data of such a solution (displacement), and by $\psi_i = n_j C_{ijkl}\partial_k u_l|_{\partial\Omega}$ $i = 1, 2, 3$, the Neumann data (traction), where $n = (n_1, n_2, n_3)^t$ is the outer normal on $\partial\Omega$. We will consider two cases:

i) the solution u has Dirichlet data $\phi_i = \frac{1}{\sqrt{3}}x_i$, $i = 1, 2, 3$.

ii) the solution u has Neumann data $\psi_i = n_i$, $i = 1, 2, 3$.

Our main result is

Theorem 1.1. *There exists constants $C_i, C_{ii} \geq 0$, $C_i \leq 1$ depending (explicitly) on Ω, the values of $\mathbb{C}$ and the boundary Dirichlet and Neumann data of the solutions in case i) for C_i and case ii) for C_{ii}, such that $C_{ii} \leq f_1 \leq C_i$. Furthermore, $C_i = 1$ only if $f_1 = 1$ and $C_{ii} = 0$ only if $f_1 = 0$. The explicit forms of the constants C_i and C_{ii} are the ones appearing in equations* (4) *and* (5) *respectively.*

The method of proof is inspired by the work of Kang-Milton [8] for the 3D conductivity equation. In section 2.1 we will "translate" the minimum principle for the elasticity equation by a null-Lagrangian, i.e. a quantity that may be determined form boundary data. We will then extend the class of fields over which we minimize to obtain inequalities that would lead to an upper estimate of f_1. In section 2.2 we will proceed similarly, except that in this case we will "translate" the minimum principle by a quasi-convex quadratic. This quadratic form is not determined by the boundary data, but it's size can nevertheless be controlled by it. The result will be the lower estimate on f_1.

We would like to compare this to Milton's result in [14]. With the same special boundary conditions, Milton also derives bounds of the volume fraction for the two-phase isotropic elasticity in two and three dimensions. Some of these bounds require multiple measurements, though bounds derived from single measurements are also derived. The bounds, which are given in implicit form and may be dificult to be put into explicit forms, are obtained from the bounds of elastic response tensors.

1.2 Main results – complex conductivity

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let $\sigma : \Omega \to \{\sigma_1, \sigma_2\} \subset \mathbb{C}$ be a conductivity function with two values. For any complex valued quantity a we will write $a' = \mathfrak{Re}\, a$ and $a'' = \mathfrak{Im}\, a$. Then, for example, we have

$$\sigma = \sigma' + \sqrt{-1}\sigma''.$$

We will assume that the following gap conditions hold

$$\begin{array}{c} \sigma_1' > \sigma_2' > 0, \\ \frac{\sigma_1'^2 + \sigma_1''^2}{\sigma_1'} > \frac{\sigma_2'^2 + \sigma_2''^2}{\sigma_2'}. \end{array} \tag{2}$$

As above, let Ω_1, Ω_2, be the domains of the two phases and let $f_1 := |\Omega_1|/|\Omega|$, $f_2 := |\Omega_2|/|\Omega|$.

Let u_1, u_2, u_3 be three $\mathbb{R}$-linearly independent solutions of

$$\nabla \cdot (\sigma \nabla u) = 0, \quad \text{in } \Omega. \tag{3}$$

Denote $\phi_j := u_j''|_{\partial\Omega}$, $\psi_j = n \cdot (\sigma\nabla u)'|_{\partial\Omega}$, $j = 1, 2, 3$. We will consider three cases

i) $\phi_j = 0$ (i.e. *real* Dirichlet data) and the matrix

$$\left(\int_{\partial\Omega} x_i \psi_j \right)_{i,j \in \{1,2,3\}}$$

is invertible. Since the ψ_j can be any set of three linearly independent Neumann data, this condition is generic.

ii) $\psi_j = 0$ (i.e. *imaginary* Neumann data) and the matrix

$$\left(\int_{\partial\Omega} n_i \phi_j\right)_{i,j\in\{1,2,3\}}$$

is invertible. Likewise, this condition is generic.

iii) $\phi_j = 0$, $\psi_j = n_j$, $j = 1, 2, 3$. This is a particular instance of the first case.

Our main theorem is stated as follows.

Theorem 1.2. *In each of the three cases, there exist constants $c, C \geq 0$, depending (explicitly) on Ω, σ_1, σ_2, $u_j|_{\partial\Omega}$, $n\cdot(\sigma\nabla u_j)|_{\partial\Omega}$ such that $c \leq f_1 \leq C$. For the first case the bounds are the ones in* (9)*, for the second case they are the ones in* (10)*, and for the third case the ones in* (11)*.*

This result is obtained using two different versions of the translation method, adapted from the work of Kang-Milton [8]. In both instances we begin with a variational principle of Cherkaev-Gibiansky (see [4]). In the first and second case, we perturb the minimized quantity by a null-Lagrangian. Extending the class of fields over which we are taking the minimum, we obtain inequalities that eventually lead to upper and lower bounds for f_1. We carry out the detailed computations in section 3.1 for the first case and section 3.2 for the second case.

In the third case we perturb the minimum principle by a quasi-convex quadratic form. This method is explained in detail in section 3.3.

2 Estimates for elasticity

2.1 Upper bound

Using the notation

$$\langle \cdot \rangle := \frac{1}{|\Omega|}\int_\Omega \cdot \, dx,$$

we define

$$W = \langle \nabla u : \mathbb{C} : \nabla u \rangle = \langle \partial_i u_j C_{ijkl} \partial_k u_l \rangle.$$

It is known that

$$W = \min_{\underline{u}|_{\partial\Omega}=\phi} \langle \nabla \underline{u} : \mathbb{C} : \nabla \underline{u} \rangle.$$

We note that, using integration by parts,

$$W = \frac{1}{|\Omega|}\int_{\partial\Omega} u_i n_j C_{ijkl} \partial_k u_l,$$

that is, W is determined by the Dirichlet data ϕ and the Neumann data ψ.

We denote by $\mathbb{T}$ the tensor with coefficients

$$T_{ijkl} = \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}.$$

Note that, for $\underline{u}$ that shares the same Dirichlet data as u,

$$(\nabla\mathbb{T} : \nabla\underline{u})_j = -\partial_i T_{ijkl}\partial_k \underline{u}_l = -\partial_j\partial_k\underline{u}_k + \partial_i\partial_j\underline{u}_i = 0,$$

and therefore

$$\langle\nabla\underline{u} : \mathbb{T} : \nabla\underline{u}\rangle = \frac{1}{|\Omega|}\int_{\partial\Omega} n_i\underline{u}_i\partial_k\underline{u}_k - n_i\underline{u}_k\partial_k\underline{u}_i = \frac{1}{|\Omega|}\int_{\partial\Omega} \underline{u}_i(n_i\partial_k - n_k\partial_i)\underline{u}_k.$$

As the vector $v_j := n_i\delta_{jk} - n_k\delta_{ij}$ satisfies $n_j v_j = 0$, we conclude that $n_i\partial_k - n_k\partial_i$ is a tangential derivative. It follows that $\langle\nabla\underline{u} : \mathbb{T} : \nabla\underline{u}\rangle$ is determined by the boundary data ϕ, i.e. it is a null-Lagrangian.

We define, for $c > 0$,

$$W_c := \langle\nabla u : \mathbb{C} : \nabla u\rangle + c\langle\nabla u : \mathbb{T} : \nabla u\rangle.$$

Since the extra term is determined by ϕ, we have that

$$W_c = \min_{\underline{u}|_{\partial\Omega}=\phi} \left(\langle\nabla\underline{u} : \mathbb{C} : \nabla\underline{u}\rangle + c\langle\nabla\underline{u} : \mathbb{T} : \nabla\underline{u}\rangle\right).$$

Any 3×3 matrice may be written in vector form by ordering the pairs of indices (ij) as follows: (11), (21), (31), (12), $\ldots$. Then ∇u becomes

$$\mathbf{v} := \begin{pmatrix}\partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 & \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 & \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3\end{pmatrix}^t.$$

Using this notation we can write

$$W_c = \mathbf{v}\cdot\mathcal{L}\mathbf{v},$$

where

$$\mathcal{L} = \begin{pmatrix}
\lambda+2\mu & 0 & 0 & 0 & \lambda+c & 0 & 0 & 0 & \lambda+c \\
0 & \mu & 0 & \mu-c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 & 0 & \mu-c & 0 & 0 \\
0 & \mu-c & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\
\lambda+c & 0 & 0 & 0 & \lambda+2\mu & 0 & 0 & 0 & \lambda+c \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & \mu-c & 0 \\
0 & 0 & \mu-c & 0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu-c & 0 & \mu & 0 \\
\lambda+c & 0 & 0 & 0 & \lambda+c & 0 & 0 & 0 & \lambda+2\mu
\end{pmatrix}.$$

Let J be the orthogonal matrix

$$J := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, we have

$$J^t \mathcal{L} J = \begin{pmatrix} \lambda+2\mu & \lambda+c & \lambda+c & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda+c & \lambda+2\mu & \lambda+c & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda+c & \lambda+c & \lambda+2\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & \mu-c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu-c & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & \mu-c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu-c & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & \mu-c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu-c & \mu \end{pmatrix}.$$

The upper left block is of particular interest. We denote it by

$$B := \begin{pmatrix} \lambda+2\mu & \lambda+c & \lambda+c \\ \lambda+c & \lambda+2\mu & \lambda+c \\ \lambda+c & \lambda+c & \lambda+2\mu \end{pmatrix}.$$

This block may be fully diagonalized by conjugation with the orthogonal matrix

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix},$$

that is

$$C := S^t B S = \begin{pmatrix} 2\mu-c & 0 & 0 \\ 0 & 2\mu-c & 0 \\ 0 & 0 & 3\lambda+2\mu+2c \end{pmatrix}.$$

Here we may also note that each block $\begin{pmatrix} \mu & \mu-c \\ \mu-c & \mu \end{pmatrix}$ can be diagonalized by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^t \begin{pmatrix} \mu & \mu-c \\ \mu-c & \mu \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2\mu-c & 0 \\ 0 & c \end{pmatrix}.$$

We see then that as long as $0 < c < 2\mu$, then $\mathcal{L} > 0$. By (1), the condition is that $0 < c < 2\mu_2$.

If $\underline{u} : \Omega \to \mathbb{R}^3$ is such that $\underline{u}|_{\partial\Omega} = \phi$, then

$$\langle \partial_i \underline{u}_j \rangle = \frac{1}{|\Omega|} \int_{\partial\Omega} n_i \phi_j.$$

It then follows that

$$W_c \geq \min_{\langle \underline{\mathbf{v}} \rangle = \langle \mathbf{v} \rangle} \langle \underline{\mathbf{v}} \cdot \mathcal{L} \underline{\mathbf{v}} \rangle > 0.$$

Let $\hat{\mathbf{v}}$ be an element for which the minimum is realized. Then for any φ s.t. $\langle \varphi \rangle = 0$,

$$\langle \varphi \cdot \mathcal{L} \hat{\mathbf{v}} \rangle = \frac{1}{2} \frac{d}{dt}\bigg|_{t=0} \langle (\hat{\mathbf{v}} + t\varphi) \cdot \mathcal{L}(\hat{\mathbf{v}} + t\varphi) \rangle = 0,$$

so there must be a constant $m \in \mathbb{R}^9$ such that

$$(\mathcal{L}\hat{\mathbf{v}})(x) = \mu, \quad x \in \Omega.$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the values $\mathcal{L}$ takes on Ω_1 and Ω_2 respectively. Clearly $\hat{\mathbf{v}}$ is constant on Ω_1 and Ω_2. It follows that

$$W_c \geq \min_{f_1\mathbf{v}_1 + f_2\mathbf{v}_2 = \langle \mathbf{v} \rangle} \left(f_1 \mathbf{v}_1 \cdot \mathcal{L}_1 \mathbf{v}_1 + f_2 \mathbf{v}_2 \cdot \mathcal{L}_2 \mathbf{v}_2 \right),$$

where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^9$ are constants. By [7, Lemma 3.1] we conclude that

$$W_c \geq \langle \mathbf{v} \rangle \cdot \langle \mathcal{L}^{-1} \rangle^{-1} \langle \mathbf{v} \rangle,$$

where

$$\langle \mathcal{L}^{-1} \rangle = f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1}.$$

Since C is diagonal it is easy to determine that

$$\langle C^{-1} \rangle_{11}^{-1} = \langle C^{-1} \rangle_{22}^{-1} = (2\mu_2 - c) \left(1 - f_1 \frac{2(\mu_1 - \mu_2)}{2\mu_1 - c} \right)^{-1},$$

and

$$\langle C^{-1} \rangle_{33}^{-1} = (3\lambda_2 + 2\mu_2 + 2c) \left(1 - f_1 \frac{3(\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2)}{3\lambda_1 + 2\mu_1 + 2c} \right)^{-1},$$

the other matrix elements being zero.

If we choose Dirichlet data $\phi_j = \frac{1}{\sqrt{3}} x_j|_{\partial\Omega}$, then we have $\langle \partial_i u_j \rangle = \frac{1}{\sqrt{3}} \delta_{ij}$. Then

$$W_c \geq \langle \mathbf{v} \rangle \cdot \langle \mathcal{L}^{-1} \rangle^{-1} \langle \mathbf{v} \rangle = \langle C^{-1} \rangle_{33}^{-1}.$$

If we take the limit as $c \nearrow 2\mu_2$, we obtain the inequality

$$f_1 \leq \frac{3\lambda_1 + 2\mu_1 + 4\mu_2}{3(\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2)} \left(1 - \frac{1}{T}\right), \tag{4}$$

where

$$T := \frac{\lim_{c \nearrow 2\mu_2} W_c}{3\lambda_2 + 6\mu_2}.$$

With the choice of Dirichlet data we are making, by the minimum principle, we have

$$W_{2\mu_2} \leq \frac{1}{3} \left(\langle \mathbf{I_{3\times3}} : \mathbb{C} : \mathbf{I_{3\times3}} \rangle + 2\mu_2 \langle \mathbf{I_{3\times3}} : \mathbb{T} : \mathbf{I_{3\times3}} \rangle \right) \leq 3\lambda_1 + 2\mu_1 + 4\mu_2,$$

with equality attained only if $f_1 = 1$. Then we have that

$$1 - \frac{1}{T} \leq \frac{3(\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2)}{3\lambda_1 + 2\mu_1 + 4\mu_2},$$

and it follows that the upper bound in (4) satisfies

$$\frac{3\lambda_1 + 2\mu_1 + 4\mu_2}{3(\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2)} \left(1 - \frac{1}{T}\right) \leq 1,$$

with equality attained only if $f_1 = 1$. The bound is therefore non-trivial.

Remark 2.1. *Here we have used one solution with specifically chosen Dirichlet boundary data, but a similar bound may be obtained for almost any solution. To see this, note that there is an orthonormal basis in which* $\lim_{c \nearrow 2\mu_2} \langle \mathcal{L}^{-1} \rangle^{-1}$ *is*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \langle C^{-1} \rangle_{33}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu_2 \end{pmatrix}.$$

Then in this basis we would have

$$\langle C^{-1} \rangle_{33}^{-1} \langle \mathbf{v} \rangle_3^2 \leq W_{2\mu_2} - 2\mu_2 \left(\langle \mathbf{v} \rangle_5^2 + \langle \mathbf{v} \rangle_7^2 + \langle \mathbf{v} \rangle_9^2 \right),$$

which would produce the same type of bound as (4), as long as $\langle \mathbf{v} \rangle_3 \neq 0$. *However, in this case, there is no guarantee that the upper bound obtained for this data is non-trivial.*

2.2 Lower bound

Let

$$\alpha := -\frac{\lambda}{2\mu(3\lambda+2\mu)}, \quad \beta := \frac{1}{4\mu},$$

and $\mathbf{D}$ be the (compliance) tensor with components

$$D_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We note that α and β also satisfy the strong convexity conditions, and that

$$\beta_1 < \beta_2, \quad 3\alpha_1 + 2\beta_1 < 3\alpha_2 + 2\beta_2.$$

Let $J := \mathbb{C} : \nabla u$, then

$$J = J^t, \quad \nabla J = 0,$$

$$\mathbf{D} : J = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right),$$

and therefore we have

$$W = \langle \nabla u : \mathbb{C} : \nabla u \rangle = \langle J : \mathbf{D} : J \rangle.$$

If $\underline{J}$ is symmetric, satisfies $\nabla \underline{J} = 0$ and $\underline{J}n|_{\partial\Omega} = \psi$, then clearly

$$\langle (\underline{J} - J) : \mathbf{D} : J \rangle = \mathrm{Tr}\, \langle (\underline{J} - J)\nabla u \rangle = 0.$$

It then follows easily that

$$W = \min_{\nabla \underline{J}=0, \underline{J}n|_{\partial\Omega}=\psi} \langle \underline{J} : \mathbf{D} : \underline{J} \rangle.$$

In this section we assume that the solution u has Neumann data $\psi_j = n_j$. In this case

$$\langle J_{ij} \rangle = \frac{1}{|\Omega|}\int_{\partial\Omega} x_i n_k J_{kj} = \frac{1}{|\Omega|}\int_{\partial\Omega} x_i \psi_j = \delta_{ij}.$$

The tensors Λ^h, Λ^s, Λ^a defined below are orthogonal projections acting on $M_{3\times3}$. Their components are

$$\Lambda^h_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl},$$

$$\Lambda^s_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl},$$

$$\Lambda^a_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

With this notation we notice that

$$\mathbf{D} = (3\alpha + 2\beta)\Lambda^h + 2\beta\Lambda^s.$$

Let

$$\mathbb{T}' := 2\Lambda^s - \Lambda^h.$$

It can be shown (see [8]) that

$$P : \mathbb{T}' : P \geq 0, \quad \forall\, P \text{ of rank at most 2.}$$

Using methods borrowed from homogenization theory, in [8, (4.20)] it is shown that, when we choose boundary data as we have done,

$$g := \min_{\nabla \underline{J}=0,\, \underline{J}n|_{\partial\Omega}=\psi} \langle \underline{J} : \mathbb{T}' : \underline{J} \rangle = -3.$$

We define

$$W_c' := \langle J : (\mathbf{D} - c\mathbb{T}') : J \rangle = \langle J : \big[(3\alpha + 2\beta + c)\Lambda^h + 2(\beta - c)\Lambda^s\big] : J \rangle,$$

where $c > 0$. We can estimate W_c' from above by

$$W_c' \leq \min_{\nabla \underline{J}=0,\, \underline{J}n|_{\partial\Omega}=\psi} \langle \underline{J} : \mathbf{D} : \underline{J} \rangle - c \min_{\nabla \underline{J}=0,\, \underline{J}n|_{\partial\Omega}=\psi} \langle \underline{J} : \mathbb{T}' : \underline{J} \rangle \leq \langle \mathbf{I_{3\times 3}} : \mathbf{D} : \mathbf{I_{3\times 3}} \rangle + 3c$$
$$\leq 3(3\alpha_2 + 2\beta_2 + c).$$

Now if $\underline{J}n|_{\partial\Omega} = \psi$, then $\langle \underline{J} \rangle = \langle J \rangle$. Therefore

$$W_c' \geq \min_{\langle \underline{J} \rangle = \langle J \rangle} \big\langle (3\alpha + 2\beta + c)\underline{J} : \Lambda^h : \underline{J} + 2(\beta - c)\underline{J} : \Lambda^s : \underline{J} \big\rangle > 0,$$

as long as $c < \beta_1$. In this case we may drop the second term to obtain

$$W_c' \geq \frac{1}{3} \min_{\langle \underline{J} \rangle = \langle J \rangle} \big\langle (3\alpha + 2\beta + c)(\mathrm{Tr}\, \underline{J})^2 \big\rangle.$$

By Jensen's inequality we have

$$\big\langle (3\alpha + 2\beta + c)(\mathrm{Tr}\, \underline{J})^2 \big\rangle \geq (3\alpha_1 + 2\beta_1 + c)\frac{1}{f_1}\left(\frac{1}{|\Omega|}\int_{\Omega_1} \mathrm{Tr}\, \underline{J}\right)^2$$
$$+ (3\alpha_2 + 2\beta_2 + c)\frac{1}{f_2}\left(\frac{1}{|\Omega|}\int_{\Omega_2} \mathrm{Tr}\, \underline{J}\right)^2.$$

Let

$$z := \frac{1}{|\Omega|}\int_{\Omega_1} \mathrm{Tr}\, \underline{J} \in [0, 3].$$

Then

$$W'_c \geq \frac{1}{3}\left[(3\alpha_1+2\beta_1+c)\frac{1}{f_1}z^2+(3\alpha_2+2\beta_2+c)\frac{1}{f_2}(3-z)^2\right],$$

and minimizing over $z \in [0,3]$ we have that

$$W'_c \geq \frac{3}{\frac{f_1}{3\alpha_1+2\beta_1+c}+\frac{f_2}{3\alpha_2+2\beta_2+c}}.$$

Taking the limit $c \nearrow \beta_1$, we may rewrite this estimate as

$$\left(1+f_1\frac{3(\alpha_2-\alpha_1)+2(\beta_2-\beta_1)}{3(\alpha_1+\beta_1)}\right)^{-1} \leq T',$$

where

$$T' := \frac{W'_{\beta_1}}{3(3\alpha_2+2\beta_2+\beta_1)} \geq 1,$$

with equality holding only when $f_1 = 0$. We conclude that

$$f_1 \geq \frac{3(\alpha_1+\beta_1)}{3(\alpha_2-\alpha_1)+2(\beta_2-\beta_1)}\left(\frac{1}{T'}-1\right). \tag{5}$$

As we have observed above, this lower bound is non-trivial and equals zero only when $f_1 = 0$.

3 Estimates for complex conductivity

3.1 The first case – real general Dirichlet data

We adopt some notations from [7] and [8]. Define

$$\mathbf{e}_i := -\nabla u_i, \quad \mathbf{j}_i := \sigma\mathbf{e}_i = -\sigma\nabla u_i,$$
$$\mathbf{e}_i =: \mathbf{e}'_i+\sqrt{-1}\mathbf{e}''_i, \quad \mathbf{j}_i =: \mathbf{j}'_i+\sqrt{-1}\mathbf{j}''_i, \quad \sigma =: \sigma'+\sqrt{-1}\sigma'',$$
$$\mathbf{E} := (E_{ij})_{ij}, \quad E_{ij} := -\partial_i u_j, \quad \mathbf{J} := \sigma\mathbf{E},$$
$$\mathbf{E} =: \mathbf{E}'+\sqrt{-1}\mathbf{E}'', \quad \mathbf{J} =: \mathbf{J}'+\sqrt{-1}\mathbf{J}'',$$
$$\mathbf{V} := \begin{pmatrix}\mathbf{J}'\\ \mathbf{E}''\end{pmatrix}.$$

Let $\mathbf{d}$ be a 6×6 matrix given by

$$\mathbf{d} = \begin{pmatrix} d_{11}\mathbf{I_{3\times 3}} & d_{12}\mathbf{I_{3\times 3}} \\ d_{21}\mathbf{I_{3\times 3}} & d_{22}\mathbf{I_{3\times 3}}\end{pmatrix} := \frac{1}{\sigma'}\begin{pmatrix}\mathbf{I_{3\times 3}} & \sigma''\mathbf{I_{3\times 3}} \\ \sigma''\mathbf{I_{3\times 3}} & (\sigma'^2+\sigma''^2)\mathbf{I_{3\times 3}}\end{pmatrix}$$

and denote

$$\mathbf{A} := \langle \mathbf{V}^t \mathbf{d} \mathbf{V} \rangle.$$

It is easy to check that

$$\begin{pmatrix} \mathbf{e}'_i \\ \mathbf{j}''_i \end{pmatrix} = \mathbf{d} \begin{pmatrix} \mathbf{j}'_i \\ \mathbf{e}''_i \end{pmatrix},$$

and therefore, using integration by parts,

$$A_{ij} = \langle \begin{pmatrix} \mathbf{j}'_i \\ \mathbf{e}''_i \end{pmatrix} \cdot \mathbf{d} \begin{pmatrix} \mathbf{j}'_j \\ \mathbf{e}''_j \end{pmatrix} \rangle = \langle \begin{pmatrix} \mathbf{j}'_i \\ \mathbf{e}''_i \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e}'_j \\ \mathbf{j}''_j \end{pmatrix} \rangle = \frac{1}{|\Omega|} \int_{\partial\Omega} \left(u'_j n \cdot (\sigma \nabla u_i)' + u''_i n \cdot (\sigma \nabla u_j)'' \right).$$

We see then that $\mathbf{A}$ is determined by boundary measurements for the three solutions we are considering.

Furthermore, notice that for any

$$\underline{\mathbf{V}} = \begin{pmatrix} \underline{\mathbf{J}}' \\ \underline{\mathbf{E}}'' \end{pmatrix},$$

it holds that

$$\underline{\mathbf{V}}^t \mathbf{d} \underline{\mathbf{V}} = \frac{1}{\sigma'} (\underline{\mathbf{J}}' + \sigma'' \underline{\mathbf{E}}'')^t (\underline{\mathbf{J}}' + \sigma'' \underline{\mathbf{E}}'') + \sigma' (\underline{\mathbf{E}}'')^t \underline{\mathbf{E}}'' \geq 0, \tag{6}$$

meaning positive-definite. We again use the tensor $\mathbb{T}$ with components

$$T_{ijkl} = \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}.$$

Then

$$\langle \underline{\mathbf{E}}'' : \mathbb{T} : \underline{\mathbf{E}}'' \rangle = \frac{1}{|\Omega|} \int_{\partial\Omega} n_i \underline{u}''_i \partial_k \underline{u}''_k - n_i \underline{u}''_k \partial_k \underline{u}''_i = \frac{1}{|\Omega|} \int_{\partial\Omega} \underline{u}''_i (n_i \partial_k - n_k \partial_i) \underline{u}''_k.$$

As remarked above, $n_i\partial_k - n_k\partial_i$ is a tangential derivative. We can then conclude that $\langle \underline{\mathbf{E}}'' : \mathbb{T} : \underline{\mathbf{E}}'' \rangle$ is determined by the boundary data ϕ_j for any $\underline{\mathbf{E}}''$ as in (8). Thus, $\mathbb{T}$ defines a null Lagrangian for our problem. We also note that

$$P : \mathbb{T} : P \geq -\text{Tr}\,(P^t P), \quad \forall P \in M_{3\times 3}. \tag{7}$$

For any k, l, let $\mathbf{I}^{kl}$ be the 3×3 matrix given by $\mathbf{I}^{kl} := (\delta_{ik}\delta_{jl})_{ij}$, then $(\mathbf{E}''\mathbf{I}^{kl})_{ij} = -\partial_i u''_k \delta_{jl}$ and $\partial_i T_{ijkl} (\mathbf{E}''\mathbf{I}^{mn})_{kl} = 0$. The tensor $\mathbb{M}$ with components

$$\begin{aligned} M_{ijkl} &:= \langle (\mathbf{E}''\mathbf{I}^{ij}) : \mathbb{T} : (\mathbf{E}''\mathbf{I}^{kl}) \rangle = \langle (\mathbf{E}''\mathbf{I}^{ij})_{\alpha\beta} T_{\alpha\beta\gamma\delta} (\mathbf{E}''\mathbf{I}^{kl})_{\gamma\delta} \rangle \\ &= \langle \partial_j u''_i \partial_l u''_k - \partial_l u''_i \partial_j u''_k \rangle \end{aligned}$$

can also be determined from boundary measurements. For any $K \in M_{3\times 3}(\mathbb{R})$, let $\mathbf{b}^i$ be the product of the normalized eigenvector of KK^t and the associated singular value of K, then $KK^t = \sum_{i=1}^{3} \mathbf{b}^i(\mathbf{b}^i)^t$. In view of the variational principle of Cherkaev and Gibiansky [4], we have that

$$W_K := \mathrm{Tr}\ \langle K^t\mathbf{V}^t\mathbf{d}\mathbf{V}K\rangle = \mathrm{Tr}\ (K^t\mathbf{A}K) = \sum_{i=1}^{3}(\mathbf{b}^i)^t\mathbf{A}\mathbf{b}^i = \min_{\underline{\mathbf{V}}\in\mathcal{C}_1} \mathrm{Tr}\ \langle K^t\underline{\mathbf{V}}^t\mathbf{d}\underline{\mathbf{V}}K\rangle,$$

where Tr stands for the trace of a matrix. Here the admissible space $\mathcal{C}_1$ is given by

$$\mathcal{C}_1 = \left\{ \underline{\mathbf{V}} = \begin{pmatrix} \underline{\mathbf{J}}' \\ \underline{\mathbf{E}}'' \end{pmatrix} : \underline{E}''_{ij} = -\partial_i \underline{u}''_j, \partial_i \underline{J}'_{ij} = 0, \underline{u}''_j|_{\partial\Omega} = \phi_j, \underline{J}'_{ij}n_i|_{\partial\Omega} = \psi_j \right\}. \tag{8}$$

We also have that

$$\langle(\mathbf{E}''K) : \mathbb{T} : (\mathbf{E}''K)\rangle = K : \mathbb{M} : K.$$

Next, we "translate" W_K by a null Lagrangian

$$\begin{aligned} W_{c,K} &:= \mathrm{Tr}\ \langle K^t\mathbf{V}^t\mathbf{d}\mathbf{V}K\rangle + c\langle(\mathbf{E}''K) : \mathbb{T} : (\mathbf{E}''K)\rangle \\ &= \mathrm{Tr}\ (K^t\mathbf{A}K) + cK : \mathbb{M} : K. \end{aligned}$$

By (6), (7) we have that for any $\underline{\mathbf{V}}$

$$\mathrm{Tr}\ \langle\underline{\mathbf{V}}^t\mathbf{d}\underline{\mathbf{V}}\rangle + c\langle\underline{\mathbf{E}}'' : \mathbb{T} : \underline{\mathbf{E}}''\rangle \geq \frac{1}{\sigma'}(\underline{\mathbf{J}}' + \sigma''\underline{\mathbf{E}}'') : (\underline{\mathbf{J}}' + \sigma''\underline{\mathbf{E}}'') + (\sigma' - c)\,\underline{\mathbf{E}}'' : \underline{\mathbf{E}}'',$$

and it is positive as long as $0 < c < \sigma'$. Taking (2) into consideration, this condition reduces to

$$0 < c < \sigma'_2.$$

Also notice that

$$W_{c,K} = \min_{\underline{\mathbf{V}}\in\mathcal{C}_1} \left(\mathrm{Tr}\ \langle K^t\underline{\mathbf{V}}^t\mathbf{d}\underline{\mathbf{V}}K\rangle + c\langle(\underline{\mathbf{E}}''K) : \mathbb{T} : (\underline{\mathbf{E}}''K)\rangle\right).$$

As we have done above, we will put 3×3 matrices in vector form by ordering the pairs of indices (ij) as follows: (11), (21), (31), (12), ... K would then become

$$\mathbf{k} := \begin{pmatrix} k_{11} & k_{21} & k_{31} & k_{12} & k_{22} & k_{32} & k_{13} & k_{23} & k_{33} \end{pmatrix}^t.$$

Extending this idea to 6×3 matrices, $\mathbf{V}$ becomes

$$\mathbf{v} := \begin{pmatrix} \mathbf{j}_1' \\ \mathbf{j}_2' \\ \mathbf{j}_3' \\ \mathbf{e}_1'' \\ \mathbf{e}_2'' \\ \mathbf{e}_3'' \end{pmatrix},$$

while $\mathbf{V}K$ becomes

$$\mathbf{v}_K := \begin{pmatrix} k_{11}\mathbf{j}_1' + k_{21}\mathbf{j}_2' + k_{31}\mathbf{j}_3' \\ k_{12}\mathbf{j}_1' + k_{22}\mathbf{j}_2' + k_{32}\mathbf{j}_3' \\ k_{13}\mathbf{j}_1' + k_{23}\mathbf{j}_2' + k_{33}\mathbf{j}_3' \\ k_{11}\mathbf{e}_1'' + k_{21}\mathbf{e}_2'' + k_{31}\mathbf{e}_3'' \\ k_{12}\mathbf{e}_1'' + k_{22}\mathbf{e}_2'' + k_{32}\mathbf{e}_3'' \\ k_{13}\mathbf{e}_1'' + k_{23}\mathbf{e}_2'' + k_{33}\mathbf{e}_3'' \end{pmatrix}.$$

With this notation we have

$$W_K = \mathbf{k} \cdot \begin{pmatrix} \mathbf{A} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \mathbf{A} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & \mathbf{A} \end{pmatrix} \mathbf{k}$$

and

$$W_{c,K} = \mathbf{k} \cdot \mathcal{D}_c \mathbf{k},$$

where

$$\mathcal{D}_c := \begin{pmatrix} \mathbf{A} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \mathbf{A} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & \mathbf{A} \end{pmatrix} + c\mathbf{M}$$

and $\mathbf{M}$ is defined via $\mathbf{k} \cdot \mathbf{M}\mathbf{k} := K : \mathbb{M} : K$. Equivalently, we can write

$$W_{c,K} = \langle \mathbf{v}_K \cdot \mathcal{L}\mathbf{v}_K \rangle = \min_{\mathbf{V} \in \mathcal{C}_1} \langle \underline{\mathbf{v}}_K \cdot \mathcal{L}\underline{\mathbf{v}}_K \rangle,$$

where

$$\mathcal{L} = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12} \\
\hline
d_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{22} & 0 & 0 & 0 & c & 0 & 0 & 0 & c \\
0 & d_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{22} & 0 & -c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{22} & 0 & 0 & 0 & -c & 0 & 0 \\
\hline
0 & 0 & 0 & d_{21} & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & d_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{21} & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & d_{22} & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & d_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{22} & 0 & -c & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & d_{21} & 0 & 0 & 0 & 0 & -c & 0 & 0 & 0 & d_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{21} & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & d_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{21} & c & 0 & 0 & 0 & c & 0 & 0 & 0 & d_{22}
\end{array}\right).$$

3.1.1 Construction of a lower bound for $W_{c,K}$

Note that $W_{c,K}$ is completely determined by the boundary measurements provided c and K are given. Here we would like to derive a lower bound for $W_{c,K}$. If $\underline{\mathbf{V}} \in \mathcal{C}_1$ integrating by parts we get

$$\langle \underline{E}''_{ij} \rangle = -\frac{1}{|\Omega|} \int_{\partial\Omega} n_i \phi_j, \quad \langle \underline{J}'_{ij} \rangle = \frac{1}{|\Omega|} \int_{\partial\Omega} x_i n_k \underline{J}'_{kj} = \frac{1}{|\Omega|} \int_{\partial\Omega} x_i \psi_j.$$

Let $\underline{\mathbf{J}}', \underline{\mathbf{E}}''$ be any 3×3 matrix-valued functions. We define the set

$$\mathcal{C}_2 = \left\{ \underline{\mathbf{V}} = \begin{pmatrix} \underline{\mathbf{J}}' \\ \underline{\mathbf{E}}'' \end{pmatrix} : \langle \underline{\mathbf{V}} \rangle = \langle \mathbf{V} \rangle \right\}$$

Then it is easily seen that $\mathcal{C}_1 \subset \mathcal{C}_2$ and

$$W_{c,K} \geq \min_{\langle \underline{\mathbf{v}} \rangle = \langle \mathbf{v} \rangle} \langle \underline{\mathbf{v}}_K \cdot \mathcal{L} \underline{\mathbf{v}}_K \rangle > 0,$$

the second inequality holding provided $0 < c < \sigma'_2$. Let $\hat{\mathbf{V}} = \begin{pmatrix} \hat{\mathbf{J}}' \\ \hat{\mathbf{E}}'' \end{pmatrix} \in \mathcal{C}_2$ be an element for which the minimum is realized. Then

$$0 = \left. \frac{d}{dt} \right|_{t=0} \langle (\hat{\mathbf{v}} + t\psi)_K \cdot \mathcal{L}(\hat{\mathbf{v}} + t\psi)_K \rangle = 2 \langle \psi_K \cdot \mathcal{L} \hat{\mathbf{v}}_K \rangle$$

for any ψ s.t. $\langle \psi \rangle = 0$. There is therefore a constant $\mu \in \mathbb{R}^{18}$ s.t.

$$(\mathcal{L} \hat{\mathbf{v}}_K)(x) = \mu, \quad x \in \Omega.$$

We denote by $\mathcal{L}_1$ and $\mathcal{L}_2$ the values $\mathcal{L}$ takes on Ω_1 and Ω_2 respectively. Clearly $\hat{\mathbf{v}}_K$ is constant on Ω_1 and Ω_2. It follows that

$$W_{c,K} \geq \min_{f_1 \mathbf{v}_1 + f_2 \mathbf{v}_2 = \langle \mathbf{v}_K \rangle} \left(f_1 \mathbf{v}_1 \cdot \mathcal{L}_1 \mathbf{v}_1 + f_2 \mathbf{v}_2 \cdot \mathcal{L}_2 \mathbf{v}_2 \right),$$

where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{18}$ are constants. We can use, for example, [7, Lemma 3.1] to conclude

$$W_{c,K} \geq \langle \mathbf{v}_K \rangle \cdot \langle \mathcal{L}^{-1} \rangle^{-1} \langle \mathbf{v}_K \rangle,$$

where

$$\langle \mathcal{L}^{-1} \rangle = f_1 \mathcal{L}_1^{-1} + f_2 \mathcal{L}_2^{-1}.$$

3.1.2 Matrix manipulations

In this subsection, we will derive an interval bound for f_1. Let J be the orthogonal matrix

$$J = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Then, denoting $d_1 := d_{11}$, $d_2 := d_{12}$, $d_3 := d_{22}$,

$$J^t \mathcal{L} J = \begin{pmatrix}
d_1 & 0 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_1 & 0 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_1 & 0 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_2 & 0 & 0 & d_3 & c & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_2 & 0 & c & d_3 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_2 & c & c & d_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_1 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_2 & 0 & d_3 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 & -c & d_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_1 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_1 & 0 & d_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 & 0 & d_3 & -c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 & -c & d_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_1 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_1 & 0 & d_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 & 0 & d_3 & -c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_2 & -c & d_3
\end{pmatrix}$$

In order to investigate $\langle \mathcal{L}^{-1} \rangle^{-1}$, we may consider the individual blocks separately. Let

$$B = \begin{pmatrix} d_1 & 0 & 0 & d_2 & 0 & 0 \\ 0 & d_1 & 0 & 0 & d_2 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 \\ d_2 & 0 & 0 & d_3 & c & c \\ 0 & d_2 & 0 & c & d_3 & c \\ 0 & 0 & d_2 & c & c & d_3 \end{pmatrix}$$

be the first of the diagonal blocks. We define another orthogonal matrix

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Then

$$C := S^t B S = \begin{pmatrix} d_1 & d_2 & 0 & 0 & 0 & 0 \\ d_2 & d_3 - c & 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 & d_2 & 0 & 0 \\ 0 & 0 & d_2 & d_3 - c & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & 0 & d_2 & d_3 + 2c \end{pmatrix}.$$

The inverse of C is

$$C^{-1} = \begin{pmatrix} \frac{d_3-c}{1-cd_1} & -\frac{d_2}{1-cd_1} & 0 & 0 & 0 & 0 \\ -\frac{d_2}{1-cd_1} & \frac{d_1}{1-cd_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d_3-c}{1-cd_1} & -\frac{d_2}{1-cd_1} & 0 & 0 \\ 0 & 0 & -\frac{d_2}{1-cd_1} & \frac{d_1}{1-cd_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{d_3+2c}{1+2cd_1} & -\frac{d_2}{1+2cd_1} \\ 0 & 0 & 0 & 0 & -\frac{d_2}{1+2cd_1} & \frac{d_1}{1+2cd_1} \end{pmatrix}$$

$$=: \begin{pmatrix} X_1 & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & X_2 & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & X_3 \end{pmatrix}$$

We will write

$$\delta_1 := \frac{1}{\sigma_2'}, \quad \delta_2 := \frac{\sigma_2''}{\sigma_2'}, \quad \delta_3 := \frac{\sigma_2'^2 + \sigma_2''^2}{\sigma_2'},$$

$$\Delta_1 := \frac{1}{\sigma_1'}, \quad \Delta_2 := \frac{\sigma_1''}{\sigma_1'}, \quad \Delta_3 := \frac{\sigma_1'^2 + \sigma_1''^2}{\sigma_1'}.$$

We want to look into what happens when $c \nearrow 1/\delta_1 = \sigma_2'$. So we can write X_1, X_2, X_3 as

$$X_1 = X_2 = \frac{1}{1 - cd_1} \begin{pmatrix} \frac{d_2^2}{d_1} + \frac{1-cd_1}{d_1} & -d_2 \\ -d_2 & d_1 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} \frac{d_2^2+3}{3d_1} + \frac{2d_2}{3d_1(1+2cd_1)}(1 - cd_1) & -\frac{d_2}{3} - \frac{2d_2}{3(1+2cd_1)}(1 - cd_1) \\ -\frac{d_2}{3} - \frac{2d_2}{3(1+2cd_1)}(1 - cd_1) & \frac{d_1}{3} + \frac{2d_1}{3(1+2cd_1)}(1 - cd_1) \end{pmatrix}.$$

Then

$$\det\langle X_1\rangle = \left[f_2^2 + \frac{f_1 f_2}{\delta_1 - \Delta_1}\left[\delta_1(\Delta_3\delta_1 - 1) + \delta_2(\Delta_1\delta_2 - \delta_1\Delta_2)\right]\right](1-c\delta_1)^{-1} + \mathcal{O}(1-c\delta_1)$$

and

$$\langle X_1\rangle^{-1} = \left[f_2 + f_1\frac{\delta_1(\Delta_3\delta_1 - 1) + \delta_2(\Delta_1\delta_2 - \delta_1\Delta_2)}{\delta_1 - \Delta_1}\right]^{-1} \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_2 & \frac{\delta_2^2}{\delta_1} \end{pmatrix} + \mathcal{O}(1-c\delta_1).$$

In the case of X_3, we can see that $\det\langle X_3\rangle > 0$ and satisfies

$$\begin{aligned} \det\langle X_3\rangle &= f_2^2\frac{1}{3} + f_1^2\frac{\delta_1}{2\Delta_1 + \delta_1} \\ &\quad + f_1 f_2\frac{1}{3(2\Delta_1 + \delta_1)}[\Delta_1(\delta_2^2 + 3) + (\Delta_3\delta_1 + 2)\delta_1 - 2\Delta_2\delta_1\delta_2] + \mathcal{O}(1 - c\delta_1) \\ &= \frac{1}{3} + f_1\frac{1}{3(2\Delta_1 + \delta_1)}[\Delta_1(\delta_2^2 - 1) + \Delta_3\delta_1^2 - 2\Delta_2\delta_1\delta_2] \\ &\quad + f_1^2\frac{1}{3(2\Delta_1 + \delta_1)}[2\delta_1 - \Delta_1(\delta_2^2 + 1) - \Delta_3\delta_1^2 + 2\Delta_2\delta_1\delta_2] + \mathcal{O}(1 - c\delta_1) \\ &=: \frac{1}{3}\left[1 + f_1\alpha + f_1^2\beta\right] + \mathcal{O}(1 - c\delta_1) \end{aligned}$$

and

$$\begin{aligned} \langle X_3\rangle^{-1} &= \left[1 + f_1\alpha + f_1^2\beta\right]^{-1} \\ &\quad \times \begin{pmatrix} \delta_1 - f_1\frac{\delta_1(\delta_1-\Delta_1)}{2\Delta_1+\delta_1} & \delta_2 - f_1\left(\delta_2 - \frac{3\Delta_2\delta_1}{2\Delta_1+\delta_1}\right) \\ \delta_2 - f_1\left(\delta_2 - \frac{3\Delta_2\delta_1}{2\Delta_1+\delta_1}\right) & \frac{\delta_2^2+3}{\delta_1} - f_1\left(\frac{\delta_2^2+3}{\delta_1} - 3\frac{\Delta_3\delta_1+2}{2\Delta_1+\delta_1}\right) \end{pmatrix} + \mathcal{O}(1 - c\delta_1), \end{aligned}$$

where

$$\alpha = \frac{1}{(2\Delta_1+\delta_1)}[\Delta_1(\delta_2^2-1)+\Delta_3\delta_1^2-2\Delta_2\delta_1\delta_2],$$

$$\beta = \frac{1}{(2\Delta_1+\delta_1)}[2\delta_1-\Delta_1(\delta_2^2+1)-\Delta_3\delta_1^2+2\Delta_2\delta_1\delta_2].$$

We would like to remark that since $\det\langle X_3\rangle > 0$, then $1+f_1\alpha+f_1^2\beta>0$.

It would be convenient to have $\beta<0$. To see that this is indeed so, we can estimate

$$(2\Delta_1+\delta_1)\frac{\beta}{\delta_1} = 2-\Delta_1\delta_3-\Delta_3\delta_1+2\Delta_2\delta_2 \le 2+\Delta_2^2+\delta_2^2-\Delta_1\delta_3-\Delta_3\delta_1$$
$$= \Delta_1\Delta_3+\delta_1\delta_3-\Delta_1\delta_3-\Delta_3\delta_1 = (\Delta_3-\delta_3)(\Delta_1-\delta_1).$$

Since by (2), $\delta_1>\Delta_1$ and $\Delta_3>\delta_3$, it follows that indeed $\beta<0$. Similarly

$$(2\Delta_1+\delta_1)\frac{\alpha}{\delta_1} = \Delta_1\delta_3-\frac{2\Delta_1}{\delta_1}+\Delta_3\delta_1-2\Delta_2\delta_2 \ge \Delta_1\delta_3+\Delta_3\delta_1-\Delta_2^2-\delta_2^2-\frac{2\Delta_1}{\delta_1}$$
$$= \Delta_1\delta_2+\Delta_3\delta_1+2-\delta_1\delta_3-\Delta_1\Delta_3-\frac{2\Delta_1}{\delta_1} = 2(1-\frac{\Delta_1}{\delta_1})+(\delta_1-\Delta_1)(\Delta_3-\delta_3)>0.$$

Now since $\langle \mathbf{J}'\rangle$ is invertible, we may choose $K=\frac{1}{\sqrt{3}}\langle \mathbf{J}'\rangle^{-1}$. Then

$$\langle \mathbf{v}_K\rangle = \frac{1}{\sqrt{3}}\left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc} 1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\end{array}\right)^t$$

and

$$\langle \mathbf{v}_K\rangle\cdot\langle\mathcal{L}^{-1}\rangle^{-1}\langle \mathbf{v}_K\rangle = \begin{pmatrix}1\\0\end{pmatrix}\cdot\langle X_3\rangle^{-1}\begin{pmatrix}1\\0\end{pmatrix}.$$

Let

$$T := \frac{1}{\delta_1}\lim_{c\nearrow 1/\delta_1} W_{c,\frac{1}{\sqrt{3}}\langle \mathbf{J}'\rangle^{-1}},$$

$$\gamma := \frac{(\delta_1-\Delta_1)}{2\Delta_1+\delta_1} > 0,$$

then we have

$$T \ge \frac{1-\gamma f_1}{1+\alpha f_1+\beta f_1^2}.$$

Equivalently,

$$1+(\alpha+\frac{\gamma}{T})f_1+\beta f_1^2 \ge \frac{1}{T}$$

and since $\beta < 0$ we get the bounds

$$\frac{-(\alpha+\frac{\gamma}{T})+\sqrt{(\alpha+\frac{\gamma}{T})^2-4\beta[1-T^{-1}]}}{2\beta} < f_1 < \frac{-(\alpha+\frac{\gamma}{T})-\sqrt{(\alpha+\frac{\gamma}{T})^2-4\beta[1-T^{-1}]}}{2\beta}. \quad (9)$$

Note that since we take $\phi_j = 0$, in this case $\mathbb{M} = 0$. Then

$$\begin{aligned} W_{c,\frac{1}{\sqrt{3}}\langle \mathbf{J}'\rangle^{-1}} &= \min_{\underline{\mathbf{V}}\in\mathcal{C}_1} \mathrm{Tr}\ \langle \frac{1}{\sqrt{3}}(\langle \mathbf{J}'\rangle^{-1})^t \underline{\mathbf{V}}^t \mathbf{d}\ \underline{\mathbf{V}} \frac{1}{\sqrt{3}}\langle \mathbf{J}'\rangle^{-1}\rangle \\ &\le \frac{1}{3}\mathrm{Tr}\ \langle(\langle \mathbf{J}'\rangle^{-1})^t \begin{pmatrix}\langle \mathbf{J}'\rangle \\ 0_{3\times 3}\end{pmatrix}^t \mathbf{d} \begin{pmatrix}\langle \mathbf{J}'\rangle \\ 0_{3\times 3}\end{pmatrix} \langle \mathbf{J}'\rangle^{-1}\rangle \\ &= \mathrm{Tr}\ \langle \begin{pmatrix}\mathbf{I_{3\times 3}} \\ 0_{3\times 3}\end{pmatrix} \mathbf{d} \begin{pmatrix}\mathbf{I_{3\times 3}} \\ 0_{3\times 3}\end{pmatrix}\rangle \le \delta_1. \end{aligned}$$

It follows that $T < 1$, which means that the lower bound in (9) is not negative.

Remark 3.1. *It appears that the upper bound in* (9) *is not neccesarily non-trivial. In fact, in the particular case of real conductivities it always is trivial.*

3.2 The second case – imaginary Neumann data

In this section, we suppose that $\psi_j = 0$, i.e. that the Neumann data is purely imaginary. We may choose $K = \frac{1}{\sqrt{3}}\langle \mathbf{E}''\rangle^{-1}$ since the matrix $\langle E''\rangle$ is invertible. Then

$$\langle \mathbf{v}_K\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \,|\, 0 & 0 & 0 \,|\, 0 & 0 & 0 \,|\, 1 & 0 & 0 \,|\, 0 & 1 & 0 \,|\, 0 & 0 & 1 \end{pmatrix}^t$$

and

$$\langle \mathbf{v}_K\rangle \cdot \langle \mathcal{L}^{-1}\rangle^{-1}\langle \mathbf{v}_K\rangle = \begin{pmatrix}0\\1\end{pmatrix} \cdot \langle X_3\rangle^{-1}\begin{pmatrix}0\\1\end{pmatrix}.$$

Let

$$\tilde{T} := \frac{\delta_1}{\delta_1\delta_3+2}\lim_{c\nearrow 1/\delta_1} W_{c,\frac{1}{\sqrt{3}}\langle \mathbf{E}''\rangle^{-1}},$$

$$\tilde{\gamma} := 1 - \frac{(3\delta_1(\Delta_3\delta_1+2)}{(\delta_1\delta_3+2)(2\Delta_1+\delta_1)} < 0,$$

then we have

$$\tilde{T} \ge \frac{1-\tilde{\gamma}f_1}{1+\alpha f_1+\beta f_1^2},$$

and we get the bounds

$$\frac{-(\alpha+\frac{\tilde{\gamma}}{\tilde{T}})+\sqrt{(\alpha+\frac{\tilde{\gamma}}{\tilde{T}})^2-4\beta[1-\tilde{T}^{-1}]}}{2\beta}<f_1$$
$$<\frac{-(\alpha+\frac{\tilde{\gamma}}{\tilde{T}})-\sqrt{(\alpha+\frac{\tilde{\gamma}}{\tilde{T}})^2-4\beta[1-\tilde{T}^{-1}]}}{2\beta}. \quad (10)$$

Remark 3.2. *In the case of real conductivities, the lower bound in* (10) *is negative, hence trivial. The upper one reduces to (see also [8])*

$$f_1 \leq \frac{1}{\gamma}\left(1-\frac{1}{\tilde{T}}\right),$$

and since in that case

$$\tilde{T} \leq \frac{\delta_1+2\Delta_1}{3\Delta_1},$$

it follows that the upper bound is nontrivial, i.e.

$$\frac{1}{\gamma}\left(1-\frac{1}{\tilde{T}}\right) \leq 1.$$

Since all the quantities that appear in the upper bound of (10) *depend continuously on the values σ_1 and σ_2, it follows that at least for certain cases with non-real conductivities,* (10) *does also provides non-trivial upper bounds.*

Remark 3.3. *A particular case of the boundary data considered in this section is $\psi_j = 0$, $\phi_j = \frac{1}{\sqrt{3}}x_j|_{\partial\Omega}$. We then see that the case treated here is analogous to the one treated in section 2.1 for the elasticity case.*

3.3 The third case – special boundary data

In this section, we derive another interval bound of the volume fraction. We will use special boundary conditions such that $\phi_j = 0$ and $\psi_j = n_j$, which implies $\langle \mathbf{J}' \rangle = \mathbf{I_{3\times3}}$. As pointed out in [8], there is no null-Lagragian corresponding to three-dimensional current fields. As in [8] and section 2.2 above, we translate the energy by a quasi-convex quadratic form. As above we use the tensor

$$\mathbb{T}' = 2\Lambda^s - \Lambda^h.$$

We define, for positive c,

$$W'_c := \min_{\underline{\mathbf{V}}\in\mathcal{C}_1} \left(\mathrm{Tr}\ \langle \underline{\mathbf{V}}^t \mathbf{d} \underline{\mathbf{V}} \rangle - c\langle \underline{\mathbf{J}}' : \mathbb{T}' : \underline{\mathbf{J}}' \rangle\right),$$

which is bounded from above by

$$W'_c \le \text{Tr } \mathbf{A} - c \min_{\underline{\mathbf{V}} \in \mathcal{C}_1} \langle \underline{\mathbf{J}}' : \mathbb{T}' : \underline{\mathbf{J}}' \rangle = \text{Tr } \mathbf{A} - cg \le 3(\delta_1 + c).$$

Now, we can estimate

$$\begin{aligned}
W'_c &\ge \min_{\underline{\mathbf{V}} \in \mathcal{C}_2} \left(\text{Tr } \langle \underline{\mathbf{V}}^t \mathbf{d} \underline{\mathbf{V}} \rangle - c \langle \underline{\mathbf{J}}' : \mathbb{T}' : \underline{\mathbf{J}}' \rangle \right) \\
&= \min_{\underline{\mathbf{V}} \in \mathcal{C}_2} \big\langle \underline{\mathbf{J}}' : ((d_{11} + c)\Lambda^h + (d_{11} - 2c)\Lambda^s + d_{11}\Lambda^a) : \underline{\mathbf{J}}' \\
&\qquad + \underline{\mathbf{J}}' : (d_{12}\Lambda^h + d_{12}\Lambda^s + d_{12}\Lambda^a) : \underline{\mathbf{E}}'' + \underline{\mathbf{E}}'' : (d_{21}\Lambda^h + d_{21}\Lambda^s + d_{21}\Lambda^a) : \underline{\mathbf{J}}' \\
&\qquad + \underline{\mathbf{E}}'' : (d_{22}\Lambda^h + d_{22}\Lambda^s + d_{22}\Lambda^a) : \underline{\mathbf{E}}'' \big\rangle \\
&= \min_{\underline{\mathbf{V}} \in \mathcal{C}_2} \langle \frac{1}{3} \left[\frac{1}{\sigma'} (\text{Tr } \underline{\mathbf{J}}' + \sigma'' \text{Tr } \underline{\mathbf{E}}'')^2 + \sigma' (\text{Tr } \underline{\mathbf{E}}'')^2 + c (\text{Tr } \underline{\mathbf{J}}')^2 \right] \\
&\qquad + \left[\frac{1}{\sigma'} (\Lambda^s : \underline{\mathbf{J}}' + \sigma'' \Lambda^s : \underline{\mathbf{E}}'')^{:2} + \sigma' (\Lambda^s : \underline{\mathbf{E}}'')^{:2} - 2c (\Lambda^s : \underline{\mathbf{J}}')^{:2} \right] \rangle,
\end{aligned}$$

where $A^{:2} := A : A$. Let $X = \Lambda^s : \underline{\mathbf{J}}'$, $Y = \Lambda^s : \underline{\mathbf{E}}''$. Then

$$\begin{aligned}
&(X + \sigma'' Y)^{:2} + \sigma'^2 Y^{:2} - 2c\sigma' X^{:2} \\
&\qquad = \left(\sqrt{1 - 2c\sigma'} X + \frac{\sigma''}{\sqrt{1 - 2c\sigma'}} Y \right)^{:2} + \left(\sigma'^2 - \frac{2c\sigma'}{1 - 2c\sigma'} \sigma''^2 \right) Y^{:2},
\end{aligned}$$

and this is non-negative provided $c \le \frac{\sigma'}{2(\sigma'^2 + \sigma''^2)}$. This holds provided $c \le \frac{\sigma'_1}{2(\sigma_1'^2 + \sigma_1''^2)}$. With this condition

$$W'_c \ge \min_{\underline{\mathbf{V}} \in \mathcal{C}_2} \langle \frac{1}{3} \left[\frac{1}{\sigma'} (\text{Tr } \underline{\mathbf{J}}' + \sigma'' \text{Tr } \underline{\mathbf{E}}'')^2 + \sigma' (\text{Tr } \underline{\mathbf{E}}'')^2 + c (\text{Tr } \underline{\mathbf{J}}')^2 \right] \rangle.$$

Using Jensen's inequality, we get

$$\langle (\text{Tr } \underline{\mathbf{J}}')^2 \rangle \ge \frac{1}{f_1} \left(\frac{1}{|\Omega|} \int_{\Omega_1} \text{Tr } \underline{\mathbf{J}}' \right)^2 + \frac{1}{f_2} \left(\frac{1}{|\Omega|} \int_{\Omega_2} \text{Tr } \underline{\mathbf{J}}' \right)^2,$$

$$\langle \sigma' (\text{Tr } \underline{\mathbf{E}}'')^2 \rangle \ge \frac{\sigma'_1}{f_1} \left(\frac{1}{|\Omega|} \int_{\Omega_1} \text{Tr } \underline{\mathbf{E}}'' \right)^2 + \frac{\sigma'_2}{f_2} \left(\frac{1}{|\Omega|} \int_{\Omega_2} \text{Tr } \underline{\mathbf{E}}'' \right)^2,$$

and

$$\begin{aligned}
\langle \frac{1}{\sigma'} (\text{Tr } \underline{\mathbf{J}}' + \sigma'' \text{Tr } \underline{\mathbf{E}}'')^2 \rangle &\ge \frac{1}{\sigma'_1 f_1} \left(\frac{1}{|\Omega|} \int_{\Omega_1} \text{Tr } \underline{\mathbf{J}}' + \sigma''_1 \text{Tr } \underline{\mathbf{E}}'' \right)^2 \\
&\qquad + \frac{1}{\sigma'_2 f_2} \left(\frac{1}{|\Omega|} \int_{\Omega_2} \text{Tr } \underline{\mathbf{J}}' + \sigma''_2 \text{Tr } \underline{\mathbf{E}}'' \right)^2.
\end{aligned}$$

Let

$$X_1 = \frac{1}{f_1|\Omega|}\int_{\Omega_1} \mathrm{Tr}\ \underline{\mathbf{J}}', \quad X_2 = \frac{1}{f_2|\Omega|}\int_{\Omega_2} \mathrm{Tr}\ \underline{\mathbf{J}}',$$

$$Y_1 = \frac{1}{f_1|\Omega|}\int_{\Omega_1} \mathrm{Tr}\ \underline{\mathbf{E}}'', \quad Y_2 = \frac{1}{f_2|\Omega|}\int_{\Omega_2} \mathrm{Tr}\ \underline{\mathbf{E}}'',$$

$$Z_1 = \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix},$$

$$\mathcal{L}' = \frac{1}{\sigma'}\begin{pmatrix} 1 + c\sigma' & \sigma'' \\ \sigma'' & (\sigma'^2 + \sigma''^2) \end{pmatrix}.$$

Then, applying again [7, Lemma 3.1], we obtain

$$W_c' \geq \frac{1}{3} \min_{f_1 Z_1 + f_2 Z_2 = \left(\substack{\langle \mathrm{Tr}\ \mathbf{J}' \rangle \\ \langle \mathrm{Tr}\ \mathbf{E}'' \rangle}\right)} (f_1 Z_1 \cdot \mathcal{L}_1' Z_1 + f_2 Z_2 \cdot \mathcal{L}_2' Z_2)$$

$$\geq \frac{1}{3}\begin{pmatrix} \langle \mathrm{Tr}\ \mathbf{J}' \rangle \\ \langle \mathrm{Tr}\ \mathbf{E}'' \rangle \end{pmatrix} \cdot \langle \mathcal{L}'^{-1} \rangle^{-1} \begin{pmatrix} \langle \mathrm{Tr}\ \mathbf{J}' \rangle \\ \langle \mathrm{Tr}\ \mathbf{E}'' \rangle \end{pmatrix}.$$

Note that

$$\mathcal{L}'^{-1} = \frac{1}{1 + c d_3}\begin{pmatrix} d_3 & -d_2 \\ -d_2 & d_1 + c \end{pmatrix},$$

so we see that

$$\det\langle \mathcal{L}'^{-1} \rangle = \frac{1}{1 + c\delta_3}\left[1 + \frac{\alpha'}{1 + c\Delta_3} f_1 + \frac{\beta'}{1 + c\Delta_3} f_1^2\right],$$

where

$$\alpha' = \delta_3(\Delta_1 + c) + \Delta_3(\delta_1 - c) - 2\delta_2\Delta_2 - 2,$$

$$\beta' = 2 + 2\delta_2\Delta_2 - \delta_3\Delta_1 - \Delta_3\delta_1.$$

We have seen already that $\beta' < 0$. Then

$$\langle \mathcal{L}'^{-1} \rangle^{-1} = (1 + c\delta_3)\left[1 + \frac{\alpha'}{1 + c\Delta_3} f_1 + \frac{\beta'}{1 + c\Delta_3} f_1^2\right]^{-1}$$

$$\times \begin{pmatrix} \frac{\delta_1 + c}{1 + c\delta_3}(1 - f_1) + \frac{\Delta_1 + c}{1 + c\Delta_3} f_1 & \frac{\delta_2}{1 + c\delta_3}(1 - f_1) + \frac{\Delta_2}{1 + c\Delta_3} f_1 \\ \frac{\delta_2}{1 + c\delta_3}(1 - f_1) + \frac{\Delta_2}{1 + c\Delta_3} f_1 & \frac{\delta_3}{1 + c\delta_3}(1 - f_1) + \frac{\Delta_3}{1 + c\Delta_3} f_1 \end{pmatrix}.$$

With boundary conditions we are considering, $\langle \mathrm{Tr}\ \mathbf{J}' \rangle = 3$, $\langle \mathbf{E}'' \rangle = 0$, so

$$W'_{\frac{1}{2\Delta_3}} \leq \mathrm{Tr}\ \mathbf{A} + \frac{3}{2\Delta_3} \leq 3\left(\delta_1 + \frac{1}{2\Delta_3}\right),$$

and

$$W'_{\frac{1}{2\Delta_3}} \geq 3\left(\delta_1 + \frac{1}{2\Delta_3}\right)\left[1 + \frac{2\alpha'}{3}f_1 + \frac{2\beta'}{3}f_1^2\right]^{-1}$$
$$\times\left[1 - f_1\left(1 - \frac{(2\Delta_3\Delta_1 + 1)(2\Delta_3 + \delta_3)}{3\Delta_3(2\Delta_3\delta_1 + 1)}\right)\right].$$

Let

$$T'' := \frac{\text{Tr}\,\mathbf{A} + \frac{3}{2\Delta_3}}{3\left(\delta_1 + \frac{1}{2\Delta_3}\right)} \leq 1,$$

$$\alpha'' := \frac{2}{3}\left[\delta_3(\Delta_1 + \frac{1}{2\Delta_3}) + \Delta_3(\delta_1 - \frac{1}{2\Delta_3}) - 2\delta_2\Delta_2 - 2\right],$$

$$\beta'' := \frac{2}{3}\left[2 + 2\delta_2\Delta_2 - \delta_3\Delta_1 - \Delta_3\delta_1\right],$$

$$\gamma'' := 1 - \frac{(2\Delta_3\Delta_1 + 1)(2\Delta_3 + \delta_3)}{3\Delta_3(2\Delta_3\delta_1 + 1)}.$$

With this notation

$$1 + (\alpha'' + \frac{\gamma''}{T''})f_1 + \beta'' f_1^2 \geq \frac{1}{T''},$$

and since $\beta'' < 0$ we get the bounds

$$\frac{-(\alpha'' + \frac{\gamma''}{T''}) + \sqrt{(\alpha'' + \frac{\gamma''}{T''})^2 - 4\beta''[1 - T''^{-1}]}}{2\beta''} < f_1$$
$$< \frac{-(\alpha'' + \frac{\gamma''}{T''}) - \sqrt{(\alpha'' + \frac{\gamma''}{T''})^2 - 4\beta''[1 - T''^{-1}]}}{2\beta''}. \quad (11)$$

Note that since $T'' \leq 1$, the lower bound is again nonnegative.

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