ON THE CHARACTERIZATION OF NON-RADIATING SOURCES FOR
THE ELASTIC WAVES IN ANISOTROPIC INHOMOGENEOUS MEDIA

PU-ZHAO KOW AND JENN-NAN WANG

ABSTRACT. In this paper, we would like to characterize non-radiating volume and surface 
(faulting) sources for the elastic waves in anisotropic inhomogeneous media. Each type of the 
source can be decomposed into a radiating part and a non-radiating part. The radiating part 
can be uniquely determined by an explicit formula containing the near-field measurements. On 
the other hand, the non-radiating part does not induce scattered waves at a certain frequency. 
In other words, such non-radiating source cannot be detected by measuring field at one single 
frequency in a region outside of the domain where the source is located.

1. Motivation of study and mathematical setup

Seismic waves in earth are typically generated by two types of sources. One is the external 
sources including winds, volcanic eruptions, vented explosions, meteorite impacts, etc. The 
other is the internal sources such as earthquakes or underground explosions. Seismic waves 
generated by external sources are usually foreseeable, while those induced by internal sources 
are hard to predict and often cause massive destruction. In this paper, we are interested in 
characterizing internal sources in terms of the scattering theory.

Internal sources can be grouped into two categories: volume sources and surface sources. In 
the seismic terminology, surface sources are called faulting sources resulting from slips across 
fracture planes [AR02, Chapter 3]. It was generally recognized that earthquakes are due 
to waves radiated from spontaneous slippage on active geological faults. According to U. S. 
there are more than a hundred of significant earthquakes around the world each year. Seismic waves 
generated by non-radiating sources are only confined in a bounded region. In this sense, the existence of non-radiating sources poses little threats to the nature 
environment. The main theme of this paper is to characterize non-radiating internal sources, 
both volume and surface sources. As a byproduct, we also derive reconstruction formulae 
of determining radiating volume and surface sources by the near-field measurements at one 
single frequency. For our problem, we will consider the stationary elastic wave equation in 
anisotropic inhomogeneous media.
To set up our mathematical problem, let $\Omega \subset \mathbb{R}^3$ be a domain with Lipschitz boundary $\partial \Omega$ and $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. Before introducing the elasticity tensor, we first define

$$
\begin{align*}
(A : B)_{ijkt} &:= \sum_{p,q=1}^{3} A_{ijpq} B_{pqkt} & \text{for two tensors } A \text{ and } B, \\
A : B &:= \sum_{i,j=1}^{3} a_{ij} b_{ij} & \text{for two matrices } A, B, \\
|A|^2 &:= A : \overline{A} & \text{for arbitrary matrix } A.
\end{align*}
$$

Assumption 1.1 (Assumptions on the elasticity tensor). Let $C(x) = (C_{ijkl}(x))_{1 \leq i,j,k,\ell \leq 3}$ be a real-valued elasticity tensor such that each entry $C_{ijkl} \in C^\infty(\mathbb{R}^3)$ satisfies symmetry properties

$$
C_{ijkt}(x) = C_{klij}(x), \quad C_{ijkl}(x) = C_{jikl}(x) \quad \text{in } \mathbb{R}^3,
$$

for all $1 \leq i,j,k,\ell \leq 3$. Moreover, assume that the strong ellipticity holds, that is, there exist constants $0 < \kappa_1 < \kappa_2$ such that

$$(1.1) \quad \kappa_1 |A|^2 \leq A : C(x) : \overline{A} \leq \kappa_2 |A|^2 \quad \text{for all } x \in \mathbb{R}^3,$$

and for all (complex-valued) matrix $A$. In addition, we assume that $C$ is isotropic and homogeneous outside $\Omega$, with Lamé constants $\lambda$ and $\mu$, that is,

$$(1.2) \quad C_{ijkl}(x) = \lambda \delta_{ij} \delta_{k\ell} + \mu (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$ 

Remark 1.2. In $\mathbb{R}^3 \setminus \overline{\Omega}$, (1.1) holds whenever the Lamé constants in (1.2) satisfy $\mu > 0$ and $3\lambda + 2\mu > 0$. Moreover, from (1.2), we have

$$
\nabla \cdot (C(x) : \nabla u) = \mathcal{L}^{\lambda,\mu} u := \mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u) \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.
$$

In the following section, we will describe the non-radiating sources, both volume and surface ones, in detail.

2. Main results and consequences

The theme of this section is to give precise definitions of non-radiating volume and surface sources and statements of main theorems. In view of main theorems, we then derive some interesting consequences in the case of homogeneous isotropic media. The proofs of theorems will be deferred to the later sections.

2.1. Volume sources. Suppose $f \in [L^2(\mathbb{R}^3)]^3$ with supp $(f) \subset \Omega$ (possibly complex-valued), denoted by $[L^2_\Omega(\mathbb{R}^3)]^3$. Let $\omega > 0$ be a frequency and consider the following time-harmonic elasticity equation:

$$(2.1) \quad \begin{cases}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u = -f & \text{in } \mathbb{R}^3, \\
u \text{ satisfies the Kupradze radiation condition at } |x| \to \infty
\end{cases}
$$

with $u \in [H^1_{loc}(\mathbb{R}^3)]^3$. We shall explain the Kupradze radiation condition in Definition 3.2 later.
Definition 2.1. We say that \( f \in [L^2_\Omega(\mathbb{R}^3)]^3 \) is a non-radiating volume source, if there exists a number \( R > 0 \) be such that \( u(x) = 0 \) for all \( |x| > R \).

Remark 2.2. By Rellich’s lemma, \( f \) is non-radiating if and only if the far-field pattern of \( u \) in (2.1) vanishes identically. Indeed, non-radiating sources also can be written in the form of interior transmission problem, see Appendix A.

Let us denote
\[
\mathcal{E}(\Omega) := \{ \mathbf{v} \in (H^1(\Omega))^3 \mid \nabla \cdot (\mathbf{C}(x) : \nabla \mathbf{v}) + \omega^2 \mathbf{v} = 0 \text{ in } \Omega \}
\]
and \( \mathbb{E}(\Omega) \) be the completion of \( \mathcal{E}(\Omega) \) in \([L^2(\Omega)]^3\). Now we state the first main result of the paper.

Theorem 2.3. \( f \in [L^2_\Omega(\mathbb{R}^3)]^3 \) is a non-radiating volume source if and only if \( f \in \mathbb{E}(\Omega) \), that is,
\[
\iint_{\Omega} f \cdot \mathbf{v} \, dx = 0 \quad \text{for all } \mathbf{v} \in \mathbb{E}(\Omega).
\]

For a general volume source \( f \in [L^2_\Omega(\mathbb{R}^3)]^3 \), let \( f = f_{\parallel} + f_{\perp} \), where \( f_{\perp} \) is the \((L^2(\Omega))^3\)-orthogonal projection of \( f \) onto \( \mathbb{E}(\Omega)^\perp \). Then \( f_{\parallel} \in \mathbb{E}(\Omega) \) is uniquely determined by a single measurement of Cauchy data \((u|_{\partial \Omega}, (\mathbf{C}(x) : \nabla u)|_{\partial \Omega})\) with the explicit formula
\[
\iint_{\Omega} f_{\parallel} \cdot \mathbf{v} \, dx = \iint_{\partial \Omega} (\mathbf{C}(x) : \nabla u) \cdot \mathbf{v} \cdot \nu \, ds(x) + \iint_{\partial \Omega} u \cdot (\mathbf{C}(x) : \nabla \nu_\perp) \nu \, ds(x)
\]
for all \( \mathbf{v} \in \mathcal{E}(\Omega) \), where \( \nu_\perp(x) = \lim_{y \in \Omega, y \rightarrow x} \mathbf{v}(y) \) for all \( x \in \partial \Omega \), and \( \nu \) is the unit outward normal vector on \( \partial \Omega \).

Remark 2.4. For the unique determination of \( f_{\parallel} \), one only needs to measure \( u|_{\partial \Omega} \) since \( u \) in \( \mathbb{R}^3 \setminus \Omega \) is uniquely determined by \( u|_{\partial \Omega} \) and the radiation condition. In other words, \((\mathbf{C}(x) : \nabla u)|_{\partial \Omega}\) is determined by \( u|_{\partial \Omega} \).

Remark 2.5. We want to elaborate (2.2) in a simple case. Suppose that \( \mathbf{C} \) is isotropic and homogeneous with Lamé constants \( \lambda, \mu \) throughout \( \mathbb{R}^3 \). Let
\[
\mathbf{x} = (|\mathbf{x}| \sin \theta \cos \varphi, |\mathbf{x}| \sin \theta \sin \varphi, |\mathbf{x}| \cos \theta)
\]
be the spherical coordinates and \( \mathbf{\hat{x}} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \). We consider the vector spherical harmonics (VSH)
\[
\mathbf{P}^m_n(\mathbf{\hat{x}}) := \mathbf{\hat{x}} Y^m_n(\mathbf{\hat{x}}), \quad \mathbf{C}^m_n(\mathbf{\hat{x}}) := \frac{1}{\sqrt{n(n+1)}} \left( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} - \frac{\varphi}{\sin \theta} \frac{\partial}{\partial \theta} \right) Y^m_n(\mathbf{\hat{x}}),
\]
\[
\mathbf{B}^m_n(\mathbf{\hat{x}}) := \frac{1}{\sqrt{n(n+1)}} \left( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} + \frac{\varphi}{\sin \theta} \frac{\partial}{\partial \theta} \right) Y^m_n(\mathbf{\hat{x}}),
\]
where \( \{ Y^m_n \mid n = 0, 1, 2, \cdots, |m| \leq n \} \) are the standard spherical harmonics, see [DR95, (28)-(30)] or [BEG85] (or see [Han33] for the earliest result).

The set of VSH forms a complete orthogonal basis in \([L^2(S^2)]^3\). In particular, one can write
\[
[L^2(S^2)]^3 = [L^2_r(S^2)]^3 \oplus [L^2_i(S^2)]^3,
\]
where \([L^2_r(S^2)]^3\) is the subspace spanned by \(\{P^m_n\}_{n,m}\) and \([L^2_r(S^2)]^3\) is the subspace spanned by \(\{C^m_n\}_{n,m} \cup \{B^m_n\}_{n,m}\) (see [DR95, Lemma 1]). Using [DR95, (39)-(47)], we can show that if \(v \in \mathcal{E}(\Omega)\), then \(v\) can be expressed by

\[
(2.5) \quad v(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( v^L_{nm} L^m_n(x) + v^M_{nm} M^m_n(x) + v^N_{nm} N^m_n(x) \right),
\]

where

\[
L^m_n(x) = \frac{i}{4\pi i^n} \int_{S^2} e^{ik_s x} P^m_n(\hat{\xi}) ds(\hat{\xi}), \quad k_p = \frac{\omega}{\sqrt{\lambda + 2\mu}},
\]

\[
M^m_n(x) = \frac{\sqrt{n(n+1)}}{4\pi i^n} \int_{S^2} e^{ik_s x} C^m_n(\hat{\xi}) ds(\hat{\xi}), \quad k_s = \frac{\omega}{\sqrt{\mu}},
\]

\[
N^m_n(x) = \frac{i\sqrt{n(n+1)}}{4\pi i^n} \int_{S^2} e^{ik_s x} B^m_n(\hat{\xi}) ds(\hat{\xi}),
\]

which are known as Navier eigenvectors, see [DR95, Lemma 4,5,6]. Here, we also refer to some classical monographs [MF53, Str41] for more details on the Navier eigenvectors. In other words, for \(v \in \mathcal{E}(\Omega)\), we can write

\[
v(x) = v_P(x) + v_C(x) + v_B(x),
\]

where

\[
v_P(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} v^P_{nm} \int_{S^2} e^{ik_p x} P^m_n(\hat{\xi}) ds(\hat{\xi}),
\]

\[
v_C(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} v^C_{nm} \int_{S^2} e^{ik_s x} C^m_n(\hat{\xi}) ds(\hat{\xi}),
\]

\[
v_B(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} v^B_{nm} \int_{S^2} e^{ik_s x} B^m_n(\hat{\xi}) ds(\hat{\xi}).
\]

Here \(v^P_{nm}, v^C_{nm}\), and \(v^B_{nm}\) are given by

\[
\begin{aligned}
 v^P_{nm} &= \frac{i}{4\pi i^n} v^L_{nm}, \\
 v^C_{nm} &= \frac{\sqrt{n(n+1)}}{4\pi i^n} v^M_{nm}, \\
 v^B_{nm} &= \frac{i\sqrt{n(n+1)}}{4\pi i^n} v^N_{nm}.
\end{aligned}
\]

Therefore, we only need to test (2.2) using the following choices:

\[
v^P_{nm}(x) := \int_{S^2} e^{ik_s x} P^m_n(\hat{\xi}) ds(\hat{\xi}),
\]

\[
v^C_{nm}(x) := \int_{S^2} e^{ik_s x} C^m_n(\hat{\xi}) ds(\hat{\xi}),
\]

\[
v^B_{nm}(x) := \int_{S^2} e^{ik_s x} B^m_n(\hat{\xi}) ds(\hat{\xi}).
\]
Consequently, we can derive another characterization of non-radiating volume source, namely, \( f \) is a non-radiating volume source if and only if the following three conditions hold:

\[
\begin{align*}
\left\{ \begin{array}{l}
\iint_{S^2} \hat{f}(k_p \hat{\xi}) \cdot \overline{P_n^m(\hat{\xi})} \, ds(\hat{\xi}) = 0, \\
\iint_{S^2} \hat{f}(k_s \hat{\xi}) \cdot \overline{C_n^m(\hat{\xi})} \, ds(\hat{\xi}) = 0, \\
\iint_{S^2} \hat{f}(k_s \hat{\xi}) \cdot \overline{B_n^m(\hat{\xi})} \, ds(\hat{\xi}) = 0,
\end{array} \right.
\end{align*}
\]

(2.6)

for all \( n = 0, 1, 2, \cdots \) and \( |m| \leq n \), where

\[
\hat{f}(k) = \iint_{\Omega} f(x)e^{-ik \cdot x} \, dx
\]

denotes the Fourier transform of \( f \). In view of (2.4), (2.6) implies that \( \hat{f}(k_p \hat{\xi}) \in [L^2_t(S^2)]^3 \) and \( \hat{f}(k_s \hat{\xi}) \in [L^2_t(S^2)]^3 \). In other words, the vector field \( \hat{f}(k) \) does not have the radial component at \( |k| = k_p \) and has no tangential component at \( |k| = k_s \).

**Remark 2.6.** Now we want to discuss the reconstruction formula (2.3) in the homogeneous isotropic media as in Remark 2.5. Here we further assume \( \Omega = B_1(0) := \{ x \in \mathbb{R}^3 \mid |x| < 1 \} \), and thus \( \partial \Omega = S^2 \). Note that \( f \) is uniquely expressed by (2.5) with suitable coefficients \( \nu_{nm}^L, \nu_{nm}^M, \nu_{nm}^N \). Since \( \nu = \hat{x} \) on \( S^2 \), the tractions on \( \partial \Omega \) can be written by

\[
\begin{align*}
(C : \nabla \nu_{nm}^m) \nu &= ik_p \iint_{S^2} e^{ik_p \hat{\xi} \cdot \hat{x}} \cdot C : (\hat{\xi} \otimes \overline{P_n^m(\hat{\xi})}) \, ds(\hat{\xi}), \\
(C : \nabla \nu_{nm}^c) \nu &= ik_s \iint_{S^2} e^{ik_s \hat{\xi} \cdot \hat{x}} \cdot C : (\hat{\xi} \otimes \overline{C_n^m(\hat{\xi})}) \, ds(\hat{\xi}), \\
(C : \nabla \nu_{nm}^B) \nu &= ik_s \iint_{S^2} e^{ik_s \hat{\xi} \cdot \hat{x}} \cdot C : (\hat{\xi} \otimes \overline{B_n^m(\hat{\xi})}) \, ds(\hat{\xi}).
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
- \int_{\partial \Omega} \left( C : \nabla \nu \right) \nu \cdot v_{nm}^m \, ds(x) + \int_{\partial \Omega} \nu \cdot \left( C : \nabla \nu_{nm}^m \right) \nu \, ds(x)
&= - \iint_{S^2} \iint_{S^2} e^{-ik_p \hat{\xi} \cdot \hat{x}} \left( \nabla u(\hat{x}) : C : (\hat{x} \otimes \overline{P_n^m(\hat{\xi})}) \right) \, ds(\hat{\xi}) \, ds(\hat{x}) \\
&\quad - ik_p \iint_{S^2} \iint_{S^2} e^{-ik_p \hat{\xi} \cdot \hat{x}} \left( \nabla \nu(\hat{x}) : C : (\hat{\xi} \otimes \overline{P_n^m(\hat{\xi})}) \right) \, ds(\hat{\xi}) \, ds(\hat{x}).
\end{align*}
\]

(2.7)

Similar expressions also hold for \( \nu_{nm}^c \) and \( \nu_{nm}^B \). Therefore, (2.3) is equivalent to the following three conditions:

\[
\begin{align*}
\iint_{S^2} \hat{f}(k_p \hat{\xi}) \cdot \overline{P_n^m(\hat{\xi})} \, ds(\hat{\xi}) \\
&= - \iint_{S^2} \iint_{S^2} e^{-ik_p \hat{\xi} \cdot \hat{x}} \left( \nabla u(\hat{x}) : C : (\hat{x} \otimes \overline{P_n^m(\hat{\xi})}) \right) \, ds(\hat{\xi}) \, ds(\hat{x}) \\
&\quad - ik_p \iint_{S^2} \iint_{S^2} e^{-ik_p \hat{\xi} \cdot \hat{x}} \left( \hat{x} \otimes \nabla \nu(\hat{x}) : C : (\hat{\xi} \otimes \overline{P_n^m(\hat{\xi})}) \right) \, ds(\hat{\xi}) \, ds(\hat{x}),
\end{align*}
\]

(2.8)
\[ \int \int_{S^2} \tilde{f}(k_s \hat{\xi}) \cdot \overline{B_n^m(\hat{\xi})} \, ds(\hat{\xi}) \]

\[ = - \int \int_{S^2} \int \int_{S^2} e^{-ik_s \hat{\xi} \cdot \hat{x}} \left( \nabla u(\hat{x}) : C : (\hat{x} \otimes \overline{B_n^m(\hat{\xi})}) \right) \, ds(\hat{x}) \, ds(\hat{\xi}), \]

(2.9)

\[ - ik_s \int \int_{S^2} \int \int_{S^2} e^{-ik_s \hat{\xi} \cdot \hat{x}} \left( \hat{x} \otimes u(\hat{x}) : C : (\hat{\xi} \otimes B_n^m(\hat{\xi})) \right) \, ds(\hat{x}) \, ds(\hat{\xi}), \]

and

\[ \int \int_{S^2} \tilde{f}(k_s \hat{\xi}) \cdot B_n^m(\hat{\xi}) \, ds(\hat{\xi}) \]

\[ = - \int \int_{S^2} \int \int_{S^2} e^{-ik_s \hat{\xi} \cdot \hat{x}} \left( \nabla u(\hat{x}) : C : (\hat{x} \otimes B_n^m(\hat{\xi})) \right) \, ds(\hat{x}) \, ds(\hat{\xi}), \]

(2.10)

\[ - ik_s \int \int_{S^2} \int \int_{S^2} e^{-ik_s \hat{\xi} \cdot \hat{x}} \left( \hat{x} \otimes u(\hat{x}) : C : (\hat{\xi} \otimes B_n^m(\hat{\xi})) \right) \, ds(\hat{x}) \, ds(\hat{\xi}). \]

Formulæ (2.8)-(2.10) uniquely determine \( v_{nm}^L, v_{nm}^M, v_{nm}^N \) in (2.5) and hence \( f \).

Besides the formula (2.2), we can give another explicit characterization of the non-radiating volume source for each frequency.

**Theorem 2.7.** \( f \in [L^2_{\Omega}(\mathbb{R}^3)]^3 \) and \( f \in \mathcal{E}(\Omega)^\perp \) if and only if

\[ f := \nabla \cdot (C(x) : \nabla w) + \omega^2 w \in [L^2(\Omega)]^3 \]

for some

\[ w \in \left\{ w \in [H^1_0(\Omega)]^3 \mid \begin{array}{l} \nabla \cdot (C : \nabla w) \in [L^2(\Omega)]^3 \\ (C : \nabla w) \nu = 0 \text{ on } \partial \Omega \end{array} \right\}. \]

That is, \( f \in [L^2_{\Omega}(\mathbb{R}^3)]^3 \) is non-radiating if and only if (2.11) holds.

**Remark 2.8.** As above, we will elaborate (2.11) for the homogeneous isotropic media. Due to (2.12), let \( w_{\text{ext}} \) be the zero extension of \( w \), then \( w_{\text{ext}} \in [H^1(\mathbb{R}^3)]^3 \) and

\[ \nabla \cdot (C(x) : \nabla w_{\text{ext}}) + \omega^2 w_{\text{ext}} = f \text{ in } \mathbb{R}^3. \]

In this case, we have \( \nabla \cdot (C(x) : \nabla w_{\text{ext}}) \equiv \lambda \cdot \mu w_{\text{ext}} \) throughout \( \mathbb{R}^3 \). Let \( \tilde{w}_{\text{ext}}(\xi) \) and \( \tilde{f}(\xi) \) be the Fourier transforms of \( w_{\text{ext}}(x) \) and \( f(x) \), respectively. Then (2.11) is equivalent to

\[ \tilde{f}(\xi) = - \left( \mu |\xi|^2 + (\lambda + \mu) \xi \otimes \xi \right) \tilde{w}_{\text{ext}}(\xi) + \omega^2 \tilde{w}_{\text{ext}}(\xi) \]

\[ = \left( (\omega^2 - \mu |\xi|^2) \mathbb{I} - (\lambda + \mu) \xi \otimes \xi \right) \tilde{w}_{\text{ext}}(\xi). \]

Now we observe that

\[ \tilde{f}(k_p \hat{\xi}) \cdot \hat{\xi} = \left( (\omega^2 - k_p^2 \mu) \mathbb{I} - (\lambda + \mu) k_p^2 \hat{\xi} \otimes \hat{\xi} \right) \hat{\xi} \cdot \tilde{w}_{\text{ext}}(\xi) \]

\[ = (\omega^2 - k_p^2(\lambda + 2\mu)) (\hat{\xi} \cdot \tilde{w}_{\text{ext}}(\xi)) = 0, \]

which implies \( \tilde{f}(k_p \hat{\xi}) \in [L^2_{\mathbb{I}^2}(S^2)]^3 \). Similarly, we can show that \( \tilde{f}(k_s \hat{\xi}) \in [L^2_{\mathbb{I}^2}(S^2)]^3 \). We hence obtain the same result as in Remark 2.5 for non-radiation volume sources.
2.2. Surface sources. Let $\Sigma$ be a Lipschitz closed surface in $\mathbb{R}^3$, modeling a buried fault across which discontinuities may arise. That is, displacements or traction may differ from inside and outside of $\Sigma$. Let $\Omega_0$ be an open set such that $\partial \Omega_0 = \Sigma$. Let $\alpha$ represents the jump of displacement, while $\beta$ describes the jump of traction, across the interface $\Sigma$. We remark that, for spontaneous rupture, the traction must be continuous, that is, $\beta \equiv 0$.

**Theorem 2.9.** Let $\omega > 0$. Given any $(\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3$, there exists a unique $u \in [H^1(\Omega_0)]^3 \cap [H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega_0})]^3$ satisfies the following time-harmonic equations of elasticity:

$$
\begin{align*}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u &= 0 \quad \text{in } \mathbb{R}^3 \setminus \Sigma, \\
u \text{ satisfies the Kupradze radiation condition at } |x| \to \infty, \\
[u]_\Sigma &= \alpha \\
[(C(x) : \nabla u) \nu]_\Sigma &= \beta
\end{align*}
$$

(2.13)

Here, $\nu$ denotes the unit outer normal on $\Sigma$ and

$$
\begin{align*}
[u]_\Sigma &= u_- - u_+ \\
[(C(x) : \nabla u) \nu]_\Sigma &= (C(x) : \nabla u_-) \nu - (C(x) : \nabla u_+) \nu
\end{align*}
$$

on $\Sigma$.

where

$$
\begin{align*}
u(x) &= \lim_{h \to 0^+} u(x \pm h \nu(x)) \\
(C(x) : \nabla u_{\pm}(x)) \nu(x) &= \lim_{h \to 0^+} \left( (C(x \pm h \nu(x)) : \nabla u(x \pm h \nu(x))) \right) \nu(x)
\end{align*}
$$

on $\Sigma$.

**Definition 2.10.** The pair $(\alpha, \beta)$ is called a non-radiating surface source, if there exists a number $R > 0$ such that $u(x) = 0$ for all $|x| > R$.

As in the case of volume sources, we can give a variational characterization of a non-radiating surface source. Let $\Omega = B_R(0)$ and $\overline{\Omega_0} \subset \Omega$.

**Theorem 2.11.** $(\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3$ is a non-radiating source if and only if

$$
\int_{\Sigma} \alpha \cdot (C(x) : \nabla \nu) \nu \ ds(x) = \int_{\Sigma} \beta \cdot \nu \ ds(x) \quad \text{for all } \nu \in \mathcal{E}(\Omega).
$$

(2.14)

Indeed, for the general surface source $(\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3$, the following relation holds:

$$
\int_{\Sigma} \left( \alpha \cdot (C(x) : \nabla \nu) \nu - \beta \cdot \nu \right) \ ds(x) = \int_{\partial \Omega} \left( u \cdot (C(x) : \nabla \nu) \nu - (C(x) : \nabla u) \nu \cdot \nu \right) \ ds(x)
$$

(2.15)

for all $\nu \in \mathcal{E}(\Omega)$.

Interestingly, under the assumption that the elasticity system poses the unique continuation property (UCP), a non-radiating surface source can be characterized by a more explicit formula.
**Theorem 2.12.** If \((\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3\) is a Cauchy data of \(\nabla \cdot (C(x) : \nabla u) + \omega^2 u = 0\) in \(\Omega_0\), that is, there exists a \(u \in [H^1(\Omega_0)]^3\) be such that
\[
\begin{aligned}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u = 0 \quad \text{in} \quad \Omega_0,
\end{aligned}
\]
then \((\alpha, \beta)\) must be a non-radiating.

Conversely, if we additionally assume that
\[
\nabla \cdot (C(x) : \nabla w) + \omega^2 w = 0 \quad \text{in} \quad R^3 \setminus \overline{\Omega_0},
\]
then any non-radiating source \((\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3\) is a Cauchy data of \(\nabla \cdot (C(x) : \nabla u) + \omega^2 u = 0\) in \(\Omega_0\).

**Remark 2.13.** (Explicit characterization of radiating surface sources) Under the assumption of the UCP (2.17), we can characterize a radiating surface source using Calderón’s projectors. Indeed, any \((\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3\) can be uniquely decomposed into
\[
(\alpha, \beta) = (\alpha_\perp, \beta_\perp) + (\alpha_\parallel, \beta_\parallel),
\]
with
\[
(\alpha_\perp, \beta_\perp) := C_{\text{int}}(\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3,
\]
\[
(\alpha_\parallel, \beta_\parallel) := C_{\text{ext}}(\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3,
\]
where \(C_{\text{int}}\) and \(C_{\text{ext}}\) are Calderón’s projectors introduced in (5.3). From Theorem 2.12, it is clear that \((\alpha_\perp, \beta_\perp)\) is non-radiating. The pair \((\alpha_\parallel, \beta_\parallel)\) is the radiating part satisfying that there exists \(u_{\text{ext}} \in [H^1_{\text{loc}}(R^3 \setminus \overline{\Omega_0})]^3\) such that
\[
\begin{aligned}
\nabla \cdot (C(x) : \nabla u_{\text{ext}}) + \omega^2 u_{\text{ext}} = 0 & \quad \text{in} \quad R^3 \setminus \overline{\Omega_0},
\end{aligned}
\]
\[
\begin{aligned}
u_{\text{ext}} \text{ satisfies Kupradze radiation condition at } |x| \to \infty, \\
u_{\text{ext}} = -\alpha_\parallel & \quad \text{on} \quad \Sigma, \\
(C(x) : \nabla u_{\text{ext}})\nu = -\beta_\parallel & .
\end{aligned}
\]
Therefore, \((\alpha_\parallel, \beta_\parallel)\) can be uniquely determined by the measurements \((u|_{\partial \Omega}, (C(x) : \nabla u)\nu|_{\partial \Omega}) = (u_{\text{ext}}|_{\partial \Omega}, (C(x) : \nabla u_{\text{ext}})\nu|_{\partial \Omega})\), via formula (2.15), that is,
\[
\begin{aligned}
\int_{\Sigma} \left( \alpha_\parallel(x) \cdot (C(x) : \nabla u(x))\nu - \beta_\parallel(x) \cdot v(x) \right) ds(x) \\
= \int_{\partial \Omega} \left( u(x) \cdot (C(x) : \nabla u(x))\nu - ((C(x) : \nabla u(x))\nu) \cdot v(x) \right) ds(x)
\end{aligned}
\]
for all \(v \in E(\Omega)\). Here we again remark that \((C(x) : \nabla u)\nu|_{\partial \Omega}\) is uniquely determined by \(u|_{\partial \Omega}\).

**Remark 2.14.** Under some regularity assumptions, the UCP for solutions of isotropic elasticity system (Lamé system) is known, see [DLW20a, LNUW11, LW15]. The UCP for Lamé eigen-functions also hold, see [DLW20b]. However, the UCP for general elasticity system remains open. It is worth-mentioning that the UCP does not holds for general elliptic systems, see [KW16] for counterexamples.
2.3. **Some related results.** The investigation of radiating and non-radiating sources for the acoustic and electromagnetic waves has a long history. We refer the reader to Devaney and Wolf’s work [DW73] for the early development. Later generalizations including the inverse source problem can be found in [AM06, BC77, Dev04]. Due to the non-uniqueness of the inverse source problem using only one frequency, there are growing interests in the study of the inverse source problem by the measurements at multi-frequency. It is known that the inverse source problem is ill-posed. However, in recent studies, we observe that the stability improves as we increase the frequency. Results for such increasing stability phenomena in the inverse source problems for the acoustic, electromagnetic, and elastic waves can be found in [ABF02, BLT10, BHKY18, BLZ20, CIL16, EI18, EI18a, IW20]. Nonetheless, to our best knowledge, the detailed characterization of radiating and non-radiating elastic volume and surface sources has not been studied before.

2.4. **Organization of this paper.** In addition to Section 1 and 2, the paper is organized as follows. In Section 3, we give a precise definition of Kupradze radiation condition. In Section 4, we introduce the exterior Dirichlet-to-Neumann map related to the scattering problem. Before proving several characterization results about the volume and surface sources, we first prove Theorem 2.9 in Section 5. Theorem 2.9 will be useful in our proofs. Then, characterizations of volume and surface sources are proved in detail in Section 6 and Section 7, respectively. Finally, we present some interesting observations in Appendix A and Appendix B.

3. **Kupradze radiation condition**

Before we explain the Kupradze radiation condition, we first recall the following well-known fact, which can be found in [KGBB79, Theorem III.2.2.5 (p.123)].

**Lemma 3.1.** Given any open set $D \subset \mathbb{R}^3$ with smooth boundary, if $u$ is a (smooth) solution to

\[(\mathcal{L}^{\lambda,\mu} + \omega^2)u = 0 \quad \text{in} \quad D,\]

then $u$ can be decomposed into compression and shear components, that is, it can be represented as the sum of vectors

\[u(x) = u^{(p)}(x) + u^{(s)}(x),\]

where $u^{(p)}, u^{(s)}$ satisfy Helmholtz equations

\[
\begin{cases}
(\Delta + k_p^2)u^{(p)} = 0, & \text{curl } u^{(p)} = 0 \\
(\Delta + k_s^2)u^{(s)} = 0, & \text{div } u^{(s)} = 0
\end{cases} \quad \text{in} \quad D,
\]

where $k_p^2 = \omega^2/(\lambda + 2\mu)$ and $k_s^2 = \omega^2/\mu$.

The following Definition can be found in [KGBB79, Definition III.2.2.6 (p.124)].

**Definition 3.2.** Suppose that $u$ is a solution to (3.1) with $D = \mathbb{R}^3 \setminus B_r(0)$ for some $r > 0$. Let $u^{(p)}$ and $u^{(s)}$ be given in Lemma 3.1. We say that $u$ satisfies the Kupradze radiation
Given any \( \lambda \in [H^\\frac{1}{2}(\partial \Omega)]^3 \), there exists a unique \( v \in [H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})]^3 \) such that
\[
\begin{align*}
\nabla \cdot (C(x) : \nabla v) + \omega^2 v &= \mathcal{L}^{\lambda,\mu} v + \omega^2 v = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \\
v_+ &= \lambda \\
v \text{ satisfies the Kupradze radiation condition at } |x| \to \infty.
\end{align*}
\]

Therefore, we can define the exterior Dirichlet-to-Neumann map \( \Lambda^\text{ext}_{\text{DN}} : [H^{\frac{1}{2}}(\partial \Omega)]^3 \to [H^{-\frac{1}{2}}(\partial \Omega)]^3 \) by
\[
\Lambda^\text{ext}_{\text{DN}}(\lambda) := (C(x) : \nabla v_+)\nu,
\]

where \( \nu \) is the outer unit normal on \( \partial \Omega \).

**Lemma 4.2.** \( \Lambda^\text{ext}_{\text{DN}} : [H^{\frac{1}{2}}(\partial \Omega)]^3 \to [H^{-\frac{1}{2}}(\partial \Omega)]^3 \) is self-adjoint.

**Proof.** Given any \( \eta \in [H^{\frac{1}{2}}(\partial \Omega)]^3 \) and let \( w \in [H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})]^3 \) solve
\[
\begin{align*}
\nabla \cdot (C(x) : \nabla w) + \omega^2 w &= \mathcal{L}^{\lambda,\mu} w + \omega^2 w = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \\
w_+ &= \eta \\
w \text{ satisfies the Kupradze radiation condition at } |x| \to \infty.
\end{align*}
\]
The existence and uniqueness of \( w \) follows from Lemma 4.1. Let \( \lambda, \eta \in [H^{\frac{1}{2}}(\partial \Omega)]^3 \) and \( v \) be given in Lemma 4.1, then we have
\[
\begin{align*}
\int_{\partial \Omega} \lambda \cdot (\Lambda^\text{ext}_{\text{DN}})\eta ds(x) &= \int_{\partial \Omega} \Lambda^\text{ext}_{\text{DN}}(\lambda) \cdot \eta ds(x) = \int_{\partial \Omega} (C : \nabla v_+)\nu \cdot w_+ ds(x) \\
&= -\int_{\partial (\mathbb{R}^3 \setminus \overline{\Omega})} (C : \nabla v)\nu \cdot w ds(x) \\
&= -\int_{\partial (\mathbb{R}^3 \setminus \overline{\Omega})} \nabla \cdot (C : \nabla v) \cdot w dx - \int_{\partial (\mathbb{R}^3 \setminus \overline{\Omega})} \nabla v : C : \nabla w dx \\
&= \omega^2 \int_{\mathbb{R}^3 \setminus \overline{\Omega}} v \cdot w dx - \int_{\partial (\mathbb{R}^3 \setminus \overline{\Omega})} v \cdot (C : \nabla w)\nu ds(x) + \int_{\mathbb{R}^3 \setminus \overline{\Omega}} v \cdot \nabla (C : \nabla w) dx \\
&= \int_{\partial \Omega} \lambda \cdot (C : \nabla w_+)\nu ds(x),
\end{align*}
\]
By the arbitrariness of \( \lambda, \eta \in [H^{\frac{1}{2}}(\partial \Omega)]^3 \), we obtain our desired lemma.
5. The well-posedness of the transmission problems for the system of inhomogeneous anisotropic elasticity

Now, we want to prove Theorem 2.9 by modifying the ideas in [CS90]. Let \( G(x, y) \), for \( x \neq y \), be the Green’s \( 3 \times 3 \) dyadic [DHM18], which satisfies

\[
\nabla_x \cdot (C(x) : \nabla_x G(x, y)) + \omega^2 G(x, y) = -\delta(x - y) \quad \text{in } \mathbb{R}^3,
\]

where \( \delta \) is the Dirac function.

**Uniqueness.** The uniqueness result can be proved by using the Somigliana representation formula (see [CS90, (2.7)]) and follows the ideas in [CS90, Lemma 2.2]. For brevity, we omit the detail here.

**Existence.** We will focus on the proof of the existence. We define the single layer and double layer potentials [McL00, (6.16),(6.17)] by

\[
\begin{align*}
(Sf)(x) &:= \int_{\Sigma} G(x, y) f(y) \, ds(y) \\
(Df)(x) &:= \int_{\Sigma} (B_{\nu, y} G(x, y)^*) f(y) \, ds(y)
\end{align*}
\]

for \( x \in \mathbb{R}^3 \setminus \Sigma \),

where the superscript * denotes the conjugate transpose and the traction operator \( B_{\nu, x} \) is given by

\[
B_{\nu, x} u(x) := (C(x) : \nabla u) \nu \quad \text{for } x \in \Sigma.
\]

The following lemma is a \( \mathbb{R}^3 \) special case of [McL00, Theorem 6.11].

**Lemma 5.1.** Let \( \gamma^+ \) and \( \gamma^- \) be exterior and interior trace operator, respectively. The following linear operators are bounded and satisfy the following jump relations:

\[
\begin{align*}
\gamma^{\text{SL}} : [H^{-1/2}(\Sigma)]^3 &\rightarrow [H^{1/2}(\Sigma)]^3, \quad [SL \psi]_\Sigma = 0 \quad \text{for } \psi \in [H^{-1/2}(\Sigma)]^3, \\
\gamma^{\pm} \text{DL} : [H^{1/2}(\Sigma)]^3 &\rightarrow [H^{1/2}(\Sigma)]^3, \quad [DL \phi]_\Sigma = -\phi \quad \text{for } \phi \in [H^{1/2}(\Sigma)]^3, \\
B_{\nu}^{\pm} \text{SL} : [H^{-1/2}(\Sigma)]^3 &\rightarrow [H^{-1/2}(\Sigma)]^3, \quad [B_{\nu} \text{SL} \psi]_\Sigma = \psi \quad \text{for } \psi \in [H^{-1/2}(\Sigma)]^3, \\
B_{\nu} \text{DL} : [H^{1/2}(\Sigma)]^3 &\rightarrow [H^{-1/2}(\Sigma)]^3, \quad [B_{\nu} \text{DL} \phi]_\Sigma = 0 \quad \text{for } \phi \in [H^{1/2}(\Sigma)]^3.
\end{align*}
\]

As in [McL00, p.218-219], we define the following bounded operators:

\[
\begin{align*}
S &:= \gamma^{\text{SL}} \\
T &:= 2\gamma^+ \text{DL} - \text{Id} = 2\gamma^- \text{DL} + \text{Id} \\
T^* &:= 2B_{\nu}^+ \text{SL} + \text{Id} = 2B_{\nu}^- \text{SL} - \text{Id} \\
R &:= -B_{\nu} \text{DL}
\end{align*}
\]

which satisfy the relations

\[
S = S^*, \quad R = R^*, \quad SR = \frac{1}{4}(\text{Id} - T^2), \quad ST^* = TS, \quad RT = T^*R, \quad RS = \frac{1}{4}(\text{Id} - (T^*)^2).
\]

Therefore, similar to [McL00, p.243], the interior and exterior Calderón projectors are defined by

\[
\begin{align*}
C_{\text{int}} := \begin{pmatrix} \frac{1}{2}(\text{Id} - T) & S \\ R & \frac{1}{2}(\text{Id} + T^*) \end{pmatrix} \quad \text{and} \quad C_{\text{ext}} := \begin{pmatrix} \frac{1}{2}(\text{Id} + T) & -S \\ -R & \frac{1}{2}(\text{Id} - T^*) \end{pmatrix}
\end{align*}
\]

Clearly, \( C_{\text{int}} + C_{\text{ext}} = \text{Id} \). Relations (5.2) imply that \( C_{\text{int}} C_{\text{ext}} = 0, \ C_{\text{ext}} C_{\text{int}} = 0, \ C_{\text{int}}^2 = C_{\text{int}}, \) and \( C_{\text{ext}}^2 = C_{\text{ext}} \), see also [Ces96, Def. 4.3.4, Def. 4.3.5] for analogue ideas for the Maxwell
equations. From Lemma 5.1, it is clear that
\[ C_{\text{int}} : [H^{1/2}(\Sigma)]^3 \times [H^{1/2}(\Sigma)]^3 \rightarrow [H^{1/2}(\Sigma)]^3 \times [H^{1/2}(\Sigma)]^3, \]
\[ C_{\text{ext}} : [H^{1/2}(\Sigma)]^3 \times [H^{1/2}(\Sigma)]^3 \rightarrow [H^{1/2}(\Sigma)]^3 \times [H^{1/2}(\Sigma)]^3, \]
and both operators are bounded. Indeed, combining (5.1) and (5.3), we can easily compute
\[ C_{\text{int}} \left( \psi_1 \psi_2 \right) = \left( \gamma^- (DL \psi_1 + SL \psi_2) \right) \] and \[ C_{\text{ext}} \left( \psi_1 \psi_2 \right) = \left( \gamma^+ (DL \psi_1 - SL \psi_2) \right). \]
For \( u \in [H^1(\Omega_0)]^3 \cap [H^1(\mathbb{R}^3 \setminus \Omega_0)]^3 \) satisfies (2.13), we write
\[ u = \begin{cases} u_{\text{ext}} & \text{in } \mathbb{R}^3 \setminus \Omega_0, \\ u_{\text{int}} & \text{in } \Omega_0. \end{cases} \]
Recall the Somigliana representation formula, see e.g. [CS90, (2.7)]:
\[ \begin{align*}
\mathbf{u}_{\text{int}} &= -DL \mathbf{u}_- + SL((\mathbb{C} : \nabla \mathbf{u}_-)) \nu \\
\mathbf{u}_{\text{ext}} &= DL \mathbf{u}_+ - SL((\mathbb{C} : \nabla \mathbf{u}_+)) \nu.
\end{align*} \]
Therefore, \( u \in [H^1(\Omega_0)]^3 \cap [H^1_{\text{loc}}(\mathbb{R}^3 \setminus \Omega_0)]^3 \) satisfies (2.13) if and only if
\[ \begin{align*}
\begin{cases}
\mathbf{u}_- & = C_{\text{int}} \left( \mathbf{u}_- \right) \\
(\mathbb{C} : \nabla \mathbf{u}_-) \nu & = C_{\text{int}} \left( (\mathbb{C} : \nabla \mathbf{u}_-) \nu \right), \\
\mathbf{u}_+ & = C_{\text{ext}} \left( \mathbf{u}_+ \right) \\
(\mathbb{C} : \nabla \mathbf{u}_+) \nu & = C_{\text{ext}} \left( (\mathbb{C} : \nabla \mathbf{u}_+) \nu \right), \\
\mathbf{u}_- - \mathbf{u}_+ & = \mathbf{\alpha}, \\
(\mathbb{C} : \nabla \mathbf{u}_-) \nu - (\mathbb{C} : \nabla \mathbf{u}_+) \nu & = \mathbf{\beta}.
\end{cases}
\end{align*} \]
Now we want to eliminate the unknowns \( \mathbf{u}_+ \) and \( (\mathbb{C} : \nabla \mathbf{u}_+) \nu \). Let
\[ A := C_{\text{int}} - C_{\text{ext}} = \left( \begin{array}{cc}
-T & 2S \\
2R & T^*
\end{array} \right), \]
then
\[ \begin{align*}
\begin{bmatrix} \mathbf{\alpha} \\ \mathbf{\beta} \end{bmatrix} &= C_{\text{int}} \left( \begin{bmatrix} \mathbf{u}_- \\ (\mathbb{C} : \nabla \mathbf{u}_-) \nu \end{bmatrix} \right) - C_{\text{ext}} \left( \begin{bmatrix} \mathbf{u}_+ \\ (\mathbb{C} : \nabla \mathbf{u}_+) \nu \end{bmatrix} \right) \\
&= C_{\text{int}} \left( \begin{bmatrix} \mathbf{u}_- \\ (\mathbb{C} : \nabla \mathbf{u}_-) \nu \end{bmatrix} \right) + C_{\text{ext}} \left( \begin{bmatrix} \mathbf{\alpha} \\ \mathbf{\beta} \end{bmatrix} \right) - C_{\text{ext}} \left( \begin{bmatrix} \mathbf{u}_- \\ (\mathbb{C} : \nabla \mathbf{u}_-) \nu \end{bmatrix} \right) \\
&= C_{\text{ext}} \left( \begin{bmatrix} \mathbf{\alpha} \\ \mathbf{\beta} \end{bmatrix} \right) + A \left( \begin{bmatrix} \mathbf{u}_- \\ (\mathbb{C} : \nabla \mathbf{u}_-) \nu \end{bmatrix} \right),
\end{align*} \]
that is,
\[ A \left( \begin{bmatrix} \mathbf{u}_- \\ (\mathbb{C} : \nabla \mathbf{u}_-) \nu \end{bmatrix} \right) = (\text{Id} - C_{\text{ext}}) \left( \begin{bmatrix} \mathbf{\alpha} \\ \mathbf{\beta} \end{bmatrix} \right), \]
which is equivalent to both (2.13) and (5.6), see [CS90, (2.14)]. We define the pairing \( \langle \cdot, \cdot \rangle \) by
\[ \langle \left( \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \left( \begin{bmatrix} \psi_0 \\ \phi_0 \end{bmatrix} \right) \right) \rangle := \iint_{\Sigma} \left( \phi_0 \cdot \overline{\psi} + \phi \cdot \overline{\psi_0} \right) ds. \]
for all \((\psi, \phi)\), \((\psi_0, \phi_0)\) in \([H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3\). In view of (5.1), we can write
\[
A = A_- + A_+ \quad \text{with} \quad A_\pm = \left( \frac{-\gamma \pm DL_{S\alpha}}{R} \right) S \frac{\rho \pm }{B_{\nu}^2 S_{\nu}}
\]
which is exactly the first line of the proof of [CS90, Theorem 2.6]. Therefore, following the arguments in [CS90, Theorem 2.6], we can show that
\[
(5.8) \quad \Re \left( A + \mathcal{T} \right) \left( \begin{pmatrix} \psi \\ \phi \end{pmatrix}, \begin{pmatrix} \psi \\ \phi \end{pmatrix} \right) \geq \kappa \left( \|\psi\|^2_{H^{1/2}(\Sigma)} + \|\phi\|^2_{H^{-1/2}(\Sigma)} \right)
\]
for some positive constant \(\kappa\) and some compact operator \(\mathcal{T} : [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3 \to [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3\). Therefore, \(A\) is of Fredholm of index zero. Since the solution of (2.13) (equivalently, (5.6) or (5.7)) is unique (see [CS90, Lemma 2.2]), by the Fredholm theory, we conclude the existence of the solution, which completes the proof of Theorem 2.9.

6. Characterization of non-radiating volume sources

Testing (2.1) by a function \(v \in [H^1(\Omega)]^3\) satisfying \(\nabla \cdot (C(x) : \nabla v) \in [L^2(\Omega)]^3\) gives
\[
\int \int \int f \cdot \nabla v \, dx = -\int \int \int \nabla \cdot (C(x) : \nabla u) \cdot \nabla v \, dx - \int \int \omega^2 u \cdot \nabla v \, dx
\]
\[
(6.1) \quad = -\int_{\partial \Omega} (C(x) : \nabla u) \nu \cdot \nabla v \, ds(x) + \int \int \nabla u : C(x) : \nabla v \, dx - \int \int \omega^2 u \cdot \nabla v \, dx
\]
\[
= -\int_{\partial \Omega} (C(x) : \nabla u) \nu \cdot \nabla v \, ds(x) + \int \int u \cdot (C(x) : \nabla v) \nu \, ds(x)
\]
\[
- \int \int \int u \cdot (\nabla \cdot (C(x) : \nabla v) + \omega^2 v) \, dx.
\]
Substituting \(v \in \mathcal{E}(\Omega)\) into (6.1), we immediately obtain (2.3), i.e.,

**Lemma 6.1.** For any \(v \in \mathcal{E}(\Omega)\), the following identity holds
\[
(6.2) \quad \int \int \int f \cdot \nabla v \, dx = -\int_{\partial \Omega} (C(x) : \nabla u) \nu \cdot \nabla v \, ds(x) + \int \int u \cdot (C(x) : \nabla v) \nu \, ds(x).
\]

Now we are ready to prove Theorem 2.3.

**Proof of Theorem 2.3.** Suppose that \(f\) is a non-radiating source. Since \(\mathbb{R}^3 \setminus \Omega\) is connected, by unique continuation property for isotropic elasticity system (see Remark 2.14), we have \(u = 0\) in \(\mathbb{R}^3 \setminus \overline{\Omega}\), then
\[
(C(x) : \nabla u) \nu = u = 0 \quad \text{on} \quad \partial \Omega.
\]
From (6.2), we have
\[
\int \int \int f \cdot \nabla v \, dx = 0 \quad \text{for all} \quad v \in \mathcal{E}(\Omega),
\]
which implies
\[
\int \int \int f \cdot \nabla v \, dx = 0 \quad \text{for all} \quad v \in \mathcal{E}(\Omega),
\]
that is, \(f \in \mathcal{E}(\Omega)^\perp\).
Conversely, suppose that \( f \in E(\Omega)^\perp \). It follows from (6.2) that
\[
\iiint_{\Omega} f \cdot \nabla \, dx = 0 \quad \text{for all } v \in E(\Omega),
\]
and hence
\[
\iiint_{\partial \Omega} (\mathcal{C}(x) : \nabla u) \nu \cdot \nu \, ds(x) = \iiint_{\partial \Omega} u \cdot (\mathcal{C}(x) : \nabla \nu) \nu \, ds(x) \quad \text{for all } v \in E(\Omega).
\]

Since the inhomogeneity of \( \mathcal{C} \) and \( \text{supp}(f) \) are in \( \Omega \), we can reformulate (2.1) in the following form:
\[
\begin{cases}
\nabla \cdot (\mathcal{C}(x) : \nabla u) + \omega^2 u = -f & \text{in } \Omega, \\
u \left. = u_+ \right|_{\partial \Omega} & \text{on } \partial \Omega.
\end{cases}
\]

By Theorem 2.9, we can choose \( v \in [H^1(\Omega)]^3 \cap [H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})]^3 \) be such that
\[
\begin{cases}
\nabla \cdot (\mathcal{C}(x) : \nabla v) + \omega^2 v = 0 & \text{in } \mathbb{R}^3 \setminus \partial \Omega, \\
v \text{ satisfies the Kupradze radiation condition at } |x| \to \infty, \\
v \left. \right|_{\partial \Omega} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Choose \( g = v_+ \mid_{\partial \Omega} \in [H^{1/2}(\partial \Omega)] \). Using Lemma 4.2 and (6.3), we have
\[
\iiint_{\partial \Omega} u \cdot \Lambda^{\text{ext}}_{\text{DN}}(g) \, ds(x) = \iiint_{\partial \Omega} \Lambda^{\text{ext}}_{\text{DN}}(u \mid_{\partial \Omega}) \cdot \mathbf{g} \, ds(x)
\]
\[
= \iiint_{\partial \Omega} (\mathcal{C}(x) : \nabla u) \nu \cdot \mathbf{g} \, ds(x)
\]
\[
= \iiint_{\partial \Omega} u \cdot (\mathcal{C}(x) : \nabla \nu) \nu \, ds(x).
\]

Hence,
\[
\| u \|_{L^2(\partial \Omega)}^2 = \iiint_{\partial \Omega} u \cdot \mathbf{n} \, ds(x) = \iiint_{\partial \Omega} u \cdot [\mathcal{C}(x) : \nabla \nu] \nu \, ds(x) = 0.
\]

Therefore, we conclude that \( u \equiv 0 \) on \( \partial \Omega \). Since
\[
\begin{cases}
\mathcal{L}^\lambda u + \omega^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\
u \text{ satisfies Kupradze radiation condition at } |x| \to \infty,
\end{cases}
\]
by the uniqueness result in Lemma 4.1, we conclude that \( u = 0 \) in \( \mathbb{R}^3 \setminus \overline{\Omega} \), which implies that \( f \) is a non-radiating volume source. \( \square \)

Proof of Theorem 2.7. Clearly, if \( f \) is given in (2.11), then by Theorem 2.3, such \( f \) is non-radiating.

Conversely, given any \( f \in [L^2(\Omega)]^3 \), using the Fredholm alternative, there exists a countable set \( \text{Spec}_{\text{Dir}, \Omega}(\mathcal{C}) \) (set of Dirichlet spectra in \( \Omega \)), where \( \mathcal{L}^\omega u = \nabla \cdot (\mathcal{C}(x) : \nabla u) \), such that the following holds:
\[
\omega^2 \not\in \text{Spec}_{\text{Dir}, \Omega}(\mathcal{C})
\]
if and only if there exists a unique \( w \in [H^1_0(\Omega)]^3 \) such that
\[
\nabla \cdot (\mathcal{C}(x) : \nabla w) + \omega^2 w = f \quad \text{in } \Omega.
\]
Since \( f \in [L^2(\Omega)]^3 \), we know that \( \nabla \cdot (C : \nabla w) \in [L^2(\Omega)]^3 \). If \( f \) is non-radiating, plugging \( f \) into (2.2), we can see that \( w \) belongs to the space given in (2.12). Now we consider the case when \( \omega \in \text{Spec}_{\text{Dir}, \Omega}(L^2) \). Since \( f \) is non-radiating, (2.2) implies that \( f \) is orthogonal to the eigenfunction corresponding to the eigenvalue \( \omega^2 \). Therefore, there exists \( w \in [H_0^1(\Omega)]^3 \) (but not unique) such that (6.5) holds. The proof of Theorem 2.7 is completed. □

7. Characterization of non-radiating surface sources

Similarly, we can reformulate (2.13) in the following form:

\[
\begin{align*}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
\Lambda^\text{ext}_{\text{DN}}(u_+|_{\partial \Omega}) &= (C(x) : \nabla u)\nu \\
u_+ &= u \\
[u]_\Sigma &= \alpha \\
[(C(x) : \nabla u)\nu]_\Sigma &= \beta
\end{align*}
\]

(7.1)

Testing (7.1) by a function \( v \in [H^1(\Omega)]^3 \) satisfying \( \nabla \cdot (C(x) : \nabla v) \in [L^2(\Omega)]^3 \), we have

\[
0 = -\iiint_{\Omega} \nabla \cdot (C(x) : \nabla u) \cdot v \, dx - \iiint_{\Omega} \omega^2 u \cdot v \, dx
= -\iiint_{\Omega \setminus \overline{\Omega}_0} \nabla \cdot (C(x) : \nabla u) \cdot v \, dx - \iiint_{\Omega_0} \nabla \cdot (C(x) : \nabla u) \cdot v \, dx - \iiint_{\Omega} \omega^2 u \cdot v \, dx
= -\iiint_{\partial(\Omega \setminus \overline{\Omega}_0)} (C(x) : \nabla u)\nu \cdot v \, ds(x) - \iiint_{\partial\Omega_0} (C(x) : \nabla u)\nu \cdot v \, ds(x)
\]

\[
+ \iiint_{\Omega \setminus \overline{\Omega}_0} \nabla u : C : \nabla v \, dx + \iiint_{\Omega_0} \nabla u : C : \nabla v \, dx - \iiint_{\Omega} \omega^2 u \cdot v \, dx
= -\iiint_{\partial\Omega} (C(x) : \nabla u)\nu \cdot v \, ds(x) - \iiint_{\Sigma} [(C(x) : \nabla u)\nu]_\Sigma \cdot v \, ds(x)
\]

\[
+ \iiint_{\partial(\Omega \setminus \overline{\Omega}_0)} u : (C : \nabla v)\nu \, ds(x) + \iiint_{\partial\Omega_0} u : (C : \nabla v)\nu \, ds(x)
\]

\[
- \iiint_{\Omega} u \cdot (\nabla \cdot (C(x) : \nabla v) + \omega^2 v) \, dx
= -\iiint_{\partial\Omega} (C(x) : \nabla u)\nu \cdot v \, ds(x) + \iiint_{\partial\Omega_0} u \cdot (C(x) : \nabla v)\nu \, ds(x)
\]

\[
- \iiint_{\partial(\Omega \setminus \overline{\Omega}_0)} \underbrace{[C(x) : \nabla u]\nu}_\Sigma \cdot v \, ds(x) + \iiint_{\partial\Omega_0} \underbrace{u}_\Sigma \cdot (C(x) : \nabla v)\nu \, ds(x)
\]

\[
- \iiint_{\Omega} u \cdot (\nabla \cdot (C(x) : \nabla v) + \omega^2 v) \, dx.
\]

Consequently, we obtain the following lemma, which gives a link between the surface source \((\alpha, \beta)\) and the Cauchy data \((u|_{\partial\Omega}, (C(x) : \nabla u)\nu|_{\partial\Omega})\).
Lemma 7.1. For \( \mathbf{v} \in E(\Omega) \), (2.15) holds, i.e.,
\[
\int_{\Sigma} \left( \alpha \cdot (C(x) : \nabla \mathbf{v}) - \beta \cdot \mathbf{v} \right) ds(x)
\]
(7.2)
\[
= \int_{\partial \Omega} \left( \mathbf{u} \cdot (C(x) : \nabla \mathbf{v}) \nu - (C(x) : \nabla \mathbf{u}) \nu \right) ds(x).
\]

Now we are ready to prove Theorem 2.11.

Proof of Theorem 2.11. Let \((\alpha, \beta)\) be a non-radiating source. Since \(\mathbb{R}^3 \setminus \overline{\Omega}\) is connected, by the UCP for isotropic elasticity system (see Remark 2.14), we have that
\[
\mathbf{u} = (C(x) : \nabla \mathbf{v}) \nu = 0 \text{ on } \partial \Omega.
\]

Therefore, from (7.2), it yields (2.14).

Conversely, assume that (2.14) holds. Formula (7.2) implies
\[
\int_{\partial \Omega} (C \cdot \nabla \mathbf{u}) \nu \cdot {{\mathbf{v}}_0} ds(x) = \int_{\partial \Omega} \mathbf{u} \cdot (C(x) : \nabla \mathbf{v}) \nu ds(x) \quad \text{for all } \mathbf{v} \in E(\Omega),
\]
which is exactly (6.3). Therefore, following exactly the same argument after (6.3), we obtain Theorem 2.11. \(\square\)

Now we can prove Theorem 2.12.

Proof of Theorem 2.12. Assume that \((\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3\) is a Cauchy data of \(\nabla \cdot (C(x) : \nabla \mathbf{u}) + \omega^2 \mathbf{u} = 0\) in \(\Omega_0\), that is, there exists a \(\mathbf{u} \in [H^1(\Omega_0)]^3\) be such that (2.16) holds. Given any \(\mathbf{v} \in E(\Omega)\), we can compute
\[
- \omega^2 \int \int \int_{\Omega_0} \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)} dx
= \int \int \int_{\Omega_0} \mathbf{u}(x) \cdot (\nabla \cdot (C(x) : \nabla \mathbf{u}(x)) \overline{\mathbf{v}(x)}) dx
= \int \int_{\Sigma} \mathbf{u}(x) \cdot ((C(x) : \nabla \mathbf{u}(x)) \nu) ds(x) - \int \int \int_{\Omega_0} \nabla \mathbf{u}(x) : C(x) : \nabla \mathbf{v}(x) dx
= \int \int_{\Sigma} \mathbf{u}(x) \cdot ((C(x) : \nabla \mathbf{u}(x)) \nu) ds(x) - \int \int \int_{\Sigma} ((C(x) : \nabla \mathbf{u}(x)) \nu) \cdot \overline{\mathbf{v}(x)} ds(x)
+ \int \int \int_{\Omega_0} (\nabla \cdot (C(x) : \nabla \mathbf{u}(x))) \cdot \overline{\mathbf{v}(x)} dx.
\]

By (2.16), we see that
\[
0 = - \left( \int \int \int_{\Omega_0} (\nabla \cdot (C(x) : \nabla \mathbf{u}(x)) + \omega^2 \mathbf{u}(x)) \cdot \overline{\mathbf{v}(x)} dx \right)
= \int \int_{\Sigma} \alpha(x) \cdot ((C(x) : \nabla \mathbf{u}(x)) \nu) ds(x) - \int \int_{\Sigma} \beta(x) \cdot \mathbf{v}(x) ds(x),
\]
which is nothing but (2.14). It follows from Theorem 2.11 that \((\alpha, \beta)\) is a non-radiating surface source.

Now we prove the converse, with additional UCP assumption (2.17). Assume that \((\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3\) is a non-radiating surface source. By the well-posedness assumption,
there exists a unique $\tilde{u} \in [H^1(\Omega_0)]^3 \cap [H^1_{loc}(R^3 \setminus \overline{\Omega_0})]^3$ such that

\[
\begin{cases}
\nabla \cdot (C(x) : \nabla \tilde{u}(x)) + \omega^2 \tilde{u}(x) = 0 & \text{in } R^3 \setminus \Sigma, \\
\tilde{u} \text{ satisfies the Kupradze radiation condition at } |x| \to \infty, \\
\tilde{u}_- - \tilde{u}_+ = \alpha & \text{on } \Sigma, \\
(C(x) : \nabla \tilde{u}_-) \nu - (C(x) : \nabla \tilde{u}_+) \nu = \beta & \text{on } \Sigma.
\end{cases}
\]

(7.3)

Since $(\alpha, \beta)$ is non-radiating, by (2.17), then $\tilde{u}(x) = 0$ for all $x \in R^3 \setminus \overline{\Omega_0}$, and thus $\tilde{u}_+ = (C(x) : \nabla \tilde{u}_+) \nu = 0$ on $\Sigma$.

Therefore, $\tilde{u} \in [H^1(\Omega_0)]^3$ of (7.3) satisfies

\[
\begin{cases}
\nabla \cdot (C(x) : \nabla \tilde{u}(x)) + \omega^2 \tilde{u}(x) = 0 & \text{in } \Omega, \\
\tilde{u}_- = \alpha & \text{on } \partial \Omega, \\
(C(x) : \nabla \tilde{u}_-) \nu - (C(x) : \nabla \tilde{u}_+) \nu = \beta & \text{on } \Sigma,
\end{cases}
\]

which is exactly (2.16). The proof is completed. \qed

**Appendix A. Characterization in terms of the interior transmission problem**

In this appendix, we want to present another characterization of non-radiating sources. The characterization is related to the interior transmission problem (ITP).

**A.1. Volume sources.** We can prove that $f \in [L^2(R^3)]^3$ is a non-radiating source if and only if there exists a pair $(u_1, u_2) \in [H^1(\Omega)]^3 \times [H^1(\Omega)]^3$ such that

\[
\begin{cases}
\nabla \cdot (C(x) : \nabla u_1) + \omega^2 u_1 = f & \text{in } \Omega, \\
\nabla \cdot (C(x) : \nabla u_2) + \omega^2 u_2 = 0 & \text{in } \Omega, \\
u_1 = u_2 & \text{on } \partial \Omega, \\
(C(x) : \nabla u_1) \nu = (C(x) : \nabla u_2) \nu & \text{on } \partial \Omega.
\end{cases}
\]

(A.1)

The system (A.1) is an ITP.

By definition of a non-radiating source, together with the unique continuation property, we obtain that $f$ is non-radiating if and only if there exists $u \in [H^1(\Omega)]^3$ such that

\[
\begin{cases}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
(C(x) : \nabla u) \nu = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(A.2)

Choosing $u_1 = u$ and $u_2 = 0$, it is obvious that $(u_1, u_2)$ solves (A.1). Conversely, suppose that there exists a pair $(u_1, u_2) \in [H^1(\Omega)]^3 \times [H^1(\Omega)]^3$ such that (A.1) holds. Note that $u = u_1 - u_2 \in [H^1(\Omega)]^3$ satisfies (A.2), and hence $f$ is non-radiating.
A.2. Surface sources. Similarly, we can also characterize non-radiating surface sources by an ITP. As above, assume that the UCP holds for solutions $u$ of $\nabla \cdot (C(x) \nabla u) + \omega^2 u = 0$ in $\Omega$. Then $(\alpha, \beta) \in [H^{1/2}(\Sigma)]^3 \times [H^{-1/2}(\Sigma)]^3$ is a non-radiating surface source if and only if there exists a pair $(u_1, u_2) \in [H^1(\Omega \setminus \Sigma)]^3 \times [H^1(\Omega)]^3$ such that

$$
\begin{align*}
\nabla \cdot (C(x) : \nabla u_1) + \omega^2 u_1 &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
\nabla \cdot (C(x) : \nabla u_2) + \omega^2 u_2 &= 0 \quad \text{in } \Omega, \\
[u_1]_\Sigma &= \alpha \quad \text{on } \Sigma, \\
[(C(x) : \nabla u_1) \nu]_\Sigma &= \beta \quad \text{on } \Sigma, \\
u_1 &= u_2 \\
(C(x) : \nabla u_1) \nu = (C(x) : \nabla u_2) \nu &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

(A.3)

To prove this characterization, as in the previous case, we know that $(\alpha, \beta)$ is non-radiating if and only if there exists $u \in [H^1(\Omega \setminus \Sigma)]^3$ such that

$$
\begin{align*}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
[u]_\Sigma &= \alpha \quad \text{on } \Sigma, \\
[(C(x) : \nabla u) \nu]_\Sigma &= \beta \quad \text{on } \Sigma, \\
\nabla \cdot (C(x) : \nabla u) \nu &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

(A.4)

This is exactly (A.3) with $u_1 = u$ and $u_2 \equiv 0$. Conversely, suppose that there exists a pair $(u_1, u_2) \in [H^1(\Omega \setminus \Sigma)]^3 \times [H^1(\Omega)]^3$ such that (A.3) holds. The function $u = u_1 - u_2 \in [H^1(\Omega \setminus \Sigma)]^3$ will satisfy (A.4), and hence $(\alpha, \beta)$ is non-radiating.

**Appendix B. Relation between surface sources and volume sources**

It is known to the seismologists that a surface source can be reformulated into a singular volume source. In this appendix, we would like to prove this statement rigorously. For simplicity, we consider $(0, \beta)$ with $\beta \in [H^{-1/2}(\Sigma)]^3$ as the surface source. More precisely, let $u \in [H^1(\Omega_0)]^3 \cap [H^1(R^3 \setminus \Omega_0)]^3$ satisfy

$$
\begin{align*}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u &= 0 \quad \text{in } R^3 \setminus \Sigma, \\
u \text{ satisfies Kupradze radiation condition at } |x| \to \infty, \\
[u]_\Sigma &= 0 \quad \text{on } \Sigma, \\
[(C(x) : \nabla u) \nu]_\Sigma &= \beta \quad \text{on } \Sigma,
\end{align*}
$$

equivalently, $u \in [H^1(R^3)]^3$ and satisfies

$$
\begin{align*}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
u \text{ satisfies Kupradze radiation condition at } |x| \to \infty, \\
[(C(x) : \nabla u) \nu]_\Sigma &= \beta \quad \text{on } \Sigma,
\end{align*}
$$

(B.1)

**Lemma B.1.** $u \in [H^1(R^3)]^3$ satisfies (B.1) if and only if

$$
\int \int \int_{R^3} \nabla u : C(x) : \nabla w \ dx - \omega^2 \int \int \int_{R^3} u \cdot w \ dx = \int \int_{\Sigma} \beta \cdot w \ ds(x)
$$

(B.2)

for all $w \in [H^1(R^3)]^3$. 


Proof. Testing \( w \) on (B.1), we have

\[
-\omega^2 \iiint_{\mathbb{R}^3} u \cdot w \, dx = \iiint_{\mathbb{R}^3} \nabla \cdot (\nabla u : \nabla) \cdot w \, dx
\]

\[
= \iiint_{\mathbb{R}^3 \setminus \Omega_0} \nabla \cdot (\nabla u : \nabla) \cdot w \, dx + \iiint_{\Omega_0} \nabla \cdot (\nabla u : \nabla) \cdot w \, dx
\]

\[
= -\iiint_{\Sigma} (\nabla u_+ \cdot \nu) \cdot w \, ds(x) + \iiint_{\Sigma} (\nabla u_- \cdot \nu) \cdot w \, ds(x)
\]

\[
- \iiint_{\mathbb{R}^3 \setminus \Omega_0} \nabla \cdot (\nabla u : \nabla) \cdot w \, dx - \iiint_{\Omega_0} \nabla \cdot (\nabla u : \nabla) \cdot w \, dx
\]

\[
= \iiint_{\Sigma} [(\nabla u) \cdot \nu]_{\Sigma} \cdot w \, ds(x) - \iiint_{\mathbb{R}^3} \nabla \cdot (\nabla u : \nabla) \cdot w \, dx.
\]

Plugging this equality into (B.2), we have

\[
\iiint_{\mathbb{R}^3} (\nabla \cdot (\nabla u : \nabla) + \omega^2 u) \cdot w \, dx
\]

\[
= \iiint_{\Sigma} [(\nabla u) \cdot \nu]_{\Sigma} - \beta \cdot w \, ds(x),
\]

which implies (B.1). □

Since \( \beta \in [H^{-1/2}(\Sigma)]^3 \), in view of trace theorem, we can define \( f \in [H^{-1}(\mathbb{R}^3)]^3 \) by

\[
\langle f, w \rangle := \iiint_{\Sigma} \beta \cdot w \, ds(x) \quad \text{for all } w \in [H^1(\mathbb{R}^3)]^3,
\]

where \( \langle \cdot, \cdot \rangle \) is the \( [H^1(\mathbb{R}^3)]^3 \times [H^{-1}(\mathbb{R}^3)]^3 \) duality pair. It is easy to see that \( \text{supp}(f) \subset \Sigma \). Formally, we can set \( f(x) = \beta(x) \delta_{\Sigma}(x) \). Hence, if we write \( \langle f, w \rangle = \iiint_{\mathbb{R}^3} f \cdot w \, dx \), then (B.2) becomes

\[
\iiint_{\mathbb{R}^3} \nabla u : \nabla \cdot w \, dx - \omega^2 \iiint_{\mathbb{R}^3} u \cdot w \, dx = \iiint_{\mathbb{R}^3} f \cdot w \, dx
\]

(B.3)
for all \( \mathbf{w} \in [H^1(\mathbb{R}^3)]^3 \). In other words, (B.3) the weak formulation of the scattering problem
\[
\begin{cases}
\nabla \cdot (C(x) : \nabla u) + \omega^2 u = -f \\
u \text{ satisfies Kupradze radiation condition at } |x| \to \infty.
\end{cases}
\]
Note that here \( f \in [H^{-1}(\mathbb{R}^3)]^3 \), which is a singular volume source.

ACKNOWLEDGMENTS

Both authors are partially supported by MOST 109-2115-M-002-001-MY3.

REFERENCES


*Department of Mathematics, National Taiwan University, Taipei 106, Taiwan.*  
*Email address: d07221005@ntu.edu.tw*

*Institute of Applied Mathematical Sciences, National Taiwan University, Taipei 106, Taiwan.*  
*Email address: jnwang@math.ntu.edu.tw*