STRICT MONOTONICITY OF EIGENVALUES AND UNIQUE CONTINUATION FOR SPECTRAL FRACTIONAL ELLIPTIC OPERATORS

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Abstract. In this paper we consider the eigenvalue problem for a weighted spectral fractional second order elliptic operator in a bounded domain. We show that any eigenvalue is strictly monotone with respect to the weight function if the corresponding eigenfunction satisfies the unique continuation property from a measurable set of positive Lebesgue measure.

1. Introduction

In this paper we will investigate the relation between the monotonicity of eigenvalues and the unique continuation property for the spectral fractional elliptic operator. To motivate our study, we first briefly state the result for the elliptic operator. Consider the weighted eigenvalue problem in a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$:

$$\begin{cases} Au = \mu m(x) u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where A is a second order elliptic operator given by

$$Au = -\sum_{i,j=1}^{n} \partial_j (a_{ij}(x) \partial_i u) + a_0(x) u$$

with $a_0(x) \geq 0$ and $(a_{ij}(x)) \in L^\infty(\Omega)$ satisfying $a_{ij}(x) = a_{ji}(x)$, the ellipticity condition

$$\Lambda_1 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda_2 |\xi|^2, \quad 0 < \Lambda_1, \Lambda_2. \tag{1.2}$$

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Assume that $a_0, m(x) \in L^r(\Omega)$ for some $r > n/2$. It is known that the eigenvalues of (1.1), depending on m, form a countable sequence:

$$\cdots \leq \mu_{-2}(m) \leq \mu_{-1}(m) < 0 < \mu_1(m) \leq \mu_2(m) \leq \cdots .$$

If m is non-negative (or non-positive), then this sequence is bounded below (or bounded above). In view of the variational characterization of eigenvalues, we can observe that each μ_k, $k \in \mathbb{Z} \setminus \{0\}$, is nonincreasing in the weight function m, i.e., if $m(x) \leq \hat{m}(x)$ a.e., then $\mu_k(\hat{m}) \leq \mu_k(m)$. It was proved in [FG92] that $\mu_k(m)$ is strictly decreasing in m if and only if the corresponding eigenfunction enjoys the unique continuation property from a set of positive measure. We say that $\mu_k(m)$ is strictly monotonically decreasing in m if $m(x) \leq \hat{m}(x)$ a.e. and $\{x : \hat{m}(x) - m(x) > 0\}$ has positive measure, then $\mu_k(\hat{m}) < \mu_k(m)$. For the corresponding eigenfunction $u_k(x)$, we say that $u_k(x)$ has the measurable unique continuation property (MUCP) if $u = 0$ identically in Ω whenever $u_k(x) = 0$ in $E \subset \Omega$ with the Lebesgue measure of E, $|E| > 0$. A similar result was proved for the biharmonic operator Δ^2 in [TCRD12].

The equivalence of strict monotonicity of eigenvalues and MUCP was further extended to nonlocal operators in [FI19] where the authors considered the eigenvalue problem

$$\begin{cases} L_K u = \mu m(x) u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.3}$$

where L_K is a nonlocal operator of the following general form

$$L_K u(x) = \text{p.v.} \int_{\mathbb{R}^n} (u(x) - u(y)) K(x - y)\, dy, \tag{1.4}$$

where the kernel K satisfies

(K1) $\rho(x) K \in L^1(\mathbb{R}^n)$, where $\rho(x) = \min\{|x|^2, 1\}$;
(K2) $K(x) \geq \alpha |x|^{-(n+2s)}$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $s \in (0, 1)$;
(K3) $K(-x) = K(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

In particular, when $K(x) = |x|^{-(n+2s)}$, L_K is known to be the fractional Laplacian and (1.3) is the eigenvalue problem for the regional fractional Laplacian. Since L_K is nonlocal, to define $L_K u(x)$ for $x \in \Omega$, $u = 0$ in $\mathbb{R}^n \setminus \Omega$ serves as the zero Dirichlet condition.

The main theme of this work is to establish the equivalence of strict monotonicity of eigenvalues and MUCP for the spectral elliptic operator. To define the operator, let us denote the second order elliptic operator

$$Lu(x) = -\sum_{i,j=1}^{n} \partial_j(a_{ij}(x)\partial_i u(x)).$$

It is a standard result that there exists a sequence of positive Dirichlet eigenvalues and orthonormal eigenfunctions $\{\lambda_k, \phi_k\}_{k=0}^{\infty}$ with $\phi_k \in H_0^1(\Omega)$ for L in Ω. For $s \in (0,1)$, we define the spectral fractional elliptic operator

$$L^s u(x) := \sum_{k=0}^{\infty} \lambda_k^s u_k \phi_k(x) \quad \text{in} \quad \Omega, \tag{1.5}$$

where $u(x) = \sum_{k=0}^{\infty} u_k\phi_k \in H_0^1(\Omega)$, i.e., $u_k = (u, \phi_k)$. We also consider spectral fractional elliptic operator L^γ with the fractional power $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$ (see the precise definition of L^γ in Section 2). In this work, we will not discuss the classical case where $\gamma \in \mathbb{N}$. Let $\mu_k(m)$ be the eigenvalue of

$$L^\gamma \psi_k(x) = \mu_k(x) m(x) \psi_k(x) \quad \text{in} \quad \Omega$$

with the corresponding eigenfunction $\psi_k(x)$ belonging to a certain function space (see Section 3). We then prove that μ_k is strictly monotonically decreasing in m if and only if ψ_k enjoys MUCP in Ω. We want to point out the spectral elliptic operator L^γ can not be written in the form of L_K in (1.4) with a suitable kernel K. One can easily observe that if such kernel K exists for L^γ, then it cannot satisfy Property (K2) of K given above. In other words, our result here does not follow from that in [FI19].

Even though L^γ is defined in a bounded domain Ω, it is a nonlocal operator. The proof of the uniqueness continuation property is highly nontrivial. However, it was shown in [ST10] that L^s for $s \in (0,1)$ can be expressed as the Dirichlet-to-Neumann map of an extension problem in the spirit of the fractional Laplacian $(-\Delta)^s$ established in [CS07]. The operator in the extension problem is a local, but degenerate, elliptic operator. Combining [ST10] and [Yan13], the spectral fractional elliptic operator L^γ can also be described as the Dirichlet-to-Neumann map of an extension problem. Having established the extension problem for L^γ, we can prove the MUCP using some results from [GR19] involving Carleman estimates.

The paper is organized as follows. In Section 2, we discuss the definition of the spectral fractional elliptic operator L^γ in detailed. We also describe the corresponding extension problem, especially the case of $\gamma > 1$. In Section 3, we state and prove main results of the paper. We will discuss the unique continuation property for L^γ in Section 4.

2. The Fractional Operator L^γ

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. We now give a formal definition of L^γ for $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$. Let $\lfloor\gamma\rfloor$ be the integer part of γ and $s := \gamma - \lfloor\gamma\rfloor$, that is, we write

$$\gamma = \lfloor\gamma\rfloor + s \quad \text{with} \quad \lfloor\gamma\rfloor \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad s \in (0,1).$$

To consider the fractional elliptic operator of higher power, we need to impose a higher regularity on the coefficients, namely,

$$(a_{ij}) \in C^{2\lfloor\gamma\rfloor,1}(\Omega). \tag{2.1}$$

Before giving the precise definition of L^γ, we first discuss some special Sobolev spaces [Gr16]. For $s > 0$, we define the space $H^s(\Omega)$ as the restriction of $H^s(\mathbb{R}^n)$ to Ω. Let us denote

$$\widetilde{H}^s(\Omega) = \begin{cases} u \in H^s(\Omega), \quad 0 < s < 1/2, \\ u \in H^s(\Omega), \;\; u = 0 \;\text{ on }\; \partial\Omega, \quad 1/2 < s < 5/2, \\ u \in H^s(\Omega), \;\; u = Lu = \cdots = L^k u = 0 \;\text{ on }\; \partial\Omega, \quad 2k+1/2 < s < 2k+5/2, \\ u \in H^s(\Omega), \;\; u = Lu = \cdots = L^{k-1} u = 0 \;\text{ on }\; \partial\Omega, \;\; L^k u \in H^{1/2}(\mathbb{R}^n) \;\text{ with }\; \operatorname{supp} u \in \overline{\Omega} \\ \quad s = 2k + 1/2. \end{cases}$$

Now we define L^γ for $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$ as

$$L^\gamma u(x) := \sum_{k=0}^{\infty} \lambda_k^\gamma (u, \phi_k) \phi_k(x) \quad \text{in} \quad \Omega \tag{2.2}$$

for $u \in \operatorname{dom}(L^\gamma) = \widetilde{H}^{2\gamma}(\Omega)$. Thus, we have $L^\gamma : \operatorname{dom}(L^\gamma) \to L^2(\Omega)$. We can see that for $u \in \operatorname{dom}(L^\gamma)$

$$\begin{aligned} L^\gamma u &= \sum_{k=0}^{\infty} \lambda_k^\gamma (u, \phi_k)\phi_k = \sum_{k=0}^{\infty} \lambda_k^{s+\lfloor\gamma\rfloor-1} (u, \lambda_k\phi_k)\phi_k = \sum_{k=0}^{\infty} \lambda_k^{s+\lfloor\gamma\rfloor-1} (Lu, \phi_k)\phi_k \\ &= \sum_{k=0}^{\infty} \lambda_k^{s+\lfloor\gamma\rfloor-2} (Lu, \lambda_k\phi_k)\phi_k = \sum_{k=0}^{\infty} \lambda_k^{s+\lfloor\gamma\rfloor-2} (L^2u, \phi_k)\phi_k = \cdots \\ &= L^s(L^{\lfloor\gamma\rfloor} u). \end{aligned}$$

On the other hand, we also have that for $u \in \operatorname{dom}(L^\gamma)$

$$L^\gamma u = \sum_{k=0}^{\infty} \lambda_k^\gamma (u, \phi_k)\phi_k = \sum_{k=0}^{\infty} \lambda_k^{\lfloor\gamma\rfloor} (u, \lambda_k^s\phi_k)\phi_k = \sum_{k=0}^{\infty} \lambda_k^{\lfloor\gamma\rfloor} (L^s u, \phi_k)\phi_k = L^{\lfloor\gamma\rfloor}(L^s u),$$

which immediately implies

$$L^\gamma u = L^s(L^{\lfloor\gamma\rfloor} u) = L^{\lfloor\gamma\rfloor}(L^s u). \tag{2.3}$$

3. An Eigenvalue Problem: Strict Monotonicity and Unique Continuation

For $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$ we consider the eigenvalue problem

$$\begin{cases} L^\gamma u = \mu m(x) u \quad \text{in} \quad \Omega, \\ u \in \operatorname{dom}(L^\gamma), \end{cases} \tag{3.1}$$

where $m \in L^\infty(\Omega)$. We are interested in the connection between the strict monotonicity of the eigenvalues $\mu(m)$ and the weight function m. The classical case $\gamma = 1$ was studied by de Figueredo and Gossez in [FG92].

We first discuss the existence of discrete eigenvalues of (3.1). Since m is an indefinite weight, we will follow the approach used in [Fig82] (or [FI19]). In view of (2.3), L^γ is a self-adjoint operator in $L^2(\Omega)$ with domain $\operatorname{dom}(L^\gamma)$. The eigenvalue problem (3.1) can be expressed in the variational form:

$$a[u, v] = \mu \int_\Omega muv, \ \text{ for all } v \in \operatorname{dom}(L^\gamma),\ u \in \operatorname{dom}(L^\gamma), \tag{3.2}$$

where $a[\cdot, \cdot] : \operatorname{dom}(L^\gamma) \times \operatorname{dom}(L^\gamma) \to \mathbb{R}$ is the bilinear form (inner product) defined by

$$a[u, v] := \int_\Omega (L^\gamma u(x)) v(x)\, dx.$$

Clearly, $\|u\|_\gamma = a[u, u]^{1/2}$ induces a norm on $\operatorname{dom}(L^\gamma)$. However, $\operatorname{dom}(L^\gamma)$ is, in general, not complete in $\| \cdot \|_\gamma$. Thus, we need to consider a suitable extension of L^γ. Since L^γ is semibounded, L^γ can be extended in the Friedrichs sense to $\mathcal{H}$, the completion of $\operatorname{dom}(L^\gamma)$ in $\| \cdot \|_\gamma$. Observe that

$$a[u, v] = \int_\Omega (L^{\gamma/2} u)(L^{\gamma/2} v)\, dx \tag{3.3}$$

for $u, v \in \operatorname{dom}(L^\gamma)$. Thus, we have

$$\mathcal{H} = \widetilde{H}^\gamma(\Omega).$$

To abuse the notation, we still denote its Friedrichs extension by L^γ. Note that the Friedrichs extension of L^γ remains self-adjoint on $\widetilde{H}^\gamma(\Omega)$.

For fixed $u \in \widetilde{H}^\gamma(\Omega)$, the map $v \mapsto \displaystyle\int_\Omega muv$ is a bounded linear functional in $\widetilde{H}^\gamma(\Omega)$. By the Riesz-Fréchet representation theorem, there exists a unique element in $\widetilde{H}^\gamma(\Omega)$, says Tu, such that

$$a[Tu, v] = \int muv \quad \text{for all } v \in \widetilde{H}^\gamma(\Omega). \tag{3.4}$$

We can see that T is self-adjoint and bounded in $\widetilde{H}^\gamma(\Omega)$. We can further prove that

Lemma 3.1. *The operator* $T : \widetilde{H}^\gamma(\Omega) \to \widetilde{H}^\gamma(\Omega)$ *is compact.*

Proof. Let $\{u_n\}$ be a bounded sequence in $\widetilde{H}^\gamma(\Omega)$. Then there exists a subsequence, still denoted $\{u_n\}$, such that

$$u_n \to u \quad \text{weakly in } \widetilde{H}^\gamma(\Omega).$$

The compact embedding of $\widetilde{H}^\gamma(\Omega) \hookrightarrow L^2(\Omega)$ (see, for example, [DPV12]), implies

$$u_n \to u \quad \text{strongly in } L^2(\Omega).$$

Substituting $u = u_n - u$ and $v = Tu_n - Tu$ in (3.4), we obtain

$$\|Tu_n - Tu\|_\gamma^2 \leq \|m\|_{L^\infty(\Omega)} \|Tu_n - Tu\|_{L^2(\Omega)} \|u_n - u\|_{L^2(\Omega)} \to 0.$$

□

Consequently, T has a set of countably many real eigenpairs $\{\tilde{\lambda}_k, u_k\}$ in which $\tilde{\lambda}_k$ can only accumulate at 0. Therefore, from (3.3), we have that

$$\int_\Omega m u_k v dx = a[Tu_k, v] = \tilde{\lambda}_k a[u_k, v] = \tilde{\lambda}_k \int_\Omega (L^{\gamma/2} u_k)(L^{\gamma/2} v) dx,$$

i.e.,

$$\int_\Omega (L^{\gamma/2} u_k)(L^{\gamma/2} v) dx = \mu_k \int_\Omega m u_k v dx$$

for all $v \in \widetilde{H}^\gamma(\Omega)$, where $\mu_k = 1/\tilde{\lambda}_k$. In other words, the eigenvalue problem (3.1) has a double sequence of eigenvalues

$$\cdots \leq \mu_{-2} \leq \mu_{-1} < 0 < \mu_1 \leq \mu_2 \leq \cdots,$$

with the corresponding eigenfunctions $\{u_k\}$ in the weak sense. However, since $\mu_k m u_k \in L^2(\Omega)$, we have that $L^\gamma u_k \in L^2(\Omega)$, which implies that $u_k \in \widetilde{H}^{2\gamma}(\Omega)(= \mathrm{dom}(L^\gamma))$. In other words, (μ_k, u_k) solves (3.1) in the strong sense.

Repeating the arguments in [Fig82] (or in [FI19]), we can derive the following variational characterization of eigenvalues. For the sake of completeness, we will provide its proof in Appendix A.

Proposition 3.2. *The sequence of eigenvalues*

$$\cdots \leq \mu_{-2} \leq \mu_{-1} < 0 < \mu_1 \leq \mu_2 \leq \cdots,$$

can be characterized by

$$\frac{1}{\mu_n(m)} = \max_{F_n} \inf \left\{ \int_\Omega m u^2 : \|u\|_\gamma = 1, u \in F_n \right\},$$

$$\frac{1}{\mu_{-n}(m)} = \min_{F_n} \sup \left\{ \int_\Omega m u^2 : \|u\|_\gamma = 1, u \in F_n \right\}, \tag{3.5}$$

where F_n varies over all n-dimensional subspaces of $\widetilde{H}^\gamma(\Omega)$. In particular, we have

$$\frac{1}{\mu_k(m)} = \int_\Omega m u_k^2 \quad \textit{with} \quad \|u_k\|_\gamma = 1. \tag{3.6}$$

Following exactly the same argument as in [Fig82], we obtain the following result, which shows some properties of the eigenvalues.

Proposition 3.3. *Let $\Omega_\pm := \{x \in \Omega : m(x) \gtrless 0\}$, then*

(i) $|\Omega_+| = 0 \implies$ *there is no positive μ_n.*
(ii) $|\Omega_-| = 0 \implies$ *there is no negative μ_{-n}.*
(iii) $|\Omega_+| > 0 \implies$ *there is a sequence of positive $\mu_n \to +\infty$.*
(iv) $|\Omega_-| > 0 \implies$ *there is a sequence of negative $\mu_{-n} \to -\infty$.*

Here, $|\cdot|$ denotes the Lebesgue measure of the set.

Proposition 3.4. *Let $m, \hat{m} \in L^\infty(\Omega)$ such that $m(x) \leq \hat{m}(x)$ for $x \in \Omega$. For a given $n \in \mathbb{Z} \setminus \{0\}$, if the eigenvalues $\mu_n(m)$ and $\mu_n(\hat{m})$ exist, then $\mu_n(m) \geq \mu_n(\hat{m})$.*

Proposition 3.5. *$\mu_n(m)$ is a continuous function of m in the norm of $L^\infty(\Omega)$.*

Proposition 3.4 and Proposition 3.5 are immediate consequences of (3.6).

Now we use $\leq\not\equiv$ to denote that the inequality holds a.e. with strict inequality on a set of positive measure. The following results can be easily proved following the ideas in [FG92].

Proposition 3.6. *Let m and $\hat{m}$ be two weights with $m \leq\not\equiv \hat{m}$. For any $j \in \mathbb{N}$, if the eigenfunction associated with $\mu_j(m)$ satisfies the MUCP, then the strict inequality $\mu_j(m) > \mu_j(\hat{m})$ holds.*

Proposition 3.7. *Let m be a given weight. Assume that $\mu(m)$ is an eigenvalue of* (3.1) *and the corresponding eigenfunction $u(m)$ does not* *satisfy the MUCP. Denote* $\mathcal{N} = \{x \in \Omega : u(x) = 0\}$*. Note that* $|\mathcal{N}| > 0$*. Then for any weight* $\hat{m}$ *satisfying*

$$\{x \in \Omega : |\hat{m}(x) - m(x)| > 0\} \subseteq \mathcal{N},$$

we obtain that $u(m)$ *is also an eigenfunction of some eigenvalue* $\mu(\hat{m})$ *of* (3.1) *with weight* $\hat{m}$ *and* $\mu(\hat{m}) = \mu(m)$*.*

Remark 3.8. Observe that $\mu_j(m) = -\mu_{-j}(-m)$, we can obtain an analogue result for negative eigenvalues.

Remark 3.9. Proposition 3.7 implies that for some $j \in \mathbb{N}$, if the eigenfunction corresponding to $\mu_j(m)$ does not satisfy the MUCP, then $\mu_j(m) = \mu_\ell(\hat{m})$ for some $\ell \in \mathbb{N}$.

To make the paper self contained, we present the proofs of Proposition 3.6 and 3.7 here.

Proof of Proposition 3.6. By Proposition 3.2, there exists $F_j \subset \mathrm{dom}(L^\gamma)$ with $\dim(F_j) = j$ such that

$$\frac{1}{\mu_j(m)} = \inf_{u \in F_j, \|u\|_\gamma = 1} \int_\Omega m|u|^2. \tag{3.7}$$

Pick any $u \in F_j$ with $\|u\|_\gamma = 1$.

Case 1. If u achieves the infimum in (3.7), then u is an eigenfunction corresponding to $\mu_j(m)$, and by the MUCP assumption, we have that $u > 0$ a.e. and thus

$$\frac{1}{\mu_j(m)} = \int_\Omega m|u|^2 < \int_\Omega \hat{m}|u|^2.$$

Case 2. If u does not achieve the infimum in (3.7), then we have

$$\frac{1}{\mu_j(m)} < \int_\Omega m|u|^2 \leq \int_\Omega \hat{m}|u|^2.$$

In view of both cases, we conclude that

$$\frac{1}{\mu_j(m)} < \int_\Omega \hat{m}|u|^2, \quad \text{for all } u \in F_j \text{ with } \|u\|_\gamma = 1.$$

Since $\dim(F_j) = j < \infty$, by a compactness argument, we then obtain

$$\frac{1}{\mu_j(m)} < \inf_{u \in F_j, \|u\|_\gamma = 1} \int_\Omega \hat{m}|u|^2 \leq \max_{F_j} \inf_{u \in F_j, \|u\|_\gamma = 1} \int_\Omega \hat{m}|u|^2 = \frac{1}{\mu_j(\hat{m})},$$

which leads to the desired result. ☐

Proof of Proposition 3.7. Let $u \in \mathrm{dom}(L^\gamma)$ be an eigenfunction associated with eigenvalue $\mu(m)$ which vanishes on a set of positive measure $\mathcal{N}$. In other words, we have and

$$L^\gamma u = \mu(m) m u = \mu(m)\hat{m} u \quad \text{in} \quad \Omega,$$

that is, $\mu(m)$ is an eigenvalue of (3.1) with weight $\hat{m}$.

□

4. Remark on Unique Continuation Property

In this section, we would like to discuss the MUCP for the spectral fractional operator L^γ. Following the exactly same ideas in Appendix A2 of [GR19] and [ST10], $L^\gamma u$ with $u \in \mathrm{dom}(L^\gamma)$ can be determined by an extension problem. Firstly, we recall that the heat semigroup of L is defined by

$$e^{-tL}u := \sum_{k=0}^{\infty} e^{-t\lambda_k}(u,\phi_k)\phi_k(x). \tag{4.1}$$

Also, we define the operator

$$L_b := x_{n+1}^{-b}(\partial_{x_{n+1}} x_{n+1}^b \partial_{x_{n+1}} - x_{n+1}^b L)$$

and the iterated operator

$$L_b^j := (L_b)^j \quad \text{for} \;\; j \in \mathbb{N}.$$

Proposition 4.1. *Let $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$ and let $u \in \mathrm{dom}(L^\gamma)$. Then the Caffarelli-Silvestre-type extension of $u(x)$, $\tilde{u}(x, x_{n+1})$, satisfies the system*

(4.2)

$$\begin{aligned}
L_{1-2s}^{\lfloor\gamma\rfloor+1}\tilde{u}(x,x_{n+1}) &= 0 \;\; \textit{in} \;\; \Omega\times(0,\infty),\\
\lim_{x_{n+1}\to 0}\tilde{u}(x,x_{n+1}) &= u(x) \;\; \textit{for all} \;\; x\in\Omega,\\
\lim_{x_{n+1}\to 0} L_{1-2s}^k\tilde{u}(x,x_{n+1}) &= c_{n,\gamma,k}L^k u(x) \;\; \textit{in} \;\; \Omega, \;\; \textit{for all} \;\; k=1,\cdots,\lfloor\gamma\rfloor,\\
L^k\tilde{u}(x,x_{n+1}) &= 0 \;\; \textit{on} \;\; \partial\Omega\times(0,\infty), \;\; \textit{for all} \;\; k=0,\cdots,\lfloor\gamma\rfloor,\\
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}L_{1-2s}^{\lfloor\gamma\rfloor}\tilde{u}(x,x_{n+1}) &= c_{n,\gamma}L^\gamma u(x) \quad \textit{in} \;\; \Omega,\\
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}L_{1-2s}^{k}\tilde{u}(x,x_{n+1}) &= 0 \;\; \textit{in} \;\; \Omega, \;\; \textit{for all} \;\; k=0,\cdots,\lfloor\gamma\rfloor-1.
\end{aligned}$$

In fact, $\tilde{u}(x,x_{n+1})$ can be expressed explicitly by

$$\tilde{u}(x,x_{n+1}) := c_\gamma x_{n+1}^{2\gamma}\int_0^\infty e^{-tL}u(x)e^{-\frac{x_{n+1}^2}{4t}}\frac{dt}{t^{1+\gamma}} \in C^{2(\lfloor\gamma\rfloor+1),1}(\Omega\times(0,\infty)).$$

On the other hand, the extension solution $\tilde{u}(x, x_{n+1})$ *can also be written as*

$$\tilde{u}(x, x_{n+1}) := \tilde{c}_\gamma \int_0^\infty e^{-tL} L^\gamma u(x) e^{-\frac{x_{n+1}^2}{4t}} \frac{dt}{t^{1-\gamma}}.$$

Setting $\tilde{u}_0(x, x_{n+1}) := \tilde{u}(x, x_{n+1})$*, we can rewrite system (4.2) as the following one*

(4.3)

$$\begin{aligned} L_{1-2s}\tilde{u}_{\lfloor\gamma\rfloor} &= 0 \ \ \textit{in} \ \ \Omega\times(0,\infty), \\ L_{1-2s}\tilde{u}_j &= \tilde{u}_{j+1} \ \ \textit{in} \ \ \Omega\times(0,\infty), \ \ \textit{for all} \ \ j=0,\cdots,\lfloor\gamma\rfloor-1, \\ \lim_{x_{n+1}\to 0}\tilde{u}_j(x',x_{n+1}) &= c_{n,\gamma,j}L^j u(x) \ \ \textit{in} \ \ \Omega, \ \ \textit{for all} \ \ j=0,\cdots,\lfloor\gamma\rfloor, \\ \tilde{u}_j &= 0 \ \ \textit{on} \ \ \partial\Omega\times(0,\infty), \ \ \textit{for all}\ j=0,\cdots,\lfloor\gamma\rfloor. \\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}_{\lfloor\gamma\rfloor} &= c_{n,\gamma}L^\gamma u \ \ \textit{in} \ \ \Omega, \\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}_j &= 0 \ \ \textit{in} \ \ \Omega, \ \ \textit{for all} \ \ j=0,\cdots,\lfloor\gamma\rfloor-1. \end{aligned}$$

All boundary conditions in (4.2) *and* (4.3) *hold in* L^2 *sense.*

In [GR19], the authors study the fractional operator L^γ in $\mathbb{R}^n$ for $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, where the fractional operator L^γ is defined in terms of the spectral decomposition. Precisely, we write

$$(Lf, g) = \int_0^\infty \lambda dE_{f,g}(\lambda), \ \text{ for all } \ f \in \mathrm{dom}(L), \ \ g \in L^2(\mathbb{R}^n),$$

where $\mathrm{dom}(L) = \{f \in L^2(\mathbb{R}^n) : \int_0^\infty \lambda^2 dE_{f,f}(\lambda) < \infty\}$ and $dE_{f,g}(\lambda)$ is the spectral measure corresponding to L. The fractional operator L^γ is now defined by

$$(L^\gamma f, g) = \int_0^\infty \lambda^\gamma dE_{f,g}(\lambda), \ \text{ for all } \ f \in \mathrm{dom}(L^\gamma), \ \ g \in L^2(\mathbb{R}^n),$$

where $\mathrm{dom}(L^\gamma) = \{f \in L^2(\mathbb{R}^n) : \int_0^\infty \lambda^{2\gamma} dE_{f,f}(\lambda) < \infty\}$. The extension problem related to $L^\gamma u$ for $u \in \mathrm{dom}(L^\gamma)$ is similar to (4.2) and (4.3) except that Ω is replaced by $\mathbb{R}^n$ and no boundary restrictions

$$L^k \tilde{u}(x, x_{n+1}) = 0 \ \text{ on } \ \partial\Omega \times (0,\infty), \text{ for all } \ k = 0, \cdots, \lfloor\gamma\rfloor$$

and

$$\tilde{u}_j = 0 \ \text{ on } \ \partial\Omega \times (0,\infty), \ \text{ for all } j = 0, \cdots, \lfloor\gamma\rfloor$$

are required.

In [GR19, Theorem 4], relying on the extension problem, the MUCP is established for the equation

$$|L^\gamma u| \leq \sum_{j=0}^{\lfloor\gamma\rfloor} |q_j(x)||\nabla^j u| \quad \text{in} \quad \mathbb{R}^n \tag{4.4}$$

under suitable assumptions on $a_{ij}(x)$ and q_j. Besides of the extension problem, another key ingredient in the proof of MUCP for (4.4) is the Carleman estimate for the extension problem. Since the extension problem is local, the same Carleman estimate can be used to prove the MUCP for

$$L^{\gamma}u = q(x)u \ \text{ in } \ \Omega \tag{4.5}$$

with $q \in L^{\infty}(\Omega)$. To state the MUCP result for (4.5), we first give the assumptions imposed on a_{ij}:

- (C1) $(a_{ij}) : \Omega \to \mathbb{R}^{n\times n}$ is symmetric, strictly positive definite and bounded;
- (C2) $(a_{ij}) \in C^{2\lfloor\gamma\rfloor,1}(\Omega, \mathbb{R}^{n\times n}_{sym})$ with $\sum_{k=1}^{2\lfloor\gamma\rfloor+1} \|\nabla^k a_{ij}\|_{L^{\infty}(\Omega)} \ll \delta$ for some sufficiently small parameter $\delta > 0$;
- (C3) $a_{ij}(0) = \delta_{ij}$.

Repeating the proof of Theorem 4 in [GR19], we can prove that

Theorem 4.2 (MUCP for spectral fractional operator L^{γ})**.** *Let $u \in$ dom(L^{γ}) satisfy*

$$L^{\gamma}u(x) = q(x)u(x) \ \text{ in } \ \Omega,$$

where a_{jk} satisfies the conditions (C1)–(C3) *and $q \in L^{\infty}(\Omega)$. If there exists a measurable set $E \subset \Omega$ with $|E| > 0$ such that $u = 0$ in E, then $u \equiv 0$ in Ω.*

For other UCP results for the fractional operators, we refer the reader to [FF14], [Rül15], [Yu17], etc. and references therein.

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Appendix A. The Min-Max Principle of Eigenvalues of Compact Operators

In this section, we shall prove the min-max principle in Proposition 3.2. The content of this section can be found in [Fig82] or [FG92]. Let H be a Hilbert space, and $T : H \to H$ be a compact symmetric linear operator. First of all, we recall a well-known facts about the compact linear operators.

Proposition A.1 (Existence of orthonormal eigenfunctions)**.** *If*

$$\lambda_n = \sup\{(Tx, x) : \|x\| = 1, x \perp \phi_1, \cdots, \phi_{n-1}\} > 0,$$

then there exists $\phi_n \in H$ *with* $\|\phi_n\| = 1$ *and* $\phi_n \perp \phi_1, \cdots, \phi_{n-1}$ *such that*

$$(T\phi_n, \phi_n) = \lambda_n \quad \textit{and} \quad T\phi_n = \lambda_n \phi_n.$$

Similarly, if

$$\lambda_{-n} = \sup\{(Tx, x) : \|x\| = 1, x \perp \phi_{-1}, \cdots, \phi_{-(n-1)}\} < 0,$$

then there exists $\phi_{-n} \in H$ *with* $\|\phi_{-n}\| = 1$ *and* $\phi_{-n} \perp \phi_{-1}, \cdots, \phi_{-(n-1)}$ *such that*

$$(T\phi_{-n}, \phi_{-n}) = \lambda_{-n} \quad \textit{and} \quad T\phi_{-n} = \lambda_{-n} \phi_{-n}.$$

Using Proposition A.1, we can obtain the following min-max principle.

Proposition A.2. *For each positive integer* n, $\lambda_{\pm n}$ *can be characterized as*

$$\lambda_n = \max_{F_n} \inf\{(Tx, x) : \|x\| = 1, x \in F_n\}, \tag{A.1}$$

$$\lambda_{-n} = \min_{F_n} \sup\{(Tx, x) : \|x\| = 1, x \in F_n\}, \tag{A.2}$$

where the maximum (minimum) is taken over all subspaces F_n *of* H *with* $\dim(F_n) = n$.

Proof. Here we only prove (A.1). The proof of (A.2) is similar.

Given any subspace F_n of H with $\dim(F_n) = n$, choose $x \in F_n$ with $\|x\| = 1$ and $x \perp \phi_1, \cdots, \phi_{n-1}$. By Proposition A.1, we have $(Tx, x) \leq \lambda_n$. By arbitrariness of such x, we reach

$$\inf\{(Tx, x) : \|x\| = 1, x \in F_n\} \leq \lambda_n \quad \text{for all subspace } F_n \text{ of } H \text{ with } \dim(F_n) = n.$$

This implies

$$\sup_{F_n} \inf\{(Tx, x) : \|x\| = 1, x \in F_n\} \leq \lambda_n. \tag{A.3}$$

By Proposition A.1, we can choose $\tilde{F}_n := \text{span}\{\phi_1, \cdots, \phi_n\}$. For each $x \in \tilde{F}$ with $\|x\| = 1$, we can write

$$x = \sum_{i=1}^{n} x_i \phi_i \quad \text{with} \quad \sum_{i=1}^{n} x_i^2 = 1.$$

For such x, we have

$$(Tx, x) = \sum_{i=1}^{n} x_i^2 \lambda_i \geq \sum_{i=1}^{n} x_i^2 \lambda_n = \lambda_n.$$

This shows that the supremum in (A.3) is attained by $\tilde{F}_n$. So we can write "max" rather than "sup". □

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