CONSISTENCY OF THE BAYES METHOD FOR THE INVERSE SCATTERING PROBLEM

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Abstract. In this work, we consider the inverse scattering problem of determining an unknown refractive index from the far-field measurements using the nonparametric Bayesian approach. We use a collection of large "samples", which are noisy discrete measurements taking from the scattering amplitude. We will study the frequentist property of the posterior distribution as the sample size tends to infinity. Our aim is to establish the consistency of the posterior distribution with an explicit contraction rate in terms of the sample size. We will consider two different priors on the space of parameters. The proof relies on the stability estimates of the forward and inverse problems. Due to the ill-posedness of the inverse scattering problem, the contraction rate is of a logarithmic type. We also show that such contraction rate is optimal in the statistical minimax sense.

Contents

1. Introduction

In this work, we study the Bayes method for solving the inverse medium scattering problem. Our aim is to prove the consistency property of the posterior distribution. Let the refractive index $n \geq 0$ and $1-n$ be a compactly supported function in $\mathbb{R}^3$ with $\mathrm{supp}(1-n) \subset D$, where D is an open bounded smooth domain, and having suitable regularity, which will be specified later. Let $u = u^{\mathrm{inc}} + u_n^{\mathrm{sca}}$ satisfy

$$\Delta u + \kappa^2 n u = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{1.1a}$$

and

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial u_n^{\mathrm{sca}}}{\partial |x|} - \mathbf{i}\kappa u_n^{\mathrm{sca}} \right) = 0. \tag{1.1b}$$

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Assume that u^{inc} is the plane incident field, i.e. $u^{\mathrm{inc}} = e^{\mathbf{i}\kappa x\cdot\theta}$ with $\theta \in \mathcal{S}^2$. Then the scattered field u_n^{sca} satisfies

$$u_n^{\mathrm{sca}}(x,\theta) = \frac{e^{\mathbf{i}\kappa|x|}}{|x|}u_n^{\infty}(\theta',\theta) + o(|x|^{-1}) \quad \text{as} \quad |x| \to \infty, \tag{1.1c}$$

where $\theta' = x/|x|$, see for example, [Ser17, Page 232]. The inverse scattering problem is to determine the medium perturbation $1-n$ from the knowledge of the *scattering amplitude* $u_n^{\infty}(\theta',\theta)$ for all $\theta',\theta \in \mathcal{S}^2$ at one fixed energy κ^2.

It was known that the scattering amplitude $u_n^{\infty}(\theta',\theta)$ uniquely determines the near-field data of (1.1a) on ∂D, which in turn, determines the Dirichlet-to-Neumann map of (1.1a) provided κ^2 is not an Dirichlet eigenvalue of $-\Delta$ on D, for example, see [Nac88]. Combining this fact and the uniqueness results proved in [SU87], one can show that the scattering amplitude $u_n^{\infty}(\theta',\theta)$ for all $\theta',\theta \in \mathcal{S}^2$ uniquely determines n, at least when n is essentially bounded. For the stability, a log-type estimate was derived in [HH01].

In this paper, we would like to apply the Bayes approach to the inverse scattering problem. To describe the method, we first introduce the measurement model. Let μ be the uniform distribution on $\mathcal{S}^2\times\mathcal{S}^2$, i.e., $\mu = \mathrm{d}\omega/|\mathcal{S}^2|^2$, where $\mathrm{d}\omega$ is the product measure on $\mathcal{S}^2\times\mathcal{S}^2$, that is, $\int_{\mathcal{S}^2\times\mathcal{S}^2}\mathrm{d}\omega = |\mathcal{S}^2|^2$. We also write $\mu = \mathrm{d}\xi$ and hence $\int_{\mathcal{S}^2\times\mathcal{S}^2}\mathrm{d}\xi = 1$. Consider the iid random variables $X_i \sim \mu$, $i = 1,2,\cdots,N$ with $N \in \mathbb{N}$. In other words, $\{X_i\}_{i=1}^N$ is a sequence of independent samples of μ on $\mathcal{S}^2\times\mathcal{S}^2$. Denote

$$G(n)(X_i) = u_n^{\infty}(\theta',\theta), \tag{1.2}$$

where (θ',θ) is a realization of X_i. The observation of the scattering amplitude $G(n)(X_i)$ is polluted by the measurement noise which is assumed to be a Gaussian random variable. Since $G(n)(X_i)$ is a complex-valued function, we treat it as a $\mathbb{R}^2$-valued function. For convenience, we slightly abuse the notation by writing

$$G(n)(X_i) = \begin{pmatrix}\mathrm{Re}\{G(n)(X_i)\}\\ \mathrm{Im}\{G(n)(X_i)\}\end{pmatrix}.$$

Consequently, the statistical model of the scattering problem is given as

$$Y_i = G(n)(X_i) + \sigma W_i,\ \ W_i \overset{iid}{\sim} \mathcal{N}(0,I_2),\ \ i = 1,\cdots,N, \tag{1.3}$$

where $\sigma > 0$ is the noise level, I_2 is the 2×2 unit matrix. We also assume that $W^{(N)} := \{W_i\}_{i=1}^N$ and $X^{(N)} := \{X_i\}_{i=1}^N$ are independent.

The theme of this paper is to consider the inference of n from the observational data $(Y^{(N)}, X^{(N)})$ with $Y^{(N)} = \{Y_i\}_{i=1}^N$ by the Bayes method. In particular, we are interested in the asymptotic behavior of the posterior distribution induced from a large class of Gaussian process priors on n as $N \to \infty$. The aim is to establish the statistical consistency theory of recovering n in (1.1c) with an explicit convergence rate as the number of measurements N increases, i.e. the contraction rate of the posterior distribution to the "ground truth" n_0 when the observation data is indeed generated by n_0. Gaussian process priors are often used in applications in which efficient numerical simulations can be carried out based on modern MCMC algorithms such as the pCN (preconditioned Crank-Nicholson) method [CRSW13].

The study of inverse problems in the Bayesian inversion framework has recently attracted much attention since Stuart's seminal article [Stu10] (also see [DS17]). The setting of the problem considered in this paper is closely related to the ones studied in [GN20] and [Kek22]. In [GN20], Gaussian process priors were used in the Bayesian approach to study the recovery of the diffusion coefficient in the elliptic equation by measuring the solution at randomly chosen interior points with uniform distribution. It was shown that the posterior distribution concentrates around the true parameter at a rate $N^{-\lambda}$ for some $\lambda > 0$ as $N \to \infty$, where N is the number of measurements (or sample size). Previously, (frequentist) consistency of

Bayesian inversion in the elliptic PDE considered in [GN20] was derived in [Vol13]. However, the contraction rates obtained in [Vol13] were only implicitly given. Based on the method in [GN20], similar results were proved in [Kek22] for the parabolic equation where the aim is to recover the absorption coefficient by the interior measurements of the solution. For further results on the Bayesian inverse problems in the non-linear settings, we refer the reader to other interesting papers [Abr19, NS17, NS19, MNP21, Nic20, NP21, NW20]. On the other hand, for linear inverse problems, the statistical guarantees of nonparametric Bayesian methods with Gaussian priors have been extensively studied and well understood, see for example [ALS13, KLS16, KvdVvZ11, MNP19, Ray13] and references therein.

The ideas used in proving the consistency of the Bayesian inversion for the inverse scattering problem studied here are similar to those used in [GN20] and [Kek22] in which main ideas are from [MNP21]. Unlike the polynomial contraction rates derived in [GN20], [Kek22], the posterior distribution $\Pi(\cdot|Y^{(N)}, X^{(N)})$ of $n|(Y^{(N)}, X^{(N)})$ contracts at the true refractive index n_0 as $N \to \infty$ with a logarithmic rate. The logarithmic rate is due to the ill-posedness of the inverse scattering medium problem by the knowledge of the scattering amplitude at one fixed energy, see [HH01].

This paper is organized as follows. In Section 2, we describe the statistical model arising from the scattering problem. We state main consistency theorems with contraction rates assuming re-scaled Gaussian processes priors and re-scaled Gaussian sieve priors. The proofs of theorems are given in Section 3. In Appendix A, we derive some estimates for the forward scattering problem, and in Appendix B, we prove the optimality of the logarithmic stability estimate in the inverse medium scattering problem based on Mandache's idea. Similar instability estimate was also derived in [Isa13]. To make the paper self contained, we present our own proof and slightly refine the estimate obtained in [Isa13].

2. The statistical inverse scattering problem

2.1. **Some function spaces and notations.** Throughout this paper, we shall use the symbol $\lesssim$ and $\gtrsim$ for inequalities holding up to a universal constant. For two real sequences (a_N) and (b_N), we say that $\simeq$ if both $a_N \lesssim b_N$ and $b_N \lesssim a_N$ for all sufficiently large N. For a sequence of random variables Z_N and a real sequence (a_N), we write $Z_N = O_{\mathbb{P}}(a_N)$ if for all $\varepsilon > 0$ there exists $1 \leq M_\varepsilon < \infty$ such that for all N large enough, $\mathbb{P}(|Z_N| \geq M_\varepsilon a_N) < \varepsilon$. Denote $\mathcal{L}(Z)$ the law of a random variable Z. Let $C_c^t(\mathcal{O})$ with $t \geq 0$ denote the Hölder space of order t with compact supports in the bounded smooth domain $\mathcal{O}$.

Let D be a bounded smooth domain in $\mathbb{R}^3$, let $s \geq 0$ be an integer and we consider the Hilbert space

$$H^s(D) = \left\{ f \in L^2(D) : D^j f \in L^2(D) \text{ for all } |j| \leq s \right\},$$

$$\text{with scalar product} \quad \langle f, g\rangle_{H^s(D)} = \sum_{|j|\leq s} \langle D^j f, D^j g\rangle_{L^2(D)}.$$

For non-integer s, $H^s(D)$ is defined in terms of the interpolation, see [LM72]. It is known that the restriction operator to D is a continuous linear map of $H^s(\mathbb{R}^3)$ to $H^s(D)$ [LM72, (8.6)]. The space $H_0^s(D)$ is the completion of $C_c^\infty(D)$ with respect to $H^s(D)$. Also, for $s > 1/2$ with $s \neq \mathbb{Z} + 1/2$, the zero extension of $f \in H_0^s(D)$ (extension of f by 0 outside of D) is a continuous map $H_0^s(D) \to H^s(\mathbb{R}^3)$ [LM72, Theorem 11.4]. Let $\mathcal{K}$ be a compact subset, define $H_{\mathcal{K}}^s = \{f \in H^s(\mathbb{R}^3) : \operatorname{supp}(f) \subseteq \mathcal{K}\}$. In fact, we have

$$H_{\overline{D}}^s \subset H_0^s \text{ for all } s \geq 0 \text{ and equality (up to equivalent norms) holds when } s \notin \mathbb{Z} + \frac{1}{2},$$

see [McL00, Theorem 3.29 and Theorem 3.33].

We now define the space of parameters. For $M_0 > 1$ and $s > \frac{3}{2}$, let

$$\mathcal{F}_{M_0}^s = \left\{ n \in H^s(D) : 0 < n < M_0,\, n|_{\partial D} = 1,\, \left.\frac{\partial^j n}{\partial \nu^j}\right|_{\partial D} = 0 \text{ for all } 1 \le j \le s-1 \right\},$$

where the traces are defined in the sense of [LM72, Theorem 8.3]. For each $n \in \mathcal{F}_{M_0}^s$ we extend $n \equiv 1$ in $\mathbb{R}^3 \setminus D$, still denoted by n. Then it is clear that $\operatorname{supp}(1-n) \subset D$. Note that for $n \in \mathcal{F}_{M_0}^s$, we only put the restriction on the size of n, but not on the $H^s(D)$-norm of u. As in [NvdGW20, GN20, AN19, Kek22], we will consider a re-parametrization of $\mathcal{F}_{M_0}^s$. We consider the *link function* Φ satisfying

(i) $\Phi : (-\infty, \infty) \to (0, M_0)$, $\Phi(0) = 1$, $\Phi'(z) > 0$ for all z;
(ii) for any $k \in \mathbb{N}$

$$\sup_{-\infty < z < \infty} |\Phi^{(k)}(z)| < \infty.$$

One example to satisfy (i) and (ii) is the logistic function

$$\Phi(z) = \frac{M_0}{M_0 + (M_0 - 1)(e^{-z} - 1)}, \quad -\infty < z < \infty.$$

As pointed out in [NvdGW20, Section 3], by utilizing a characterization of the space $H_0^s(D)$ (see e.g. [LM72, Theorem 11.5]), one can show that the parameter space can be realized as

$$\mathcal{F}_{M_0}^s = \{ n = \Phi(F) : F \in H_0^s(D) \}. \tag{2.1}$$

We end this subsection by emphasizing that our link function is different to those in [NvdGW20, GN20, AN19, Kek22], see (i).

2.2. **An abstract statistical model.** For each *forward map* $G : (0, M_0) \to \mathbb{R}^2$, we define the *reparametrized forward map* by

$$\mathcal{G}(F) = G(\Phi(F)) \quad \text{for all } F \in H_0^s(D), \tag{2.2a}$$

and consider the following random design regression model

$$Y_i = \mathcal{G}(F)(X_i) + \sigma W_i, \;\; W_i \overset{iid}{\sim} N(0, I_2), \;\; i = 1, \cdots, N. \tag{2.2b}$$

Assume that $\mathcal{G}$ satisfies

$$S_1 := \sup_{F \in H_0^s(D)} \|\mathcal{G}(F)\|_{L^\infty(\mathcal{S}^2 \times \mathcal{S}^2)} < \infty, \tag{2.2c}$$

and for each $F_1, F_2 \in (H^1(D))^*$ one has

$$\|\mathcal{G}(F_1) - \mathcal{G}(F_2)\|_{L^2(\mathcal{S}^2 \times \mathcal{S}^2)} \le S_2(1 + \|F_1\|_{C^t(D)}^2 \vee \|F_2\|_{C^t(D)}^2) \|F_1 - F_2\|_{(H^1(D))^*}, \tag{2.2d}$$

for some constant $S_2 > 0$ and $t \ge 1$. The statistical model (2.2b) with conditions (2.2c) and (2.2d) falls into the general framework described in [GN20]. We want to remark that the uniform boundedness of the forward map $\mathcal{G}$ (condition (2.2c)) in [GN20, NvdGW20] (elliptic boundary value problem) or in [Kek22] (parabolic initial-boundary value problem) is ensured by the positivity assumption of the coefficient and the bound S_1 is determined by the fixed boundary value or the fixed initial and boundary values. Due to these facts, the ranges of the link functions used in [GN20, NvdGW20] and [Kek22] do not required to have finite upper bounds. In the scattering problem, the boundedness requirement of the forward map (2.2c) cannot be guaranteed by the sign restriction of the potential. In this case, we choose a link with finite range (like Φ given above) to ensure (2.2c).

Remark 2.1. In view of (1.2) and (2.1), one notes that the statistical model (1.3) fits into the framework of (2.2b). For the inverse scattering problem studied here, if the forward map $G(n)$ is defined by the far-field pattern (1.2) with refractive index $n \in \mathcal{F}^s_{M_0}$ with $s > \frac{3}{2}$, from (A.9) we see that (2.2c) satisfies with $S_1 = S_1(D, \kappa, M_0)$. From (A.14) we have

$$\begin{aligned}&\|\mathcal{G}(F_1) - \mathcal{G}(F_2)\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)}\\ &\qquad \le C(1 + \|\Phi(F_1)\|_{C^1(D)} \vee \|\Phi(F_2)\|_{C^1(D)})\|\Phi(F_1) - \Phi(F_2)\|_{(H^1(D))^*},\end{aligned}$$

where $C = C(D, k, M_0)$. By using (ii) and [NvdGW20, Lemma 29], we have

$$\|\Phi(F_j)\|_{C^1(D)} \lesssim 1 + \|F_j\|_{C^1(D)} \quad \text{for all } j = 1, 2,$$

and

$$\|\Phi(F_1) - \Phi(F_2)\|_{(H^1(D))^*} \lesssim (1 + \|F_1\|_{C^1(D)} \vee \|F_2\|_{C^1(D)})\|F_1 - F_2\|_{(H^1(D))^*},$$

therefore we see that (2.2d) satisfies with $S_2 = S_2(D, \kappa, M_0)$ with $t = 1$.

Observe that the random vectors (Y_i, X_i) are iid with laws $\mathbb{P}^i_F$. It turns out the Radon-Nikodym derivative of $\mathbb{P}^i_F$ is given by

$$p_F(y, \xi) = \frac{\mathrm{d}\mathbb{P}^i_F}{\mathrm{d}y \times \mathrm{d}\xi} = \frac{1}{2\pi\sigma^2} e^{-\frac{(y-\mathcal{G}(F)(\xi))^T(y-\mathcal{G}(F)(\xi))}{2\sigma^2}}. \tag{2.3}$$

By slightly abusing the notation, now we define $\mathbb{P}^N_F = \otimes^N_{i=1}\mathbb{P}^i_F$ the joint law of the random vectors $(Y_i, X_i)^N_{i=1}$. Moreover, $\mathbb{E}^i_F$, $\mathbb{E}^N_F$ denote the expectation operators in terms of the laws $\mathbb{P}^i_F, \mathbb{P}^N_F$, respectively.

In the Bayesian approach, let Π be a Borel probability measure on the parameter space $H^s_0(D)$ supported in the Banach space $C(D)$. From the continuity property of $(F, (y, \xi)) \to p_F(y, \xi)$, the posterior distribution $\Pi(\cdot|Y^{(N)}, X^{(N)})$ of $F|(Y^{(N)}, X^{(N)})$ is given by

$$\Pi(B|Y^{(N)}, X^{(N)}) = \frac{\int_B e^{\ell^{(N)}(F)}\,\mathrm{d}\Pi(F)}{\int_{C(D)} e^{\ell^{(N)}(F)}\,\mathrm{d}\Pi(F)} \quad \text{for all Borel set } B \subset C(D), \tag{2.4a}$$

where the log-likelihood function is written as

$$\ell^{(N)}(F) = -\frac{1}{2\sigma^2}\sum_{i=1}^N (Y_i - \mathcal{G}(F)(X_i))^T (Y_i - \mathcal{G}(F)(X_i)). \tag{2.4b}$$

Finally, we end this subsection by referring to the monograph [Nic23] for a nice introduction on the above preliminaries.

2.3. **Statistical convergence rates.** In this work we would like to show that the posterior distribution arising from certain priors concentrates near sufficiently regular ground truth $\Phi(F_0)$, and derive a bound on the rate of contraction, assuming that the observation data $(Y^{(N)}, X^{(N)})$ are generated through the model (2.2a)–(2.2d) of law $\mathbb{P}^N_{F_0}$.

2.3.1. *Rescaled Gaussian priors.* We now describe explicitly Gaussian priors introduced in [GN20] (also see [Kek22]).

Assumption 2.2. Let $s > t+3/2$, $t \ge 1$, and $\mathcal{H}$ be a Hilbert space continuously embedded into $H^s_0(D)$, and let Π' be a centered Gaussian Borel probability measure on the Banach space $C^0_c(D)$ that is supported on a separable measurable linear subspace of $C^t_c(D)$. Assume that the reproducing kernel Hilbert space (RKHS) of Π' equals to $\mathcal{H}$.

Here, we refer to [GN21, Definition 2.6.4] for the definition of the RKHS, and we refer to [GN20, Example 25] for an example satisfies Assumption 2.2. For each s given in Assumption 2.2, let Π' be given in Assumption 2.2 and $F' \sim \Pi'$, we consider the *rescaled* prior

$$\Pi_N \equiv \Pi_N[s] = \mathcal{L}(F_N) \quad \text{with} \quad F_N = \frac{1}{N^{3/(4s+10)}} F'. \tag{2.5}$$

Again, Π_N defines a centered Gaussian prior on $C(D)$ and its RKHS $\mathcal{H}_N$ is still $\mathcal{H}$ but with the norm

$$\|F\|_{\mathcal{H}_N} = N^{3/(4s+10)} \|F\|_{\mathcal{H}}, \tag{2.6}$$

for all $F \in \mathcal{H}$. We now introduce the main device in our proof, concerning the posterior contraction with an explicit rate around F_0, whose proof can be found in [GN20, Theorem 14].

Theorem 2.3. *Let $(\mathcal{H}, \Pi')$ satisfies* Assumption 2.2 *with integer s, let $\Pi_N \equiv \Pi_N[s]$ be the rescaled prior given in (2.5), let $\Pi_N(\cdot|Y^{(N)}, X^{(N)})$ be the posterior distribution given in (2.4a) with $\Pi = \Pi_N$. Assume that $F_0 \in \mathcal{H}$ and the observation $(Y^{(N)}, X^{(N)})$ to be generated through model (2.2b)–(2.2d) of law $\mathbb{P}_{F_0}^N$. If we denote $\delta_N = N^{-(s+1)/(2s+5)}$, then for any $K > 0$, there exists a large $L > 0$, depending on $\sigma, F_0, K, s, t, D, S_1, S_2$, such that*

$$\begin{aligned} &\Pi_N\left(F : \|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)} > L\delta_N | Y^{(N)}, X^{(N)}\right) \\ &= O_{\mathbb{P}_{F_0}^N}\left(e^{-KN\delta_N^2}\right) \quad \text{as} \quad N \to \infty. \end{aligned} \tag{2.7a}$$

In addition, there exists a large $L' > 0$, depending on σ, K, s, t, such that

$$\Pi_N\left(F : \|F\|_{C^t} > L' | Y^{(N)}, X^{(N)}\right) = O_{\mathbb{P}_{F_0}^N}\left(e^{-KN\delta_N^2}\right) \quad \text{as} \quad N \to \infty. \tag{2.7b}$$

We now apply Theorem 2.3 to the inverse scattering problem considered here. Relying on the contraction rate (2.7a) and the regularization property (2.7b) and taking account of the stability estimate of G^{-1} (see Theorem A.1), we can show that the posterior distribution arising from the statistical inverse scattering model (1.3) contracts around n_0 in the L^∞-risk using ideas from [MNP21]. In light of the link function, we define the push-forward posterior on the refractive index n by

$$\tilde{\Pi}_N(\cdot|Y^{(N)}, X^{(N)}) := \mathcal{L}(n) \;\text{ with }\; n = \Phi \circ F : F \sim \Pi_N(\cdot|Y^{(N)}, X^{(N)}).$$

By the push-forward map, we can rewrite (2.7a) and (2.7b) in terms of n. That is, we have that for $N \to \infty$ that

$$\tilde{\Pi}_N\left(n : \|G(n) - G(n_0)\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)} > L\delta_N | Y^{(N)}, X^{(N)}\right) = O_{\mathbb{P}_{n_0}^N}(e^{-KN\delta_N^2}), \tag{2.8a}$$

and

$$\tilde{\Pi}_N\left(n : \|n\|_{C^t} > L' | Y^{(N)}, X^{(N)}\right) = O_{\mathbb{P}_{n_0}^N}(e^{-KN\delta_N^2}), \tag{2.8b}$$

where the second estimate can be derived from the estimates proved in [NvdGW20, Lemma 29] (see also [GN20, (27)]). Here L depends on $\sigma, n_0, K, s, t, D, \kappa, M_0$ and L' depends on $\sigma, K, s, t, \kappa, M_0$.

Theorem 2.4. *Let $t \geq 2$ and $s > t + 3/2$ be integers, and fix a real parameter $M_0 > 1$. We further assume that ϵ is any constant satisfying $0 < \epsilon < \frac{2t-3}{2t+3}$. Let $\Pi_N(\cdot|Y^{(N)}, X^{(N)})$, $F_0 \in \mathcal{H}$ and $\delta_N = N^{-(s+1)/(2s+5)}$ be given in* Theorem 2.3. *The ground truth refractive index is $n_0 = \Phi \circ F_0 \in \mathcal{F}_{M_0}^s$. Then for any $K > 0$, there exists a constant $C = C(\sigma, n_0, K, s, t, D, \kappa, \epsilon, M_0) > 0$ such that*

$$\tilde{\Pi}_N\left(n : \|n - n_0\|_{L^\infty(D)} > C(\ln N)^{-\frac{2t-3}{2t+3}+\epsilon} | Y^{(N)}, X^{(N)}\right) = O_{\mathbb{P}_{n_0}^N}(e^{-KN\delta_N^2}), \tag{2.9}$$

as $N \to \infty$.

It is clear that we can replace $\|n-n_0\|_{L^\infty(D)}$ by $\|n-n_0\|_{L^2(D)}$ in (2.9). Unlike the polynomial rate proved in [GN20, Theorem 5], we obtain a logarithmic contraction rate in (2.9), which is due to a log-type stability estimate of G^{-1}. To obtain an estimator of the unknown coefficient n, in view of the link function Φ, it is often convenient to derive an estimator of F. The posterior mean $\bar{F}_N := \mathbb{E}^\Pi[F|Y^{(N)}, X^{(N)}]$ of $\Pi_N(\cdot|Y^{(N)}, X^{(N)})$, which can be approximated numerically by an MCMC algorithm, is the most natural choice of estimator. In light of Theorem 2.4, we can also prove a contraction rate for the convergence $\bar{F}_N$ to F_0.

Theorem 2.5. *Assume that the hypotheses of* Theorem 2.4 *hold. Then, there exists a* $\tilde{C} = \tilde{C}(\sigma, n_0, K, s, t, D, \kappa, \epsilon, M_0) > 0$ *such that*

$$\mathbb{P}^N_{F_0}\left(\|\bar{F}_N - F_0\|_{L^\infty} > \tilde{C}(\ln N)^{-\frac{2t-3}{2t+3}+\epsilon}\right) \to 0 \quad \text{as } N \to \infty. \tag{2.10}$$

Corollary 2.6. *Assume that the hypotheses of* Theorem 2.4 *hold. Then there exists a sufficiently large* $\tilde{C}' = \tilde{C}'(\sigma, n_0, K, s, t, D, \kappa, \epsilon, M_0) > 0$ *such that*

$$\mathbb{P}^N_{n_0}\left(\|\Phi \circ \bar{F}_N - n_0\|_{L^\infty} > \tilde{C}'(\ln N)^{-\frac{2t-3}{2t+3}+\epsilon}\right) \to 0 \quad \text{as } N \to \infty. \tag{2.11}$$

The logarithmic contraction rates obtained in Theroem 2.4, Theorem 2.5, and Corollary 2.6 inherit from the log-type estimate of the inverse scattering problem. Nonetheless, in the next theorem, we will show that this contraction rate for the estimator $\hat{n} := \Phi \circ \bar{F}_N$ is optimal is the statistical minimax sense, at least up to the exponent of $\ln N$. We first define a parameter space. Let $s > 3/2$ be an integer, $\beta > 0$, and define

$$\hat{\mathcal{F}}^s_\beta = \{n = 1 + q : \ q \in H^s_{\overline{B_{1/2}}}(\mathbb{R}^3) \ \text{ with } \ \|q\|_{H^s} \le \beta\}.$$

Theorem 2.7. *For integer $s > 3/2$, there exists $\beta = \beta(s) > 0$ such that for any $\delta > \frac{5s}{3}$ and $\varepsilon \in (0,1)$, we have*

$$\inf_{\tilde{n}} \sup_{n \in \hat{\mathcal{F}}^s_\beta} \mathbb{P}^N_n\left(\|\tilde{n} - n\|_{L^\infty} > \frac{1}{2}(\ln N)^{-\delta}\right) > 1 - \varepsilon, \tag{2.12}$$

for all N large enough, where the infimum is taken over all measurable functions $\tilde{n} = \tilde{n}(Y, X)$ of the data $(Y, X) \sim \mathbb{P}^N_n$.

2.3.2. *High-dimensional Gaussian sieve priors.* From computational perspective, it is useful to consider *sieve* priors that are finite-dimensional approximations of the function space supporting the prior. Here we will use a randomly truncated Karhunen-Loéve type expansion in terms of Daubechies wavelets considered in [GN20, Appendix B] or [GN21, Chapter 4]. Let $\{\Psi_{\ell r} : \ell \ge -1, r \in \mathbb{Z}^3\}$ be the (3-dimensional) compactly supported Daubechies wavelets, which forms an orthonormal basis of $L^2(\mathbb{R}^3)$. Let $\mathcal{K}$ be a compact subset in D and let $R_\ell = \left\{r \in \mathbb{Z}^d : \operatorname{supp}(\Psi_{\ell r}) \cap \mathcal{K} \neq \emptyset\right\}$. Let $\mathcal{K}'$ be another compact subset in D such that $\mathcal{K}' \subsetneq \mathcal{K}$, and let $\chi \in C^\infty_c(D)$ be a cut-off function with $\chi = 1$ on $\mathcal{K}'$. Let $s > 1 + 3/2$ and consider the prior

$$\Pi'_J \equiv \Pi'_J[s] = \mathcal{L}(\chi F_J), \quad F_J = \sum_{\substack{-1 \le \ell \le J \\ r \in \mathcal{R}_\ell}} 2^{-\ell s} F_{\ell r} \Psi_{\ell r}, \quad F_{\ell r} \overset{\text{iid}}{\sim} \mathcal{N}(0,1), \tag{2.13}$$

where $J \in \mathbb{N}$ is the truncation level. In fact Π'_J defines a centered Gaussian prior that is supported on the finite-dimensional space

$$\mathcal{H}_J = \operatorname{span}\{\chi \Psi_{\ell r}, : -1 \le \ell \le J, \, r \in \mathcal{R}_\ell\} \subset C(D),$$

with RKHS norm satisfying [GN20, (B2)]. As above, we consider the "re-scaled" prior Π_N defined in (2.5) with $F' \sim \Pi'_J$.

In analogy to Theorem 2.3, we will derive a contraction rate for the statistical model (2.2b) with the prior Π_N defined above. As in Theorem 2.3, we obtain the same contraction rate in the L^2-prediction risk of the regression function.

Theorem 2.8. *Let $s > t + 3/2$ with integer $t \geq 1$ and $\Pi_N \equiv \Pi_N[s]$ be the "re-scaled" prior defined in (2.5) with priors $F' \sim \Pi'_{J_N} \equiv \Pi'_{J_N}[s]$ where $2^{J_N} \simeq N^{1/(2s+5)}$. Denote $\Pi_N(\cdot|Y^{(N)}, X^{(N)})$ the posterior distribution arising from the noisy discrete measurements $(Y^{(N)}, X^{(N)})$ of (2.2b). Let $F_0 \in H^s_K(D)$ and $\delta_N = N^{-(s+1)/(2s+5)}$. Then for any $K > 0$, there exists a large $L > 0$, depending on $\sigma, F_0, K, s, D, S_1, S_2$, such that*

$$\Pi_N(F : \|\mathcal{G}(F) - \mathcal{G}(F_0)\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)} > L\delta_N|Y^{(N)}, X^{(N)}) = O_{P^N_{F_0}}(e^{-KN\delta_N^2}) \ \ \text{as} \ \ N \to \infty, \tag{2.14}$$

and for sufficiently large $L' > 0$, depending on σ, K, s, t, we have

$$\Pi_N(F : \|F\|_{C^t} > L'|Y^{(N)}, X^{(N)}) = O_{P^N_{F_0}}(e^{-KN\delta_N^2}) \ \ \text{as} \ \ N \to \infty. \tag{2.15}$$

The proof of Theorem 2.8 only requires minor modification from the proof of Theorem 2.3, and all necessary modifications are listed in [GN20, Section 3.2]. Therefore we omit the details. Having established Theorem 2.8, similar to [GN20, Proposition 7], Theorem 2.4 and Theorem 2.5 can be directly extended to the case of Gaussian sieve priors.

Remark 2.9. As in [GN20], it is likely to extend the results for the Gaussian sieve priors with deterministic truncation level to randomly truncated ones, where the truncation level J itself is an appropriate random number. However, due to the log-type stability estimate of the inverse scattering problem, such extension is highly nontrivial. We will discuss the Bayes method with randomly truncated sieve priors for the inverse scattering problem in another paper.

3. Proofs of Theorems

The main theme of this section is to prove Theorem 2.4, Theorem 2.5 and Theorem 2.7.

Proof of Theorem 2.4. For each $M > 0$ satisfies $\|1 - n\|_{H^t(D)} \vee \|1 - n_0\|_{H^t(D)} \leq M$, one has the stability estimate of G^{-1} in Theorem A.1:

$$\|n - n_0\|_{L^\infty(D)} \leq C[-\ln^-(\|G(n) - G(n_0)\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)})]^{-\alpha}, \tag{3.1}$$

where $\alpha = \frac{2t-3}{2t+3} - \epsilon > 0$ and $C = C(D, t, \kappa, M, \epsilon)$.

If $\|n\|_{C^t(D)} \leq M'$ for some $M' \geq M_0$, since t is an integer, then $\|1 - n\|_{H^t(D)} \leq CM'$ with $C = C(t, D)$. We now set the constant $M = (CM') \vee \|1 - n_0\|_{H^t(D)}$. In view of (2.8a), (2.8b), and (3.1), for any $K > 0$, there exists large constants L, M' and L' (L' is determined by L and M') such that

$$\begin{aligned}
&\tilde{\Pi}_N\left(n : \|n - n_0\|_{L^\infty(D)} > |\ln(L\delta_N)|^{-\alpha}|Y^{(N)}, X^{(N)}\right)\\
&\quad \leq \tilde{\Pi}_N\left(n : \|G(n) - G(n_0)\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)} > L'\delta_N|Y^{(N)}, X^{(N)}\right)\\
&\qquad + \tilde{\Pi}_N\left(n : \|n\|_{C^t(D)} > M'|Y^{(N)}, X^{(N)}\right)\\
&\quad = O_{\mathbb{P}^N_{n_0}}(e^{-KN\delta_N^2}),
\end{aligned}$$

which conclude our result. □

Having proved Theorem 2.4, we then establish Theorem 2.5 using the contraction rate in Theorem 2.4 and the link function Φ.

Proof of Theorem 2.5. We proof the theorem by modifying some ideas in [GN20, Theorem 6]. By the Jensen's inequality , it suffices to prove that there exists a large $\tilde{C} > 0$ such that

$$\mathbb{P}_{F_0}^N \left(\mathbb{E}^\Pi[\|F - F_0\|_{L^\infty} | Y^{(N)}, X^{(N)}] > \tilde{C} (\ln N)^{-\frac{2t-3}{2t+3}+\epsilon} \right) \to 0 \quad \text{as } N \to \infty.$$

For a large $M > 0$ to be chosen later, we write

$$\begin{aligned} \mathbb{E}^\Pi[\|F - F_0\|_{L^\infty} | Y^{(N)}, X^{(N)}] = \mathbb{E}^\Pi[\|F - F_0\|_{L^\infty} \mathbb{1}_{\|F\|_{C^t} \le M} | Y^{(N)}, X^{(N)}] \\ + \mathbb{E}^\Pi[\|F - F_0\|_{L^\infty} \mathbb{1}_{\|F\|_{C^t} > M} | Y^{(N)}, X^{(N)}]. \end{aligned} \tag{3.2}$$

Part I: Estimating $\mathbb{E}^\Pi[\|F - F_0\|_{L^\infty} \mathbb{1}_{\|F\|_{C^t} > M} | Y^{(N)}, X^{(N)}]$. By using Cauchy-Schwartz inequality, it is easy to see that

$$\begin{aligned} &\mathbb{E}^\Pi[\|F - F_0\|_{L^\infty} \mathbb{1}_{\|F\|_{C^t} > M} | Y^{(N)}, X^{(N)}] \\ &\quad \le \sqrt{\mathbb{E}^\Pi[\|F - F_0\|_{L^\infty}^2 | Y^{(N)}, X^{(N)}]} \sqrt{\Pi_N(F : \|F\|_{C^t} > M | Y^{(N)}, X^{(N)})}. \end{aligned} \tag{3.3}$$

Let $B > 0$ be a constant to be determined later. By using (2.7b), one can choose a sufficiently large $M = M(\sigma, B, s, t) > 0$ such that

$$\begin{aligned} &\mathbb{P}_{F_0}^N \left(\mathbb{E}^\Pi[\|F - F_0\|_{L^\infty}^2 | Y^{(N)}, X^{(N)}] \Pi_N(F : \|F\|_{C^t} > M | Y^{(N)}, X^{(N)}) > (\ln N)^{-\frac{2(2t-3)}{2t+3}+2\epsilon} \right) \\ &\le \mathbb{P}_{F_0}^N \left(\mathbb{E}^\Pi[\|F - F_0\|_{L^\infty}^2 | Y^{(N)}, X^{(N)}] e^{-BN\delta_N^2} > (\ln N)^{-\frac{2(2t-3)}{2t+3}+2\epsilon} \right) + o(1). \end{aligned}$$

We now define $\mathcal{B}_N$ by

$$\mathcal{B}_N = \left\{ F : \mathbb{E}_{F_0}^1 \left[\log \frac{p_{F_0}(Y_1, X_1)}{p_F(Y_1, X_1)} \right] \le \delta_N^2, \mathbb{E}_{F_0}^1 \left[\log \frac{p_{F_0}(Y_1, X_1)}{p_F(Y_1, X_1)} \right]^2 \le \delta_N^2 \right\}, \tag{3.4}$$

By using [GN20, Lemma 16 and Lemma 23], we have

$$\Pi_N(\mathcal{B}_N) \ge a e^{-AN\delta_N^2} \quad \text{for some } a, A > 0.$$

Let $\nu(\cdot) = \Pi_N(\cdot \cap \mathcal{B}_N) / \Pi_N(\mathcal{B}_N)$ and set the event

$$\mathcal{C}_N = \left\{ \int_{\mathcal{B}_N} \prod_{i=1}^N \frac{p_F}{p_{F_0}}(Y_i, X_i) \, \mathrm{d}\nu(F) \ge e^{-2N\delta_N^2} \right\}. \tag{3.5}$$

By [GN21, Lemma 7.3.2], we can show that

$$\mathbb{P}_{F_0}^N(\mathcal{C}_N) \to 1 \text{ (i.e. } \mathbb{P}_{F_0}^N(\mathcal{C}_N^c) \to 0) \text{ as } N \to \infty. \tag{3.6}$$

By using the properties (3.6) of $\mathcal{C}_N$, we obtain

$$\begin{aligned} &\mathbb{P}_{F_0}^N \left(\mathbb{E}^\Pi[\|F - F_0\|_{L^\infty}^2 | Y^{(N)}, X^{(N)}] e^{-BN\delta_N^2} > (\ln N)^{-\frac{2(2t-3)}{2t+3}+2\epsilon} \right) \\ &\le \mathbb{P}_{F_0}^N \left(\frac{\int_{C(D)} \|F - F_0\|_{L^\infty}^2 \prod_{i=1}^N p_F/p_{F_0}(Y_i, X_i) \, \mathrm{d}\Pi_N(F)}{\Pi(\mathcal{B}_N) \int_{\mathcal{B}_N} \prod_{i=1}^N p_F/p_{F_0}(Y_i, X_i) \, \mathrm{d}\nu(F)} e^{-BN\delta_N^2} > (\ln N)^{-\frac{2(2t-3)}{2t+3}+2\epsilon}, \mathcal{C}_N \right) \\ &\quad + o(1) \\ &\le \mathbb{P}_{F_0}^N \left(\int_{C(D)} \|F - F_0\|_{L^\infty}^2 \prod_{i=1}^N \frac{p_F}{p_{F_0}}(Y_i, X_i) \, \mathrm{d}\Pi_N(F) > (\ln N)^{-\frac{2(2t-3)}{2t+3}+2\epsilon} a e^{(B-A-2)N\delta_N^2} \right) \\ &\quad + o(1), \end{aligned}$$

which is bounded from above, using Markov's inequality and Fubini's theorem, by

$$\begin{aligned}(\ln N)^{\frac{2(2t-3)}{2t+3}-2\epsilon} a^{-1} e^{-(B-A-2)N\delta_N^2} \int_{C(D)} \|F-F_0\|_{L^\infty}^2 \mathbb{E}_{F_0}^N \left(\prod_{i=1}^N \frac{p_F}{p_{F_0}}(Y_i,X_i)\right) \mathrm{d}\Pi_N(F) \\ = (\ln N)^{\frac{2(2t-3)}{2t+3}-2\epsilon} a^{-1} e^{-(B-A-2)N\delta_N^2} \int_{C(D)} \|F-F_0\|_{L^\infty}^2 \,\mathrm{d}\Pi_N(F).\end{aligned} \tag{3.7}$$

By using Fernique's theorem (see e.g. [GN21, Exercises 2.1.1, 2.1.2 and 2.1.5]) one has $\mathbb{E}^{\Pi_N}\|F\|_{L^\infty}^2 < \infty$. Taking $B > A+2$, from (3.7) we conclude

$$\mathbb{P}_{F_0}^N\left(\mathbb{E}^\Pi[\|F-F_0\|_{L^\infty}\mathbb{1}_{\|F\|_{C^t}>M}|Y^{(N)},X^{(N)}] > (\ln N)^{-\frac{(2t-3)}{2t+3}+\epsilon}\right) \to 0. \tag{3.8}$$

Part II: Estimating $\mathbb{E}^\Pi[\|F-F_0\|_{L^\infty}\mathbb{1}_{\|F\|_{C^t}\le M}|Y^{(N)},X^{(N)}]$. The above discussions still valid if we replace M by a (possibly larger) $M = M(\sigma,B,s,t,F_0) > 0$ with $\|F_0\|_{L^\infty} \le M$. Since $n = \Phi\circ F$ and $n_0 = \Phi\circ F_0$, by (i), mean value theorem and inverse function theorems, there exists η lying between $n_0(x)$ and $n(x)$ such that

$$|F(x)-F_0(x)| = \frac{1}{|\Phi'(\Phi^{-1}(\eta))|}|n(x)-n_0(x)|,$$

for all $x\in D$. Since $F, F_0 \in [\Phi(-M),\Phi(M)]$, by (i) we reach

$$|F(x)-F_0(x)| \le \frac{1}{\min_{z\in[-M,M]}\Phi'(z)}|n(x)-n_0(x)| \lesssim |n(x)-n_0(x)| \quad \text{for all } x\in D.$$

Therefore we see that

$$\|F-F_0\|_{L^\infty}\mathbb{1}_{\|F\|_{C^t}\le M} \lesssim \|n-n_0\|_{L^\infty}\mathbb{1}_{\|F\|_{C^t}\le M},$$

and

$$\mathbb{E}^\Pi[\|F-F_0\|_{L^\infty}\mathbb{1}_{\|F\|_{C^t}\le M}|Y^{(N)},X^{(N)}] \lesssim \mathbb{E}^{\tilde\Pi}[\|n-n_0\|_{L^\infty}|Y^{(N)},X^{(N)}]. \tag{3.9}$$

Let $C>0$ be a constant to be determined later. From (3.9), we see that

$$\begin{aligned}&\mathbb{P}_{F_0}^N\left(\mathbb{E}^\Pi[\|F-F_0\|_{L^\infty}\mathbb{1}_{\|F\|_{C^t}\le M}|Y^{(N)},X^{(N)}] > C(\ln N)^{-\frac{(2t-3)}{2t+3}+\epsilon}\right)\\ &\quad\le C(\ln N)^{-\frac{2t-3}{2t+3}+\epsilon} + \mathbb{E}^{\tilde\Pi}[\|n-n_0\|_{L^\infty}\mathbb{1}_{\|n-n_0\|_{L^\infty}\ge C(\ln N)^{-\frac{2t-3}{2t+3}+\epsilon}}|Y^{(N)},X^{(N)}],\end{aligned}$$

We can modify the arguments as in **Part I** above by replacing the event $\{F : \|F\|_{C^t} > M\}$ (resp. (2.7b)) by the event $\left\{n : \|n-n_0\|_{L^\infty} > C(\ln N)^{-\frac{2t-3}{2t+3}+\epsilon}\right\}$ (resp. (2.9)) with $C = C(\sigma,n_0,K,s,t,D,\kappa,\epsilon,M_0) > 0$ given in Theorem 2.4 to show that

$$\mathbb{P}_{F_0}^N\left(\mathbb{E}^\Pi[\|F-F_0\|_{L^\infty}\mathbb{1}_{\|F\|_{C^t}\le M}|Y^{(N)},X^{(N)}] > C(\ln N)^{-\frac{(2t-3)}{2t+3}+\epsilon}\right) \to 0. \tag{3.10}$$

Finally, putting together (3.2), (3.8), and (3.10) yields Theorem 2.5 with $\tilde C = \max\{C,1\}$. □

Next, we prove the optimality of contraction rate in Corollary 2.6 in the minimax sense.

Proof of Theorem 2.7. We apply the method in the proof of the lower bound [AN19, Theorem 2] to our case here. The idea is to find $n_0, n_1 \in \hat{\mathcal{F}}_\beta^s$ (both are allowed to depend on N) such that, for some small ζ sufficiently small,

(a) $\|n_0-n_1\|_{L^\infty} \ge \theta_{N,\delta} := (\ln N)^{-\delta}$.
(b) $\mathrm{KL}(p_{n_0}^{\otimes N}, p_{n_1}^{\otimes N}) \le \zeta$,

where $p_n^{\otimes N}$ is the Radon Nikodym derivative of the joint law $\mathbb{P}_n^N$ and the Kullback-Leibler divergence $\mathrm{KL}(p_{n_0}^{\otimes N}, p_{n_1}^{\otimes N})$ is defined by

$$\mathrm{KL}(p_{n_0}^{\otimes N}, p_{n_1}^{\otimes N}) = \mathbb{E}_{n_0}^N\left[\log\frac{p_{n_0}^{\otimes N}}{p_{n_1}^{\otimes N}}\right]. \tag{3.11}$$

By independence, we note that

$$\mathrm{KL}(p_{n_0}^{\otimes N}, p_{n_1}^{\otimes N}) = N\mathbb{E}_{n_0}^1\left[\log\frac{p_{n_0}(Y_1,X_1)}{p_{n_1}(Y_1,X_1)}\right],$$

and [GN20, Lemma 23] implies

$$\mathrm{KL}(p_{n_0}^{\otimes N}, p_{n_1}^{\otimes N}) = \frac{N}{2\sigma^2}\|G(n_0)-G(n_1)\|^2_{L^2(\mathcal{S}^2\times\mathcal{S}^2)}, \tag{3.12}$$

then using the standard arguments as in [GN21, Section 6.3.1] (see also [Tsy09, Chapter 2]), we conclude the theorem.

For the sake of completeness, here we present the details. From condition (a), we see that $\psi = \mathbb{1}_{\{\|\tilde{n}-n_1\|_{L^\infty}<\|\tilde{n}-n_0\|_{L^\infty}\}}$ yields a test of

$$H_0: n = n_0 \quad \text{against} \quad H_1: n = n_1. \tag{3.13}$$

It follows from a general reduction principle that

$$\inf_{\tilde{n}}\sup_{n\in\hat{\mathcal{F}}^s_\beta}\mathbb{P}_n^N\left(\|\tilde{n}-n\|_{L^\infty}\geq\frac{1}{2}\theta_{N,\delta}\right)\geq\inf_\psi\max\left\{\mathbb{P}^N_{n_0}(\psi\neq0),\mathbb{P}^N_{n_1}(\psi\neq1)\right\},$$

where the second infimum is over all tests ψ of (3.13). Similar to the proof of [GN21, Theorem 6.3.2], we introduce the event

$$\Omega = \left\{\frac{p_{n_0}^{\otimes N}}{p_{n_1}^{\otimes N}}\geq\frac{1}{2}\right\}.$$

Note that

$$\mathbb{P}^N_{n_0}(\psi\neq0)\geq\mathbb{E}^N_{n_0}[\mathbb{1}_\Omega\psi]=\mathbb{E}^N_{n_1}\left[\frac{p_{n_0}^{\otimes N}}{p_{n_1}^{\otimes N}}\mathbb{1}_\Omega\psi\right]\geq\frac{1}{2}[\mathbb{P}^N_{n_1}(\psi=1)-\mathbb{P}^N_{n_1}(\Omega^c)].$$

Let $p_1=\mathbb{P}^N_{n_1}(\psi=1)$, then

$$\begin{aligned}&\max\left\{\mathbb{P}^N_{n_0}(\psi\neq0),\mathbb{P}^N_{n_1}(\psi\neq1)\right\}\\&\quad\geq\max\left\{\frac{1}{2}(p_1-\mathbb{P}^N_{n_1}(\Omega^c)),1-p_1\right\}\\&\quad\geq\inf_{p\in[0,1]}\max\left\{\frac{1}{2}(p-\mathbb{P}^N_{n_1}(\Omega^c)),1-p\right\}.\end{aligned}$$

It is clear that the infimum above is attained when $\frac{1}{2}(p-\mathbb{P}^N_{n_1}(\Omega^c))=1-p$ and has the value $\frac{1}{3}\mathbb{P}^N_{n_1}(\Omega)$. Hence,

$$\inf_{\tilde{n}}\sup_{n\in\hat{\mathcal{F}}^s_\beta}\mathbb{P}_n^N\left(\|\tilde{n}-n\|_\infty\geq\frac{1}{2}\theta_{N,\delta}\right)\geq\frac{1}{3}\mathbb{P}^N_{n_1}(\Omega). \tag{3.14}$$

Next, let's estimate

$$\begin{aligned}\mathbb{P}^N_{n_1}(\Omega)&=\mathbb{P}^N_{n_1}[p_{n_0}^{\otimes N}/p_{n_1}^{\otimes N}\leq2]=1-\mathbb{P}^N_{n_1}[\log(p_{n_0}^{\otimes N}/p_{n_1}^{\otimes N})>\log2]\\&\geq1-\mathbb{P}^N_{n_1}[|\log(p_{n_0}^{\otimes N}/p_{n_1}^{\otimes N})|>\log2]\\&\geq1-(\log2)^{-1}\mathbb{E}^N_{n_1}[|\log(p_{n_0}^{\otimes N}/p_{n_1}^{\otimes N})|]\quad\text{(by Markov's inequality)}.\end{aligned}$$

Using the second Pinsker inequality [GN21, Proposition 6.1.7b] and condition (b), we have

$$\begin{aligned}\frac{1}{3}\mathbb{P}_{n_1}^N(\Omega) &\geq \frac{1}{3}\left(1-(\log 2)^{-1}[\mathrm{KL}(p_{n_0}^{\otimes N},p_{n_1}^{\otimes N})+\sqrt{2\mathrm{KL}(p_{n_0}^{\otimes N},p_{n_1}^{\otimes N})}]\right)\\ &\geq \frac{1}{3}\left(1-(\log 2)^{-1}(\zeta+\sqrt{2\zeta})\right).\end{aligned}$$

We now choose ζ sufficiently small such that the last term above is bounded below by $1-\varepsilon$. Then the estimate (2.12) follows in view of (3.14).

The remaining task is to find n_0, n_1 satisfying conditions (a) and (b). For $\delta > \frac{5s}{3}$, setting $\theta = \theta_{N,\delta} = (\ln N)^{-\delta}$ in Theorem B.1, there exist $n_0, n_1 \in \hat{\mathcal{F}}_\beta^s$ satisfying $\|n_0 - n_1\|_\infty > \theta_{N,\delta}$ and

$$\|u_{n_0}^\infty - u_{n_1}^\infty\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)} \leq 2\exp(-\theta^{-\frac{3}{5s}}) = 2\exp(-(\ln N)^{\frac{3\delta}{5s}}).$$

To verify condition (b), we use (3.12) to conclude that

$$\begin{aligned}\mathrm{KL}(p_{n_0}^{\otimes N},p_{n_1}^{\otimes N}) &= \frac{N}{2\sigma^2}\|u_{n_0}^\infty - u_{n_1}^\infty\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)}^2\\ &\leq \frac{4N}{2\sigma^2}\exp(-2(\ln N)^{\frac{3\delta}{5s}})\\ &= \frac{2}{\sigma^2}\exp(\ln N - 2(\ln N)^{\frac{3\delta}{5s}}) \to 0 \text{ as } N\to\infty,\end{aligned}$$

since $\frac{3\delta}{5s} > 1$. Therefore, we can make ζ as small as we wish by taking N sufficiently large. $\square$

Appendix A. Inverse scattering problem

In what follows, assume that

$$n \in C^1(\mathbb{R}^3),\ \ \mathrm{supp}(1-n)\subset D,\ \ 0\leq n\leq M_0 \ \text{ with } \ M_0\geq 1. \tag{A.1}$$

We first discuss the existence and uniqueness of the scattered field u_n^{sca}. It is known that $w := u_n^{\mathrm{sca}}$ satisfies the following boundary value problem [CCH23, (1.51)–(1.52)]:

$$\begin{cases} -\Delta w - \kappa^2 n w = \kappa^2(n-1)u^{\mathrm{inc}} & \text{in } B_R,\\ \partial w/\partial r = S_R(w|_{\partial B_R}) & \text{on } \partial B_R,\end{cases} \tag{A.2}$$

where B_R is a ball of radius R such that $\bar{D}\subset B_R$. Here $S_R : H^{1/2}(\partial B_R)\to H^{-1/2}(\partial B_R)$ is the Dirichlet-to-Neumann map, defined for $g\in H^{1/2}(\partial B_R)$ by $S_R g = (\partial u_g/\partial r)|_{\partial B_R}$, where u_g is the solution of the Helmholtz equation satisfying the Sommerfeld radiation condition in $R^3\setminus B_R$ and the Dirichlet condition $u_g = g$ on ∂B_R. It has been shown that

$$\mathrm{Re}\langle S_R(v),v\rangle \leq 0 \quad\text{and}\quad \mathrm{Im}\langle S_R(v),v\rangle\geq 0,\ \ \forall\ v\in H^{1/2}(\partial B_R), \tag{A.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\partial B_R)$ and $H^{1/2}(\partial B_R)$, see, for example, [CCH23, Definition 1.36 and (1.50)].

To proceed further, let us replace the right-hand side of the first equation in (A.2) by a general source term f with $\mathrm{supp}(f)\subset B_R$, i.e.,

$$\begin{cases} -\Delta w - \kappa^2 n w = f & \text{in } B_R,\\ \partial w/\partial r = S_R(w|_{\partial B_R}) & \text{on } \partial B_R,\end{cases} \tag{A.4}$$

In view of the integration by parts, (A.4) is equivalent to the following variational formulation: find $w\in H^1(B_R)$ such that for all $v\in H^1(B_R)$,

$$a_1(w,v)+a_2(w,v) = F(v),$$

where

$$a_1(w,v) = \int_{B_R} \nabla w \cdot \nabla \bar{v} \, \mathrm{d}x - \kappa^2 \int_{B_R} n w \bar{v} \, \mathrm{d}x + \kappa^2 (M_0+1) \int_{B_R} w \bar{v} \, \mathrm{d}x - \langle S_R(w), \bar{v} \rangle,$$

$$a_2(w,v) = -\kappa^2 (M_0+1) \int_{B_R} w \bar{v} \, \mathrm{d}x,$$

and

$$F(v) = \int_{B_R} f \bar{v} \, \mathrm{d}x.$$

We can see that

$$\operatorname{Re} a_1(w,w) = \int_{B_R} \left(|\nabla w|^2 + k^2 (M_0 + 1 - n)|w|^2 \right) \mathrm{d}x \geq C(\kappa) \|w\|^2_{H^1(B_R)}.$$

In other words, $a_1(\cdot,\cdot)$ is strictly coercive. Combining the Riesz representation theorem and the Lax-Milgram theorem, there exists an invertible operator $\mathcal{A} : H^1(B_R) \to (H^1(B_R))^*$, where $(H^1(B_R))^*$ is the dual space of $H^1(B_R)$, such that

$$a_1(w,v) = (\mathcal{A}w, v)_{H^1(B_R)} \quad \forall \ v \in H^1(B_R).$$

Similarly, define the bounded linear operator $\mathcal{B} : H^1(B_R) \to (H^1(B_R))^*$ by $a_2(w,v) = (\mathcal{B}w, v)_{H^1(B_R)}$. It is not difficult to see that $\mathcal{B}$ is compact. Consequently, $\mathcal{A}+\mathcal{B}$ is a Fredholm operator. By the Fredholm alternative, $\mathcal{A}+\mathcal{B} : H^1(B_R) \to (H^1(B_R))^*$ is bounded invertible provided the kernel of $\mathcal{A}+\mathcal{B}$ is trivial, which follows the uniqueness of the scattered solution (by the combination of the Rellich lemma and the unique continuation property). Furthermore, we have the following estimate:

$$\|w\|_{H^1(B_R)} \leq C \|f\|_{(H^1(B_R))^*}, \tag{A.5}$$

where $C = C(D,\kappa,M_0)$. Let $f = \kappa^2(n-1)u^{\mathrm{inc}}$ and $w = u_n^{\mathrm{sca}}$, then (A.5) implies

$$\|u_n^{\mathrm{sca}}\|_{H^1(B_R)} \leq C \|1-n\|_{L^2(D)}, \tag{A.6}$$

uniformly in $\theta \in \mathcal{S}^2$. We now choose $R' < R$ such that $\overline{D} \subset B_{R'}$. By the interior estimate [GT01, Theorem 8.8], we further have

$$\|u_n^{\mathrm{sca}}\|_{H^2(B_{R'})} \leq C \|1-n\|_{L^2(D)},$$

which, by the Sobolev imbedding theorem, implies

$$\|u_n^{\mathrm{sca}}\|_{C(D)} \leq C \|1-n\|_{L^2(D)}. \tag{A.7}$$

The scattering amplitude $u_n^\infty(\theta',\theta)$ can be expressed explicitly by

$$\begin{aligned} u_n^\infty(\theta',\theta) &= -\frac{\kappa^2}{4\pi} \int_{\mathbb{R}^3} e^{-\mathbf{i}\kappa\theta'\cdot y} (1-n)(y) u(y,\theta) \, \mathrm{d}y \\ &= -\frac{\kappa^2}{4\pi} \int_{D} e^{-\mathbf{i}\kappa\theta'\cdot y} (1-n)(y) u(y,\theta) \, \mathrm{d}y, \end{aligned} \tag{A.8}$$

where $u(y,\theta) = u^{\mathrm{inc}}(y,\theta) + u_n^{\mathrm{sca}}(y,\theta)$ is the total field with $u^{\mathrm{inc}}(y,\theta) = e^{\mathbf{i}\kappa y\cdot\theta}$, see [CCH23, (1.22)] or [CK19, (8.28)] or [Ser17, Page 232]. From (A.7) and (A.8), we have

$$\|u_n^\infty\|_{L^\infty(\mathcal{S}^2\times\mathcal{S}^2)} \leq C\|1-n\|_{L^2(D)} \leq S < \infty, \tag{A.9}$$

with $S = S(D,k,M_0) = C(1+M_0)$. Since

$$-\Delta \nabla u_n^{\mathrm{sca}} - \kappa^2 n \nabla u_n^{\mathrm{sca}} = \kappa^2 \nabla n u_n^{\mathrm{sca}} + \kappa^2 \nabla n u^{\mathrm{inc}} + \kappa^2 (n-1) \nabla u^{\mathrm{inc}} \quad \text{in } B_R,$$

applying the interior estimate and the Sobolev imbedding theorem again, we have that

$$\|u_n^{\mathrm{sca}}\|_{C^1(D)} \leq C(1+\|n\|_{C^1(D)}) \|1-n\|_{L^\infty(D)} \leq C(1+\|n\|_{C^1(D)}), \tag{A.10}$$

with $C = C(D,k,M_0)$.

Next, assume that n_1, n_2 satisfy (A.1). For each open set Ω in Euclidean space, we observe that

$$\begin{aligned}\|f_1 f_2\|_{(H^1(\Omega))^*} &= \sup_{\|\varphi\|_{H^1(\Omega)}\le 1}\left|\int_\Omega f_1 f_2 \varphi\,\mathrm{d}x\right| \\ &\le \|f_1\|_{(H^1(\Omega))^*}\sup_{\|\varphi\|_{H^1(\Omega)}\le 1}\|f_2\varphi\|_{H^1(\Omega)} \le C\|f_1\|_{(H^1(\Omega))^*}\|f_2\|_{C^1(\Omega)}.\end{aligned} \tag{A.11}$$

Let $w = u_{n_2}^{\rm sca} - u_{n_1}^{\rm sca}$ and $f = \kappa^2(n_2-n_1)u_{n_1}^{\rm sca} + \kappa^2(n_2-n_1)u^{\rm inc}$, then combining (A.5), (A.7), (A.10) and (A.11), yields

$$\|u_{n_1}^{\rm sca} - u_{n_2}^{\rm sca}\|_{H^1(B_R)} \le C(1+\|n_1\|_{C^1(D)})\|n_1-n_2\|_{(H^1(D))^*}, \tag{A.12}$$

uniformly in θ and $C = C(D,k,M_0)$. Then it yields from (A.8) that

$$\begin{aligned}&\left(\int_{\mathcal{S}^2\times\mathcal{S}^2}|(u_{n_1}^\infty - u_{n_2}^\infty)(\theta',\theta)|^2\,\mathrm{d}\omega\right)^{\frac12} \\ &= \frac{\kappa^2}{4\pi}\left(\int_{\mathcal{S}^2\times\mathcal{S}^2}\left|\int_D e^{-\mathbf{i}\kappa\theta'\cdot y}[(1-n_1)(e^{\mathbf{i}\kappa\theta\cdot y}+u_{n_1}^{\rm sca}) - (1-n_2)(e^{\mathbf{i}\kappa\theta\cdot y}+u_{n_2}^{\rm sca})]\,\mathrm{d}y\right|^2\mathrm{d}\omega\right)^{\frac12} \\ &\le \frac{\kappa^2}{4\pi}\int_D\left(\int_{\mathcal{S}^2\times\mathcal{S}^2}\left|e^{-\mathbf{i}\kappa\theta'\cdot y}[(1-n_1)(e^{\mathbf{i}\kappa\theta\cdot y}+u_{n_1}^{\rm sca}) - (1-n_2)(e^{\mathbf{i}\kappa\theta\cdot y}+u_{n_2}^{\rm sca})]\right|^2\mathrm{d}\omega\right)^{\frac12}\mathrm{d}y \\ &= \frac{\kappa^2}{4\pi}\int_D\left(\int_{\mathcal{S}^2\times\mathcal{S}^2}\left|(n_2-n_1)e^{\mathbf{i}\kappa\theta\cdot y} - (n_1-n_2)u_{n_1}^{\rm sca} + (1-n_2)(u_{n_1}^{\rm sca}-u_{n_2}^{\rm sca})\right|^2\mathrm{d}\omega\right)^{\frac12}\mathrm{d}y,\end{aligned} \tag{A.13}$$

where the inequality above is due to the integral form of Minkowski's inequality. Plugging (A.6), (A.10) and (A.12) into (A.13) gives

$$\|u_{n_1}^\infty - u_{n_2}^\infty\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)} \le C(1+\|n_1\|_{C^1(D)}\vee\|n_2\|_{C^1(D)})\|n_1-n_2\|_{(H^1(D))^*}, \tag{A.14}$$

where $C = C(D,k,M_0)$.

Next, we recall the following stability estimate for the determination of the potential from the scattering amplitude.

Theorem A.1. [HH01, Theorem 1.2] *Let $t > 3/2$, $M > 0$, and $0 < \epsilon < \frac{2t-3}{2t+3}$ be given constants. Assume that $1-n_j \in H^t(\mathbb{R}^3)$ satisfying $\|1-n_j\|_{H^t(\mathbb{R}^3)} \le M$ and $\operatorname{supp}(1-n_j)\subset D$, $j=1,2$. Then*

$$\|n_1-n_2\|_{L^\infty(D)} \le C[-\ln^-(\|u_{n_1}^\infty - u_{n_2}^\infty\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)})]^{-\left(\frac{2t-3}{2t+3}-\epsilon\right)}, \tag{A.15}$$

where $C = C(D,t,k,M,\epsilon)$ and

$$\ln^-(z) = \begin{cases}\ln(z), & \text{if } t\le e^{-1},\\ -1, & \text{otherwise.}\end{cases}$$

Remark A.2. Here the constant M may be different from M_0 given above.

Appendix B. Optimality of the stability estimate

The purpose of this section is to show that the logarithmic estimate obtained in Theorem A.1 is optimal by deriving an instability estimate. Similar instability estimate was already proved in [Isa13]. To make the paper self contained, we present our own proof here and also slightly refine the estimate obtained in [Isa13]. Throughout this section, we denote $q(x) = n(x) - 1$.

Theorem B.1. *Consider the inverse scattering problem (1.1a)–(1.1c) with frequency $\kappa > 0$. Let integer $s > 0$ be a given regularizing parameter. Then there exists constants $\beta = \beta(s,\kappa) > 0$ and $\vartheta_0 = \vartheta_0(s,\kappa) > 0$ such that: for each $0 < \vartheta < \vartheta_0$ there exists non-negative $q_1, q_2 \in C^\infty(\mathbb{R}^3)$ with $\operatorname{supp}(q_j) \subset B_{\frac{1}{2}}$ satisfying the apriori bound $\|q_j\|_{C^s(\mathbb{R}^3)} \le \beta$ and*

$$\|q_1 - q_2\|_{L^\infty(\mathbb{R}^3)} \ge \vartheta, \quad \|u^\infty_{1+q_1} - u^\infty_{1+q_2}\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)} \le 2\exp(-\vartheta^{-\frac{3}{5s}}).$$

Remark B.2. From the properties of the Hilbert-Schmidt norm [Con90, Exercise IX.2.19(h)], one has

$$\begin{aligned} &\|G(1+q)\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} = \|u^\infty_{1+q}\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)}, \\ &\|G(1+q_1) - G(1+q_2)\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} = \|u^\infty_{1+q_1} - u^\infty_{1+q_2}\|_{L^2(\mathcal{S}^2\times\mathcal{S}^2)}. \end{aligned} \tag{B.1}$$

Recall $G(1+q)$ is the far-field operator given in (1.2).

Our main strategy is to modify the ideas in [DCR03] (see also [KRS21] for more details about the mechanism). Given any $\vartheta > 0$, $s \ge 0$ and $\beta > 0$, we consider the following set:

$$\mathcal{N}^\vartheta_{s\beta} := \left\{ q \ge 0 : \operatorname{supp}(q) \subset B_{1/2}, \quad \|q\|_{L^\infty(\mathbb{R}^d)} \le \vartheta, \quad \|q\|_{C^s(\mathbb{R}^d)} \le \beta \right\},$$

where B_r denotes the ball or radius r centered at the origin. The following lemma verifies the assumption (a) of [DCR03, Theorem 2.2], can be proved as in [Man01, Lemma 2] (we omit the proof), see also [Isa13, KUW21, KW22, ZZ19]. We refer [KT61] for a version in a more abstract form.

Lemma B.3. *Fix $d \in \mathbb{N}$ and $s \ge 0$. There exists a constant*[1] *$\mu = \mu(d,s) > 0$ such that the following statement holds for all $\beta > 0$ and for all $\vartheta \in (0, \mu\beta)$: there exists a ϑ-discrete subset $\mathcal{Z}_\vartheta$ of $(\mathcal{N}^\vartheta_{s\beta}, \|\cdot\|_{L^\infty(\mathbb{R}^d)})$*[2] *such that*

$$|\mathcal{Z}_\vartheta| \ge \exp\left(\left(\frac{\log 2}{2}\right)^d \left(\frac{\mu\beta}{\vartheta}\right)^{\frac{d}{s}} \right). \tag{B.2}$$

In addition, all elements in $\mathcal{Z}_\vartheta$ are in $C^\infty(\mathbb{R}^d)$.

Similar to [DCR03], the proof of Theorem B.1 is quite delicate, which is not an obvious consequence of the abstract theorem in [DCR03, Theorem 2.2]. From the asymptotic expansion (1.1c), it is easy to see that

$$|x||u^{\mathrm{sca}}_{1+q}(x,\theta)| = O(1) \text{ as } |x| \to \infty \text{ uniformly for all } \theta' = x/|x| \in \mathcal{S}^2.$$

A crucial point is to bound $|x||u^{\mathrm{sca}}_{1+q}(x,\theta)|$ for all $|x| \ge 2$, by some constant which is independent of θ and q, like [DCR03, (4.21)]. From now on, for simplicity, we restrict ourselves for the case when $d = 3$. We now prove the following lemma.

Lemma B.4. *Let $\kappa > 0$ and let $\|q\|_{L^\infty(\mathbb{R}^3)} \le 1$ with $\operatorname{supp}(q) \subset B_{1/2}$. Then there exists a constant $C = C(\kappa) > 0$ such that the following uniform decay estimate holds:*

$$\sup_{\theta\in\mathcal{S}^2} |u^{\mathrm{sca}}_{1+q}(x,\theta)| \le C|x|^{-1} \quad \textit{for all } x \in \mathbb{R}^3 \setminus \overline{B_2}.$$

Proof. This lemma is an easy consequence of (A.6) and [Ron03, Lemma 3.2] (with $R = 1$). ∎

[1]In fact, one can choose $\mu = d^{-\frac{s}{2}} \|\psi\|^{-1}_{C^s(\mathbb{R}^d)}$ for some $\psi \in C^\infty_c((-1/2,1/2)^d)$ with $\|\psi\|_{L^\infty(\mathbb{R}^d)} = 1$.

[2]This means that $\|q_1 - q_2\|_{L^\infty(\mathbb{R}^d)} \ge \vartheta$ for all $q_1, q_2 \in \mathcal{Z}_\vartheta \subset \mathcal{N}^\vartheta_{s\beta}$.

As in [HH01], we introduce the following index set

$$\mathcal{M} := \{(m,j,n,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z} : |j| \leq m \text{ and } |k| \leq n\}.$$

Let $\{Y_m^j : m \in \mathbb{Z}_{\geq 0},\ |j| \leq m\}$ be the set of spherical harmonics. For each $F \in \mathrm{HS}(L^2(\mathcal{S}^2), L^2(\mathcal{S}^2))$, we denote $F_{mn}^{jk} := \left(FY_m^j, Y_n^k\right)_{L^2(\mathcal{S}^2),L^2(\mathcal{S}^2)}$. Accordingly, we write

$$G(1+q)_{mn}^{jk} := \left(G(1+q)Y_m^j, Y_n^k\right)_{L^2(\mathcal{S}^2),L^2(\mathcal{S}^2)}.$$

Lemma B.5. *Let $\kappa > 0$ and let $\|q\|_{L^\infty(\mathbb{R}^3)} \leq 1$ with $\operatorname{supp}(q) \subset B_{1/2}$. Then there exist constants $c_{\rm abs} = c_{\rm abs}(\kappa) > 0$ and $C_{\rm abs} = C_{\rm abs}(\kappa) > e$ such that*

$$|G(1+q)_{mn}^{jk}| \leq C_{\rm abs} e^{-c_{\rm abs}\max\{m,n\}} \quad \text{for all } (m,j,n,k) \in \mathcal{M}.$$

Proof. By [CK19, Theorem 2.15], we can write

$$u_{1+q}^{\rm sc}(x,\theta) = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} \left(u_{1+q}^{\rm sca}\right)_m^j(\theta) h_m^{(1)}(\kappa|x|) Y_m^j(\theta'),$$

where $h_m^{(1)}$ is the spherical Hankel function of the first kind or order m. In fact,

$$\left(u_{1+q}^{\rm sca}\right)_m^j(\theta) h_m^{(1)}(\kappa r) = \int_{\mathcal{S}^2} u_{1+q}^{\rm sca}(r\theta',\theta)\overline{Y_m^j(\theta')}\,\mathrm{d}\theta' \quad \text{for all } r > 1,$$

see the proof of [CK19, Theorem 2.16]. For each $\theta \in \mathcal{S}^2$, also from [CK19, Theorem 2.16], we have the following expansion

$$u_{1+q}^{\infty}(\theta',\theta) = \sum_{m=0}^{\infty} \sum_{j=-m}^{m} \frac{1}{\mathbf{i}^{m+1}} \left(u_{1+q}^{\rm sca}\right)_m^j(\theta) Y_m^j(\theta'),$$

which converges uniformly in $\theta' \in \mathcal{S}^2$, and hence

$$\int_{\mathcal{S}^2} u_{1+q}^{\infty}(\theta',\theta)\overline{Y_m^j(\theta')}\,\mathrm{d}\theta' = \frac{1}{\mathbf{i}^{m+1}}\left(u_{1+q}^{\rm sc}\right)_m^j(\theta) = |h_m^{(1)}(\kappa r)|^{-1}\int_{\mathcal{S}^2} u_{1+q}^{\rm sca}(r\theta',\theta)\overline{Y_m^j(\theta')}\,\mathrm{d}\theta', \tag{B.3}$$

for all $r > 1$. We now combine (B.3) with Lemma B.4 to obtain

$$\left|\int_{\mathcal{S}^2} u_{1+q}^{\infty}(\theta',\theta)\overline{Y_m^j(\theta')}\,\mathrm{d}\theta'\right| \leq Cr^{-1}|h_m^{(1)}(\kappa r)|^{-1},$$

uniformly in $\theta' \in \mathcal{S}^2$. We now choose $\kappa r = 2$ and reach

$$\left|\int_{\mathcal{S}^2} u_{1+q}^{\infty}(\theta',\theta)\overline{Y_m^j(\theta')}\,\mathrm{d}\theta'\right| \leq C|h_m^{(1)}(2)|^{-1}. \tag{B.4}$$

From [KW22, (48)], we see that

$$|h_m^{(1)}(2)|^{-1} \leq 2\sqrt{\pi}\frac{1}{\Gamma(m+\frac{1}{2})} \quad \text{for all } m = 0,1,2,\cdots. \tag{B.5}$$

In view of the quantitative Stirling's formula [Rob55], we obtain

$$\begin{aligned} |h_m^{(1)}(2)|^{-1} &\leq 2\sqrt{\pi}\frac{1}{(m-1)!} \leq \sqrt{2}(m-1)^{-(m-\frac{1}{2})}e^{m-1}e^{-\frac{1}{12m-11}} \\ &\leq C(m-1)^{-(m-1)}e^{m-1} \quad \text{for all } m = 6,7,8,\cdots. \end{aligned} \tag{B.6}$$

From (B.5), it is clearly that

$$|h_m^{(1)}(2)|^{-1} \leq C \quad \text{for } m = 0,1,2,\cdots,5.$$

Combining (B.4) and (B.6) implies

$$\begin{aligned}|G(1+q)^{kj}_{nm}| &= \left|\int_{\mathcal{S}^2}\int_{\mathcal{S}^2} u^\infty_{1+q}(\theta',\theta)\overline{Y^j_m(\theta')}Y^k_n(\theta)\,\mathrm{d}\theta'\,\mathrm{d}\theta\right| \\ &\leq |\mathcal{S}^2|^{\frac12}\left|\int_{\mathcal{S}^2} u^\infty_{1+q}(\theta',\theta)\overline{Y^j_m(\theta')}\,\mathrm{d}\theta'\right| \leq Ce^{-cm} \quad \text{for all } (m,j,n,k)\in\mathcal{M}.\end{aligned}$$

Finally, by the first reciprocity relation [CK19, Theorem 8.8], we have

$$|G(1+q)^{jk}_{mn}| = |G(1+q)^{kj}_{nm}| \quad \text{for all } (m,j,n,k)\in\mathcal{M},$$

and the lemma is proved. □

We also need a technical lemma.

Lemma B.6. *Consider the normed space*

$$\mathrm{HS}' = \left\{F\in \mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2)) : \|F\|_{\mathrm{HS}'}<\infty\right\},$$

where

$$\|F\|_{\mathrm{HS}'} := \sup_{(m,j,n,k)\in\mathcal{M}} |F^{jk}_{mn}|(1+\max\{m,n\})^3.$$

Then

$$\|F\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} \leq C_0\|F\|_{\mathrm{HS}'}, \quad C_0 = 4\left(\sum_{n=0}^\infty \frac{1}{(1+n)^3}\right)^{\frac12}<\infty.$$

Proof. Since

$$\|F\|^2_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} = \sum_{(m,j,n,k)\in\mathcal{M}} |F^{jk}_{mn}|^2, \tag{B.7}$$

we can estimate

$$\begin{aligned}\|F\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} &= \left(\sum_{(m,j,n,k)\in\mathcal{M}} |F^{jk}_{mn}|^2\frac{(1+\max\{m,n\})^6}{(1+\max\{m,n\})^6}\right)^{\frac12} \\ &\leq \left(\sum_{(m,j,n,k)\in\mathcal{M}} \frac{1}{(1+\max\{m,n\})^6}\right)^{\frac12}\|F\|_{\mathrm{HS}'} \\ &\leq 2\left(\sum_{m,n=0,1,2,\cdots} \frac{1}{(1+\max\{m,n\})^4}\right)^{\frac12}\|F\|_{\mathrm{HS}'} \\ &\leq 2\left(\left(\sum_{n\geq m\geq 0}+\sum_{m\geq n\geq 0}\right)\frac{1}{(1+\max\{m,n\})^4}\right)^{\frac12}\|F\|_{\mathrm{HS}'} \\ &= 4\left(\sum_{n=0}^\infty\sum_{m=0}^n \frac{1}{(1+n)^4}\right)^{\frac12}\|F\|_{\mathrm{HS}'} = 4\left(\sum_{n=0}^\infty \frac{1}{(1+n)^3}\right)^{\frac12}\|F\|_{\mathrm{HS}'},\end{aligned}$$

which implies the lemma. □

We now construct a δ-net by following the procedures in [DCR03, Lemma 2.3].

Lemma B.7. *Let* $\mathcal{N}^1 := \{q \geq 0 : \mathrm{supp}\,(q) \subset B_{1/2}, \quad \|q\|_{L^\infty(\mathbb{R}^3)} \leq 1\}$. *Then there exists a constant* $C'_{\rm abs} = C'_{\rm abs}(\kappa) > e$ *such that: for each* $0 < \delta < 1/e$ *we can find a* δ*-net*[3] $\mathcal{Y}_\delta$ *for* $\big(G(1+\mathcal{N}^1), \|\cdot\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))}\big)$ *such that*

$$|\mathcal{Y}_\delta| \leq C'_{\rm abs} \exp(-\log\delta)^5. \tag{B.8}$$

Proof. Fix $\delta \in (0, 1/e)$. Let $\tilde{\ell}$ be the smallest positive integer such that

$$C_{\rm abs} e^{-c_{\rm abs}(t-1)}(1+t)^3 \leq \frac{\delta}{2C_0} \quad \text{for all } t \geq \tilde{\ell}, \tag{B.9}$$

where $C_{\rm abs} > e$ and $c_{\rm abs} > 0$ are the absolute constants appeared in Lemma B.5. One sees that there exists an absolute constant $C > 0$ such that

$$\tilde{\ell} \leq C \log\delta^{-1}. \tag{B.10}$$

We define a finite subset of the complex plane $\mathbb{C}$ by

$$\Psi_\delta := (\delta'\mathbb{Z} \cap [-C_{\rm abs}, C_{\rm abs}]) + \mathbf{i}\,(\delta'\mathbb{Z} \cap [-C_{\rm abs}, C_{\rm abs}])\,, \quad \delta' = \frac{(1+\tilde{\ell})^{-3}}{2C_0}\delta.$$

It is easy to see that

$$|\Psi_\delta| \leq \left(\frac{2C_{\rm abs}+1}{\delta'}\right)^2 = \left(\frac{2C_0(2C_{\rm abs}+1)(1+\tilde{\ell})^3}{\delta}\right)^2. \tag{B.11}$$

We now define the set

$$\mathcal{Y}_\delta := \left\{ F \in \mathrm{HS}(L^2(\mathcal{S}^2), L^2(\mathcal{S}^2)) : \begin{array}{l} F_{mn}^{jk} \in \Psi_\delta \text{ if } \max\{m,n\} \leq \tilde{\ell} \\ \text{and } F_{mn}^{jk} = 0 \text{ otherwise} \end{array} \right\}.$$

Let ℓ_* be the number of $(m,j,n,k) \in \mathcal{M}$ such that $\max\{m,n\} \leq \tilde{\ell}$, then we have

$$|\mathcal{Y}_\delta| = |\Psi_\delta|^{\ell_*} \leq |\Psi_\delta|^{16(1+\tilde{\ell})^4}. \tag{B.12}$$

Plugging (B.10) and (B.11) into (B.12) yields

$$|\mathcal{Y}_\delta| \leq \left(2C_0(2C_{\rm abs}+1)\frac{(C\log\delta^{-1})^3}{\delta}\right)^{16(C\log\delta^{-1})^4},$$

and, furthermore, from the trivial estimate $\log(1+t) \leq t$ for all $t \geq 0$, we see that

$$\begin{aligned} \log|\mathcal{Y}_\delta| &\leq 16(C\log\delta^{-1})^4 \log\left(2C_0(2C_{\rm abs}+1)\frac{(C\log\delta^{-1})^3}{\delta}\right) \\ &= 16(C\log\delta^{-1})^4 \left(\log(2C_0(2C_{\rm abs}+1)) + 3\log(C\log\delta^{-1}) + \log\delta^{-1}\right) \\ &\leq 16(C\log\delta^{-1})^4 \left(\log(2C_0(2C_{\rm abs}+1)) + 3C\log\delta^{-1} + \log\delta^{-1}\right) \\ &\leq C(\log\delta^{-1})^5, \end{aligned}$$

which gives (B.8).

The remaining task now is to verify that the set $\mathcal{Y}_\delta$ constructed above is a δ-net for $\big(G(1+\mathcal{N}^1), \|\cdot\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))}\big)$. Fix $q \in \mathcal{N}^1$ and consider the far-field pattern $G(1+q)$. For each $(m,j,n,k) \in \mathcal{M}$ with $\max\{m,n\} \leq \tilde{\ell}$, we choose $F_{mn}^{jk} \in \Psi_\delta$ be the closest element to $G(1+q)_{mn}^{jk}$; otherwise for those $(m,j,n,k) \in \mathcal{M}$ with $\max\{m,n\} > \tilde{\ell}$, we simply choose $F_{mn}^{jk} = 0$. We define the operator $F \in \mathcal{Y}_\delta$ by

$$\left(FY_m^j, Y_m^k\right)_{L^2(\mathcal{S}^2),L^2(\mathcal{S}^2)} := F_{mn}^{jk} \quad \text{for all } (m,j,n,k) \in \mathcal{M}.$$

[3]This means that for each operator $F_0 \in G(1+\mathcal{N}^1)$ there exists $F \in \mathcal{Y}_\delta$ which is approximate F_0 in the sense that $\|F_0 - F\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} \leq \delta$.

By Lemma B.5, we see that

$$G(1+\mathcal{N}^1) \subset B_{C_{\mathrm{abs}}} \subset \mathbb{C}.$$

Therefore, if $\max\{m,n\} \le \tilde{\ell}$, we have that

$$|G(1+q)^{jk}_{mn} - F^{jk}_{mn}|(1+\max\{m,n\})^3 \le \sqrt{2}\delta'(1+\tilde{\ell})^3 = \frac{\delta}{\sqrt{2}C_0}.$$

Otherwise, when $t := \max\{m,n\} > \tilde{\ell}$, from Lemma B.5 and (B.9), we estimate

$$|G(1+q)^{jk}_{mn} - F^{jk}_{mn}|(1+\max\{m,n\})^3 = |G(1+q)^{jk}_{mn}|(1+t)^3 \le C_{\mathrm{abs}}e^{-c_{\mathrm{abs}}t}(1+t^3) \le \frac{\delta}{2C_0}.$$

Consequently, by Lemma B.6, we finally obtain

$$\|G(1+q) - F\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} \le C_0 \sup_{(m,j,n,k)\in\mathcal{M}} |G(1+q)^{jk}_{mn} - F^{jk}_{mn}|(1+\max\{m,n\})^3 \le \delta.$$

□

We are now ready to prove the main instability estimate by combining Lemma B.3 and Lemma B.7.

Proof of Theorem B.1. Let $s > \frac{3}{2}$ and $C'_{\mathrm{abs}} > e$ be the constant obtained in Lemma B.7. We choose $\beta = \beta(s) > 0$ such that

$$1 \le \frac{1}{2}\left(\frac{\log 2}{2}\right)^d (\mu\beta)^{\frac{3}{s}}, \tag{B.13}$$

where $\mu = \mu(s)$ is the constant given in Lemma B.3 (with $d = 3$). Let us pick ϑ satisfying

$$0 < \vartheta < \min\left\{\mu\beta, \left(\left(\frac{\log 2}{2}\right)^d \frac{1}{2\log C'_{\mathrm{abs}}}\right)^{\frac{s}{3}}(\mu\beta), 1\right\}.$$

Assume that the set $\mathcal{Z}_\vartheta$ is given in Lemma B.3 (also take $d = 3$). Next we choose

$$\delta := \exp(-\vartheta^{-\frac{3}{5s}}) \in (0, 1/e),$$

and construct the $\mathcal{Y}_\delta$ described in Lemma B.7. Since $\mathcal{Z}_\vartheta \subset \mathcal{N}^\vartheta_{s\beta} \subset \mathcal{N}^1_{s\beta}$, it is clear that $\mathcal{Y}_\delta$ is also a δ-net for $\big(G(1+\mathcal{Z}_\vartheta), \|\cdot\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))}\big)$. Moreover, we can check that

$$\vartheta \le \left(\left(\frac{\log 2}{2}\right)^d \frac{1}{2\log C'_{\mathrm{abs}}}\right)^{\frac{s}{3}} \mu\beta \iff \log C'_{\mathrm{abs}} \le \frac{1}{2}\left(\frac{\log 2}{2}\right)^d (\mu\beta)^{\frac{3}{s}}\vartheta^{-\frac{3}{s}}.$$

Combining (B.2), (B.8) and (B.13) implies

$$|\mathcal{Y}_\delta| \le C'_{\mathrm{abs}}\exp(\vartheta^{-\frac{3}{s}}) = \exp(\vartheta^{-\frac{3}{s}} + \log C'_{\mathrm{abs}}) < \exp\left(\left(\frac{\log 2}{2}\right)^d (\mu\beta)^{\frac{3}{s}}\vartheta^{-\frac{3}{s}}\right) \le |\mathcal{Z}_\vartheta|.$$

This enables us to choose two different $q_1, q_2 \in \mathcal{Z}_\vartheta$ (by definition of $\mathcal{Z}_\vartheta$ it holds that $\|q_1 - q_2\|_{L^\infty(\mathbb{R}^3)} \ge \vartheta$) such that there exists $F \in \mathcal{Y}_\delta$ such that

$$\|G(1+q_j) - F\|_{\mathrm{HS}(L^2(\mathcal{S}^2),L^2(\mathcal{S}^2))} \le \delta = \exp(-\vartheta^{-\frac{3}{5s}}),$$

which proves the theorem by using (B.1). □

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