

# STRONG UNIQUE CONTINUATION FOR THE LAMÉ SYSTEM WITH LESS REGULAR COEFFICIENTS

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ABSTRACT. We study the strong unique continuation property (SUCP) for the Lamé system in the plane. The main contribution of our work is to prove that the SUCP holds when Lamé coefficients  $(\mu, \lambda) \in W^{2,s}(\Omega) \times L^\infty(\Omega)$  for some  $s > 4/3$ . In other words, we establish the SUCP for the Lamé system in the plane when  $\lambda$  is bounded and  $\mu$  belongs to certain Hölder classes.

## 1. INTRODUCTION

In this paper, we are interested in the strong unique continuation property (SUCP) for the Lamé system in the plane. We begin with a short description of the system. Let  $\Omega \subset \mathbb{R}^2$  be an open, connected set that contains the origin. Assume that  $\mu \in W^{2,s}(\Omega)$  and  $\lambda \in L^\infty(\Omega)$ , where

$$(1.1) \quad \begin{cases} \mu(x) \geq \delta_0, & \lambda(x) + 2\mu(x) \geq \delta_0, & \forall x \in \Omega, \\ \|\mu\|_{W^{2,s}(\Omega)} + \|\lambda\|_{L^\infty(\Omega)} \leq M_0 \end{cases}$$

for some positive constants  $\delta_0, M_0$ . Recall that

$$\|f\|_{W^{2,s}(\Omega)} = \sum_{|\alpha| \leq 2} \|\partial_x^\alpha f\|_{L^s(\Omega)}.$$

The Lamé system or isotropic elasticity equation, which represents the displacement of equilibrium, is given by

$$(1.2) \quad \operatorname{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \operatorname{div} u) = 0 \quad \text{in } \Omega,$$

where  $u = (u_1, u_2)^t$  is the real-valued displacement vector and  $(\nabla u)_{jk} = \partial_k u_j$  for  $j, k = 1, 2$ .

Without loss of generality, assume that  $0 \in \Omega$ . In the sequel,  $B_r$  denotes an open ball of radius  $r > 0$  centered at the origin, while  $B_r(x_0)$  denotes an open ball of radius  $r$  centered at  $x_0$ .

We now state the main results of the paper. Our first result is an *optimal three-ball inequality* for solutions to (1.2).

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2010 *Mathematics Subject Classification.* 35J47.

*Key words and phrases.* Carleman estimates, elliptic systems, quantitative unique continuation.

Davey is supported in part by the Simons Foundation Grant number 430198.

Lin is partially supported by the Ministry of Science and Technology of Taiwan.

Wang is partially supported by MOST 105-2115-M-002-014-MY3.

**Theorem 1.1.** *Assume that for some  $s > \frac{4}{3}$ , the estimates in (1.1) are satisfied. Then there exists an  $R_0 > 0$ , depending on  $s, M_0, \delta_0$ , such that  $B_{R_0} \subset \Omega$  and for any  $u \in H_{loc}^1(B_{R_0})$  satisfying (1.2) in  $B_{R_0}$ , if  $0 < R_1 < R_2 < R_3 \leq R_0$  and  $R_2/R_3 < 1/4$ , then*

$$(1.3) \quad \|u\|_{L^2(B_{R_2})} \leq C \|u\|_{L^2(B_{R_1})}^\tau \|u\|_{L^2(B_{R_3})}^{1-\tau},$$

where  $C = C(s, M_0, \delta_0, R_2, R_3)$  and  $\tau = \tau(R_1/R_2, R_2/R_3, R_0) \in (0, 1)$ . Moreover, for fixed  $R_2$  and  $R_3$ , the exponent  $\tau$  behaves like  $1/(-\log R_1)$  whenever  $R_1$  is sufficiently small.

We emphasize that  $C$  is independent of  $R_1$  and  $\tau$  has the asymptotic behavior like  $(-\log R_1)^{-1}$ . These facts are crucial to the derivation of the vanishing order for nontrivial solutions  $u$  to (1.2). Due to the behavior of  $\tau$ , the three-ball inequality is called optimal [11].

Our next result establishes the rate of vanishing for solutions to (1.2), thereby proving the strong unique continuation property (SUCP) in our setting.

**Theorem 1.2.** *Assume that for some  $s > \frac{4}{3}$ , (1.1) holds. Let  $R > 0$  be such that  $3R \leq R_0$ , where  $R_0$  is defined in Theorem 1.1. There exist positive constants  $K$  and  $m$ , depending on  $s, M_0, \delta_0$  and  $u$ , such that for all  $R$  sufficiently small*

$$(1.4) \quad \|u\|_{L^2(B_R)} \geq KR^m.$$

In Theorems 1.1 and 1.2, our main interest is in the case where  $s \in (4/3, 2]$ . If  $s > 2$ , then by the Sobolev embedding theorem,  $\mu$  is at least Lipschitz, so the SUCP for (1.2) has already been proved (see below). Moreover, when  $s \in (4/3, 2]$ , we have

$$\mu \in \begin{cases} C^{0,\gamma} & \text{with } \gamma = 2 - (2/s), \text{ if } s < 2 \\ C^{0,\gamma} & \text{for any } \gamma \in (0, 1), \text{ if } s = 2. \end{cases}$$

On the other hand, if  $\mu(x) = |x|^\alpha + 1$  for some  $\alpha \in (0, 1)$ , then  $\mu \in W^{2,s}$  for  $s < \frac{2}{2-\alpha}$ , but  $\mu$  is clearly not a Lipschitz function. In other words, we establish the quantitative SUCP for (1.2) when  $\lambda \in L^\infty$  and  $\mu$  is Hölder continuous.

Note that by the results of [16], [19], the unique continuation property fails for elliptic equations in  $d \geq 3$  when the coefficients are only assumed to be Hölder continuous. Therefore, our results are special to the planar setting.

Theorems 1.1 and 1.2 will be proved together. Estimates (1.3) and (1.4) are quantitative forms of the SUCP for the Lamé system (1.2). Taking advantage of the isotropy of the coefficients in (1.2), the unique continuation property for (1.2) has been studied rather thoroughly. Here we list some of the known results on the SUCP for (1.2):

- $\lambda, \mu \in C^{1,1}$ ,  $n \geq 2$  (quantitative): Alessandrini and Morassi [2].

- $\lambda, \mu \in C^{0,1}$ ,  $n = 2$  (qualitative): Lin and Wang [17].
- $\lambda \in L^\infty, \mu \in C^{0,1}$ ,  $n = 2$  (qualitative): Escauriaza [10].
- $\lambda, \mu \in C^{0,1}$ ,  $n \geq 2$  (quantitative): Lin, Nakamura, and Wang [18].
- $\mu \in C^{0,1}, \lambda \in L^\infty$ ,  $n \geq 2$  (quantitative): Lin, Nakamura, Uhlmann, and Wang [15].
- $\mu \in C^{0,1}, \lambda \in L^\infty$ ,  $n \geq 2$  (doubling inequality): Koch, Lin, and Wang [14].

We also mention a recent article by the authors [7] where we proved that a Liouville-type theorem holds for the Lamé system in the plane when  $\mu \in W^{1,2}$  and  $\|\nabla\mu\|_{L^2}$  is small. From this result, we proved a weak unique continuation property and the uniqueness of the Cauchy problem for such Lamé system (see Corollaries 1.3, 1.4 in [7]).

The main theme of this work is to prove the SUCP for (1.2) with the focus on the regularity of  $\mu$ . Recall that for scalar second-order elliptic equations in divergence or non-divergence form with  $n = 2$ , the SUCP holds even when the leading coefficients are only essentially bounded, [1], [3], [4], and [20]. However, we would like to point out that the SUCP for (1.2) may not hold when  $\mu$  is essentially bounded or even continuous, see [5] and [13]. In this paper, we establish the SUCP for (1.2) when  $\mu \in W^{2,s}$  with  $s > 4/3$  (a subclass of Hölder continuity) and  $\lambda \in L^\infty$ . An interesting open question is to prove or disprove the SUCP or even the unique continuation property for (1.2) in  $n = 2$  for general  $(\mu, \lambda) \in (W^{1,p}(\Omega), L^\infty(\Omega))$  with  $p < \infty$ .

The restriction of the range  $s > 4/3$  is dictated by the application of Carleman estimates. We need this condition in order to absorb unwanted terms, see for example (4.5) and the explanation following it. We think that the condition  $s > 4/3$  is optimal as long as we consider the radially symmetric weight function  $h$  in the Carleman estimate.

This article is organized as follows. In the next section, Section 2, we introduce the reduced system. That is, we show that (1.2) is equivalent to a system of second-order elliptic systems, and since we are working with real-valued solutions in the plane, it can also be realized as a  $\bar{\partial}$ -equation. The main Carleman estimates appear in Section 3. The first such theorem applies to first-order operators, while the second applies to second-order operators; thereby pairing with the equations derived in Section 2. The proof of the latter Carleman estimate also appears in Section 3. Section 4 is devoted to the proofs of the main theorems, Theorems 1.1 and 1.2. The technical proof of the  $\bar{\partial}$ -Carleman estimate is provided in Section 5. Finally, Section 6 states and proves the interior estimate associated to (1.2) that finds applications in the proofs of the main theorems.

**Acknowledgement.** Part of this research was carried out while the first author was visiting the National Center for Theoretical Sciences (NCTS) at National Taiwan University. The first author wishes to thank the NCTS for their financial support and their kind hospitality during her visit to Taiwan.

## 2. REDUCED SYSTEMS

Within this section, we derive reduced systems associated to (1.2). There are two new systems that we produce. First, we show that (1.2) may be written as a systems of second-order elliptic equations. Then we show that a complex-valued first-order elliptic equation may be derived from the system. These two representations will be instrumental to the Carleman estimate arguments that are used to prove our main unique continuation theorems.

With

$$(2.1) \quad \begin{aligned} v &= \frac{\lambda + 2\mu}{\mu} \operatorname{div} u = \frac{\lambda + 2\mu}{\mu} (\partial_1 u_1 + \partial_2 u_2), \\ w &= \operatorname{curl} u = \partial_2 u_1 - \partial_1 u_2, \end{aligned}$$

a straightforward computation shows that

$$(2.2) \quad \begin{aligned} \Delta u_1 &= \partial_1 \left( \frac{\mu}{\lambda + 2\mu} v \right) + \partial_2 w, \\ \Delta u_2 &= \partial_2 \left( \frac{\mu}{\lambda + 2\mu} v \right) - \partial_1 w. \end{aligned}$$

Expanding (1.2) gives

$$\begin{aligned} & \operatorname{div} \begin{bmatrix} 2\mu \partial_1 u_1 & \mu (\partial_1 u_2 + \partial_2 u_1) \\ \mu (\partial_1 u_2 + \partial_2 u_1) & 2\mu \partial_2 u_2 \end{bmatrix} + \begin{bmatrix} \partial_1 \{ \lambda (\partial_1 u_1 + \partial_2 u_2) \} \\ \partial_2 \{ \lambda (\partial_1 u_1 + \partial_2 u_2) \} \end{bmatrix} \\ &= \begin{bmatrix} \partial_1 (2\mu \partial_1 u_1) + \partial_2 \{ \mu (\partial_1 u_2 + \partial_2 u_1) \} \\ \partial_1 \{ \mu (\partial_1 u_2 + \partial_2 u_1) \} + \partial_2 (2\mu \partial_2 u_2) \end{bmatrix} + \begin{bmatrix} \partial_1 (\lambda \partial_1 u_1 + \lambda \partial_2 u_2) \\ \partial_2 (\lambda \partial_1 u_1 + \lambda \partial_2 u_2) \end{bmatrix} \\ &= 0. \end{aligned}$$

Simplifying each row shows that

$$(2.3) \quad \partial_1 [(2\mu + \lambda) \partial_1 u_1 + \lambda \partial_2 u_2] + \partial_2 [\mu (\partial_1 u_2 + \partial_2 u_1)] = 0$$

$$(2.4) \quad -\partial_2 [\lambda \partial_1 u_1 + (2\mu + \lambda) \partial_2 u_2] = \partial_1 [\mu (\partial_1 u_2 + \partial_2 u_1)].$$

For brevity, we define the operator  $\bar{\partial} = \partial_1 + i\partial_2$ . By the definition of  $v$  and  $w$ , we have

$$(2.5) \quad \begin{aligned} & \bar{\partial} (\mu v - i\mu w) \\ &= \partial_1 (\mu v) + \partial_2 (\mu w) - i\partial_1 (\mu w) + i\partial_2 (\mu v) \\ &= \partial_1 [(\lambda + 2\mu) (\partial_1 u_1 + \partial_2 u_2)] + \partial_2 [\mu (\partial_2 u_1 - \partial_1 u_2)] \\ & \quad - i\partial_1 [\mu (\partial_2 u_1 - \partial_1 u_2)] + i\partial_2 [(\lambda + 2\mu) (\partial_1 u_1 + \partial_2 u_2)]. \end{aligned}$$

From the real part of (2.5), we get

$$(2.6) \quad \begin{aligned} & \partial_1 [(\lambda + 2\mu) (\partial_1 u_1 + \partial_2 u_2)] + \partial_2 [\mu (\partial_2 u_1 - \partial_1 u_2)] \\ &= \partial_1 [(\lambda + 2\mu) \partial_1 u_1 + \lambda \partial_2 u_2] + \partial_1 [2\mu \partial_2 u_2] \\ & \quad + \partial_2 [\mu (\partial_2 u_1 + \partial_1 u_2)] + \partial_2 [-2\mu \partial_1 u_2] \\ &= 2\partial_1 [\mu \partial_2 u_2] - 2\partial_2 [\mu \partial_1 u_2] \\ &= 2\partial_1 \mu \partial_2 u_2 - 2\partial_2 \mu \partial_1 u_2, \end{aligned}$$

where we have used (2.3) to eliminate terms. For the imaginary part of (2.5), using (2.4) shows that

$$\begin{aligned}
 & \partial_1 [\mu (\partial_2 u_1 - \partial_1 u_2)] - \partial_2 [(\lambda + 2\mu) (\partial_1 u_1 + \partial_2 u_2)] \\
 &= \partial_1 [\mu (\partial_2 u_1 - \partial_1 u_2)] - \partial_2 [\lambda \partial_1 u_1 + (\lambda + 2\mu) \partial_2 u_2] - \partial_2 (2\mu \partial_1 u_1) \\
 (2.7) \quad &= \partial_1 [\mu (\partial_2 u_1 - \partial_1 u_2)] + \partial_1 [\mu (\partial_1 u_2 + \partial_2 u_1)] - \partial_2 (2\mu \partial_1 u_1) \\
 &= \partial_1 (2\mu \partial_2 u_1) - \partial_2 (2\mu \partial_1 u_1) \\
 &= 2\partial_1 \mu \partial_2 u_1 - 2\partial_2 \mu \partial_1 u_1.
 \end{aligned}$$

Substituting (2.6) and (2.7) into (2.5) then gives

$$(2.8) \quad \bar{\partial} (\mu v - i\mu w) = g_1 - ig_2,$$

where

$$g_1 = 2\partial_1 \mu \partial_2 u_2 - 2\partial_2 \mu \partial_1 u_2, \quad g_2 = 2\partial_1 \mu \partial_2 u_1 - 2\partial_2 \mu \partial_1 u_1.$$

Using the observation that  $\partial_2 u_1 = w + \partial_1 u_2$ , we rewrite  $g_1 - ig_2$  as

$$\begin{aligned}
 & g_1 - ig_2 \\
 &= 2\partial_1 \mu \partial_2 u_2 - i2\partial_1 \mu \partial_2 u_1 - 2\partial_2 \mu \partial_1 u_2 + i2\partial_2 \mu \partial_1 u_1 \\
 (2.9) \quad &= 2\partial_1 \mu (\partial_2 u_2 - iw - i\partial_1 u_2) - 2\partial_2 \mu (\partial_2 u_1 - w - i\partial_1 u_1) \\
 &= -2i\partial_1 \mu \bar{\partial} u_2 + 2i\partial_2 \mu \bar{\partial} u_1 - 2iw\bar{\partial} \mu \\
 &= \bar{\partial} (-2i\partial_1 \mu u_2) + \bar{\partial} (2i\partial_2 \mu u_1) - 2iw\bar{\partial} \mu + \bar{\partial} (2i\partial_1 \mu) u_2 - \bar{\partial} (2i\partial_2 \mu) u_1.
 \end{aligned}$$

Combining (2.8) and (2.9) then gives

$$(2.10) \quad \bar{\partial} (\mu v - i\mu w + 2i\partial_1 \mu u_2 - 2i\partial_2 \mu u_1) = g_3 + g_4.$$

where

$$(2.11) \quad g_3 = -2i\bar{\partial} \mu w, \quad g_4 = \bar{\partial} (2i\partial_1 \mu) u_2 - \bar{\partial} (2i\partial_2 \mu) u_1.$$

The equations described by (2.2) and (2.10) will be used in the proofs of the main theorems.

### 3. CARLEMAN ESTIMATES

Here we present the main Carleman estimates that will be used in the proofs of our main theorems. These Carleman estimates apply to first- and second-order elliptic operators, corresponding to the reduced systems that were derived in the previous section. Our first estimate applies to the operator  $\bar{\partial}$ . The proof of this theorem appears in a subsequent section. The second main estimate is for second-order operators and its proof relies on a similar estimate that appeared previously in [14]. The arguments that connect this theorem from [14] to our new estimate are included within this section.

With the notation  $t = \log |x|$ , define the weight function

$$h(\log |x|) = h(t) = -\beta t + \beta e^{t/4}.$$

We often abuse notation and write  $h$  as a shorthand for  $h(\log|x|)$ . The prime notation denotes differentiation with respect to the  $t$ -variable. That is,  $h' = \frac{\partial h}{\partial t}$ , and we again interpret the function  $h'$  as  $h'(\log|x|)$ . At times, we will use the notation  $h = h_\beta$  to remind ourselves of the dependence on the constant  $\beta$ .

Recall that we use the notation  $\bar{\partial} = \partial_1 + i\partial_2$ . From now on, the notation  $X \lesssim Y$  or  $X \gtrsim Y$  means that  $X \leq CY$  or  $X \geq CY$  with some constant  $C$ . We first state the following Carleman estimate for first order operators.

**Theorem 3.1.** *Let  $1 < p < 2$  and  $\tilde{R}_0 \in (0, 1)$ . Then if  $\beta$  is sufficiently large, then for all  $z \in C_0^\infty(B_{\tilde{R}_0} \setminus \{0\})$ , it holds that*

$$(3.1) \quad \beta^{1-\frac{1}{p}} \left\| |x| e^h z \right\|_2 \leq C \left\| |x|^{\tau_0} e^h \bar{\partial} z \right\|_p,$$

where  $\tau_0 = \frac{11}{4} - \frac{2}{p}$  and  $C = C(p, \tilde{R}_0)$ .

The proof of this theorem appears in Section 5. The next theorem gives a Carleman estimate for second order elliptic operators.

**Theorem 3.2.** *Let  $2 < q < q' < \infty$ . If  $\beta \in \mathbb{N} + \frac{1}{4}$  is sufficiently large, then for all  $z \in C_0^\infty(B_1 \setminus \{0\})$  and all  $f = (f_1, f_2)$  with  $f_1, f_2 \in C_0^\infty(B_1 \setminus \{0\})$ ,*

$$(3.2) \quad \begin{aligned} & \beta \left\| (1 + h'')^{1/2} e^h z \right\|_2 + \beta^{\gamma_1} \left\| |x|^{\tau_1} e^h z \right\|_q \\ & \leq C \left( \left\| e^h |x|^2 (\Delta z + \nabla f) \right\|_2 + \beta \left\| e^h |x| f \right\|_2 \right), \end{aligned}$$

where  $C = C(q, q')$ ,  $\gamma_1 = \frac{2}{q} \left(1 - \frac{q-2}{q'}\right) + \frac{1}{2}$  and  $\tau_1 = \frac{1}{8} + \left(1 - \frac{2}{q}\right) \left(\frac{q'}{q'-2}\right)$ .

To prove Theorem 3.2, we first recall the following Carleman estimate derived in [14].

**Lemma 3.3** (Corollary to Theorem 3.1 in [14]). *If  $\beta \in \mathbb{N} + \frac{1}{4}$  is sufficiently large, then for all  $z \in C_0^\infty(B_1 \setminus \{0\})$  and all  $f = (f_1, f_2)$  with  $f_1, f_2 \in C_0^\infty(B_1 \setminus \{0\})$ ,*

$$(3.3) \quad \begin{aligned} & \beta \left\| (1 + h'')^{1/2} e^h z \right\|_2 + \left\| |x| (1 + h'')^{1/2} e^h \nabla z \right\|_2 \\ & \leq C \left( \left\| e^h |x|^2 (\Delta z + \nabla f) \right\|_2 + \beta \left\| e^h |x| f \right\|_2 \right), \end{aligned}$$

where  $C$  is a universal constant.

The proof follows from Theorem 3.1 in [14] by replacing  $h$  with  $\tilde{h}(\log|x|) = \tilde{h}_\beta(t) = -\beta t + 2t + \beta e^{t/4}$ .

**Remark 3.4.** *The notation  $\nabla f$  used in [14] is interpreted as any (constant) linear combination of first order derivatives of the entries of  $f$ , as long as the resulting function takes the same algebraic structure as  $\Delta z$  (so that adding these quantities is meaningful).*

The next lemma is a straightforward consequence of Lemma 3.3.

**Lemma 3.5.** *Let  $2 < q' < \infty$ . If  $\beta \in \mathbb{N} + \frac{1}{4}$  is sufficiently large, then for all  $z \in C_0^\infty(B_1 \setminus \{0\})$  and all  $f = (f_1, f_2)$  with  $f_1, f_2 \in C_0^\infty(B_1 \setminus \{0\})$ ,*

$$\begin{aligned} & \beta^{3/2} \left\| |x|^{1/8} e^h z \right\|_2 + \beta^{1/2} \left\| |x|^{9/8} e^h \nabla z \right\|_2 + \beta^{1/2} \left\| |x|^{9/8} e^h z \right\|_{q'} \\ & \leq C \left( \left\| e^h |x|^2 (\Delta z + \nabla f) \right\|_2 + \beta \left\| e^h |x| f \right\|_2 \right), \end{aligned}$$

where  $C = C(q')$ .

*Proof.* For any  $q' \in (2, \infty)$ , the Sobolev inequality shows that

$$\begin{aligned} \left\| |x|^{9/8} e^h z \right\|_{q'} & \leq C_{q'} \left\| \nabla \left( |x|^{9/8} e^h z \right) \right\|_2 \\ & \leq C_{q'} \left\| \nabla \left( |x|^{9/8} e^h \right) z \right\|_2 + C_{q'} \left\| |x|^{9/8} e^h \nabla z \right\|_2 \\ & \leq C_{q'} \beta \left\| |x|^{1/8} e^h z \right\|_2 + C_{q'} \left\| |x|^{9/8} e^h \nabla z \right\|_2. \end{aligned}$$

Since  $h'' \sim \beta |x|^{1/4}$ , an application of Lemma 3.3 shows that

$$\begin{aligned} & \beta^{3/2} \left\| |x|^{1/8} e^h z \right\|_2 + \beta^{1/2} \left\| |x|^{9/8} e^h \nabla z \right\|_2 \\ & \lesssim \beta \left\| (1 + h'')^{1/2} e^h z \right\|_2 + \left\| |x| (1 + h'')^{1/2} e^h \nabla z \right\|_2 \\ & \lesssim \left\| e^h |x|^2 (\Delta z + \nabla f) \right\|_2 + \beta \left\| e^h |x| f \right\|_2. \end{aligned}$$

Adding these two inequalities completes the proof of the lemma.  $\square$

Lemma 3.5 in combination with the Hölder inequality leads to the proof of Theorem 3.2.

*Proof of Theorem 3.2.* Define  $\theta = \frac{q'-q}{q'-2}$  so that  $q = 2\theta + (1-\theta)q'$ . By Hölder's inequality combined with Lemma 3.5, we have

$$\begin{aligned} \left\| |x|^{\tau_1} e^h z \right\|_q & \leq C_{q,q'} \left\| |x|^{1/8} e^h z \right\|_2^{\frac{2\theta}{q}} \cdot \left\| |x|^{9/8} e^h z \right\|_{q'}^{\frac{q'(1-\theta)}{q}} \\ & \lesssim \beta^{-\tau_1} \left( \left\| e^h |x|^2 (\Delta z + \nabla f) \right\|_2 + \beta \left\| e^h |x| f \right\|_2 \right). \end{aligned}$$

Adding this estimate to (3.3) leads to the conclusion.  $\square$

#### 4. THE PROOFS OF THEOREM 1.1 AND THEOREM 1.2

With the reduced systems and associated Carleman estimates, we are now prepared to prove Theorems 1.1 and 1.2. An additional tool that we make use of is the interior estimate presented in Section 6.

We first choose  $R_0 \leq \tilde{R}_0$  such that  $B_{R_0} \subset \Omega$ . Let  $\chi \in C_0^\infty(B_{R_0} \setminus \{0\})$  be a cutoff function with the property that  $\text{supp } \chi \subset A_s \cup A_m \cup A_l$ , where  $\chi \equiv 1$  on  $A_m$  and  $\text{supp } \nabla \chi = A_s \cup A_l$ . Moreover, for all  $x, y \in A_s$ ,  $|x|/|y| \sim 1$ , and similarly for  $A_l$ . The exact definitions of these sets will be given below.

For  $\mu, \lambda$  as in (1.1),  $u_1, u_2$  from (1.2), and  $v, w$  from (2.1), define the compactly supported function

$$z = \chi(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1).$$

Applying estimate (3.1) from Theorem 3.1 to  $z$  with some  $p \in (1, 2)$  to be determined below, then using the equation (2.10) yields

$$\begin{aligned}
& \beta^{1-\frac{1}{p}} \left\| |x|e^h\chi(|v| + |w|) \right\|_2 - \beta^{1-\frac{1}{p}} \left\| |x|e^h\chi|\nabla\mu|u \right\|_2 \\
& \leq C\beta^{1-\frac{1}{p}} \left\| |x|e^h\chi(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1) \right\|_2 \\
(4.1) \quad & \leq C \left\| |x|^{\tau_0}e^h\chi(g_3 + g_4) \right\|_p \\
& + C \left\| |x|^{\tau_0}e^h\bar{\partial}\chi(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1) \right\|_p \\
& \leq C_1 \left\| |x|^{\tau_0}e^h\chi|\nabla\mu|w \right\|_p + C_1 \left\| |x|^{\tau_0}e^h\chi|D^2\mu|u \right\|_p \\
& + C_1 \left\| |x|^{\tau_0}e^h|\nabla\chi|\nabla u \right\|_p + C_1 \left\| |x|^{\tau_0}e^h|\nabla\chi||\nabla\mu|u \right\|_p,
\end{aligned}$$

where  $\tau_0 = \frac{11}{4} - \frac{2}{p}$  and we have used the structures of  $g_3$  and  $g_4$  as defined in (2.11) in the last line.

An application of (3.2) from Theorem 3.2 to the functions  $z = \chi u$  and  $f = \chi\left(\frac{\mu}{\lambda+2\mu}v, w\right)$ , where we interpret  $\nabla f = (-\partial_1 f_1 - \partial_2 f_2, -\partial_2 f_1 + \partial_1 f_2)$  (see Remark 3.4) and the values for  $2 < q < q' < \infty$  will be specified below, then using (2.2) and (2.1), shows that if  $\beta \in \mathbb{N} + \frac{1}{4}$  is sufficiently large, then

$$\begin{aligned}
(4.2) \quad & \beta \left\| (1 + h'')^{1/2}e^h\chi u \right\|_2 + \beta^{\gamma_1} \left\| |x|^{\tau_1}e^h\chi u \right\|_q \\
& \leq C_2 \left\| |x|^2e^h(|\nabla\chi||\nabla u| + |D^2\chi|u) \right\|_2 + C_2\beta \left\| |x|e^h\chi(|v| + |w|) \right\|_2.
\end{aligned}$$

Recall that  $\gamma_1 = \frac{2}{q} \left(\frac{q'-q}{q'-2}\right) + \frac{1}{2}$  and  $\tau_1 = \frac{1}{8} + \left(1 - \frac{2}{q}\right) \left(\frac{q'}{q'-2}\right)$ .

Adding  $2C_2\beta^{\frac{1}{p}} \cdot (4.1)$  to (4.2) then gives

$$\begin{aligned}
(4.3) \quad & \beta \left\| (1 + h'')^{1/2}e^h\chi u \right\|_2 + \beta^{\gamma_1} \left\| |x|^{\tau_1}e^h\chi u \right\|_q + \beta \left\| |x|e^h\chi(|v| + |w|) \right\|_2 \\
& \leq 2C_2\beta \left\| |x|e^h\chi|\nabla\mu|u \right\|_2 + 2C_1C_2\beta^{\frac{1}{p}} \left\| |x|^{\tau_0}e^h\chi|\nabla\mu|w \right\|_p \\
& + 2C_1C_2\beta^{\frac{1}{p}} \left\| |x|^{\tau_0}e^h\chi|D^2\mu|u \right\|_p \\
& + 2C_1C_2\beta^{\frac{1}{p}} \left\| |x|^{\tau_0}e^h|\nabla\chi||\nabla\mu|u \right\|_p + 2C_1C_2\beta^{\frac{1}{p}} \left\| |x|^{\tau_0}e^h|\nabla\chi|\nabla u \right\|_p \\
& + C_2 \left\| |x|^2e^h(|\nabla\chi||\nabla u| + |D^2\chi|u) \right\|_2.
\end{aligned}$$

With an appropriate choice of indices  $p, q$  and  $q'$ , the first three terms on the right of the inequality may be absorbed into the left.

First, we consider the term  $\beta \left\| |x| e^h \chi |\nabla \mu| u \right\|_2$ . From the Hölder inequality, we have that

$$\left\| |x| e^h \chi |\nabla \mu| u \right\|_2 \leq C \left\| |x|^{\tau_1} e^h \chi u \right\|_q \left\| |x|^{1-\tau_1} \right\|_\infty \left\| \nabla \mu \right\|_{\frac{2q}{q-2}}.$$

To absorb this term, we need that

$$(4.4) \quad \gamma_1 > 1, \quad \tau_1 \leq 1, \quad \frac{2q}{q-2} \leq s^*,$$

where  $s^*$  denotes the Sobolev conjugate with  $s^* = \frac{2s}{2-s}$  for  $s < 2$ ,  $s^* \in (2, \infty)$  arbitrary for  $s = 2$ , and  $s^* = \infty$  otherwise. Notice that  $s^* > 4$  when  $s > 4/3$ .

Next, we consider the term  $\beta^{\frac{1}{p}} \left\| |x|^{\tau_0} e^h \chi |\nabla \mu| w \right\|_p$ . If we restrict  $p \in (1, 4/3)$ , then from the Hölder inequality, we have that

$$\left\| |x|^{\tau_0} e^h \chi |\nabla \mu| w \right\|_p \leq C \left\| |x| e^h \chi w \right\|_2 \left\| |x|^{\frac{7}{4} - \frac{2}{p}} \right\|_{\frac{2ps^*}{2s^* - ps^* - 2p}} \left\| \nabla \mu \right\|_{s^*}.$$

Observe that  $\left(\frac{7}{4} - \frac{2}{p}\right) \frac{2ps^*}{2s^* - ps^* - 2p} > -2$  whenever  $s^* > \frac{8}{3}$ , independent of the choice of  $p$ . Thus we may absorb this term into the left if  $s^* > \frac{8}{3}$ .

Finally, using similar techniques, we have

$$\left\| |x|^{\tau_0} e^h \chi |D^2 \mu| u \right\|_p \leq C \left\| |x|^{\tau_1} e^h \chi u \right\|_q \left\| |x|^{\frac{11}{4} - \tau_1 - \frac{2}{p}} \right\|_\infty \left\| D^2 \mu \right\|_{\frac{pq}{q-p}}.$$

To handle this term, we need that

$$(4.5) \quad \tau_1 + \frac{2}{p} \leq \frac{11}{4}, \quad \frac{pq}{q-p} \leq s.$$

From the condition that  $\gamma_1 > 1$ , we require that  $q \in (2, 4)$ . This reduction to the range of  $q$  in combination with the condition (4.5) explains why we can only handle  $s > \frac{4}{3}$ . Since  $s^* > 4$ , the second term may be absorbed into the left. Moreover, since

$$\lim_{q \rightarrow 4} \frac{2q}{q-2} = 4 < s^* \quad \lim_{q \rightarrow 4, p \rightarrow 1} \frac{pq}{q-p} = \frac{4}{3} < s$$

$$\lim_{(q, q') \rightarrow (4, \infty)} \gamma_1 = 1 \quad \lim_{(p, q, q') \rightarrow (1, 4, \infty)} \left( \tau_1 + \frac{2}{p} \right) = \frac{21}{8},$$

then we can choose  $p \in (1, 4/3)$ ,  $q \in (2, 4)$  and  $q' \in (q, \infty)$  so that (4.4) and (4.5) hold.

From now on, we fix  $p, q, q'$  as described above to satisfy (4.4) and (4.5). That is, if  $\beta$  is sufficiently large then the first three terms on the righthand side of (4.3) are absorbed by the lefthand side of (4.3) to get

$$(4.6) \quad \begin{aligned} & \left\| e^h \chi u \right\|_2 \\ & \leq C_3 \left\| |x|^{\tau_0} e^h |\nabla \chi| |\nabla u| \right\|_p + C_3 \left\| |x|^{\tau_0} e^h |\nabla \chi| |\nabla \mu| u \right\|_p \\ & + C_3 \left\| |x|^2 e^h (|\nabla \chi| |\nabla u| + |D^2 \chi| u) \right\|_2 \\ & \leq C_4 \left\| |x|^2 e^h \left( |x|^{-1/4} |\nabla \chi| |\nabla u| + |D^2 \chi| |u| + |x|^{-1} |\nabla \chi| |u| \right) \right\|_2, \end{aligned}$$

where we have used the Hölder inequality and the structure of  $\text{supp } \nabla \chi$  to show that

$$\begin{aligned} \left\| |x|^{\tau_0} e^h |\nabla \chi| |\nabla u| \right\|_p &\leq C \left\| |x|^{\frac{7}{4}} e^h |\nabla \chi| |\nabla u| \right\|_2 \left\| \text{supp } \nabla \chi |x|^{1-\frac{2}{p}} \right\|_{\frac{2p}{2-p}} \\ &\leq C \left\| |x|^{\frac{7}{4}} e^h |\nabla \chi| |\nabla u| \right\|_2 \end{aligned}$$

and

$$\begin{aligned} \left\| |x|^{\tau_0} e^h |\nabla \chi| |\nabla \mu| u \right\|_p &\leq C \left\| |x| e^h |\nabla \chi| u \right\|_2 \left\| |x|^{\frac{7}{4}-\frac{2}{p}} \right\|_{\frac{2ps^*}{2s^*-ps^*-2p}} \|\nabla \mu\|_{s^*} \\ &\leq C \left\| |x| e^h |\nabla \chi| u \right\|_2 \end{aligned}$$

in the last inequality.

Define the sets  $A_s = \{R_1/3 \leq |x| \leq 2R_1/3\}$ ,  $A_m = \{2R_1/3 \leq |x| \leq R_3/3\}$ , and  $A_l = \{R_3/3 \leq |x| \leq 2R_3/3\}$ . As stated above,  $\chi$  is a smooth cut-off function supported in  $A_s \cup A_m \cup A_l$  with  $\chi \equiv 1$  in  $A_m$ . Since  $\text{supp } D^\alpha \chi \subset A_s \cup A_l$ , then for any multi-index  $\alpha$

$$(4.7) \quad \begin{cases} |D^\alpha \chi| = O\left(R_1^{-|\alpha|}\right) & \text{for all } x \in A_s \\ |D^\alpha \chi| = O\left(R_3^{-|\alpha|}\right) & \text{for all } x \in A_l. \end{cases}$$

We replace  $h(t)$  by  $h(t) + t$  in (4.6), then apply (4.7) and (6.1) with  $A_s \Subset \tilde{A}_s := \{R_1/4 \leq |x| \leq R_1\}$  and  $A_l \Subset \tilde{A}_l := \{R_3/4 \leq |x| \leq R_3\}$  to get

$$\begin{aligned} R_2^2 e^{2\tilde{h}(R_2)} \int_{\{2R_1/3 < |x| < R_2\}} |u|^2 dx &\leq \int_{\{2R_1/3 < |x| < R_2\}} |x|^2 e^{2h} |u|^2 dx \\ &\leq C_5 R_1^{3/2} e^{2\tilde{h}(R_1/3)} \int_{\tilde{A}_s} |u|^2 dx + C_5 R_3^{3/2} e^{2\tilde{h}(R_3/3)} \int_{\tilde{A}_l} |u|^2 dx, \end{aligned}$$

where we set  $\tilde{h}(a) = h(\ln a)$  and used that  $R_2 \leq R_3/4 < R_3/3$ . In the use of (6.1), we choose  $R_0$  to be smaller if necessary. Dividing through by  $R_2^2 e^{2\tilde{h}(R_2)}$  shows that

$$\begin{aligned} &\int_{\{2R_1/3 < |x| < R_2\}} |u|^2 \\ &\leq \frac{C_5}{R_2^2} \left[ R_1^{3/2} e^{2\tilde{h}(R_1/3)-2\tilde{h}(R_2)} \int_{\tilde{A}_s} |u|^2 + R_3^{3/2} e^{2\tilde{h}(R_3/3)-2\tilde{h}(R_2)} \int_{\tilde{A}_l} |u|^2 \right] \\ &\leq C_6 R_2^{-1/2} \left[ e^{2(\beta-3/4)\log\left(\frac{3R_2}{R_1}\right)} \int_{\tilde{A}_s} |u|^2 + e^{-(3/2)(\beta-1)\log\left(\frac{R_3}{3R_2}\right)} \int_{\tilde{A}_l} |u|^2 \right] \\ &\leq C_6 R_2^{-1/2} \left[ e^{2(\beta-3/4)\log\left(\frac{3R_2}{R_1}\right)} \int_{\tilde{A}_s} |u|^2 + e^{-(1/2)(\beta-3/4)\log\left(\frac{R_3}{3R_2}\right)} \int_{\tilde{A}_l} |u|^2 \right], \end{aligned}$$

since

$$\begin{aligned} 2\tilde{h}(R_1/3) - 2\tilde{h}(R_2) &= 2\beta \log\left(\frac{3R_2}{R_1}\right) - \beta\left(\sqrt{R_2} - \sqrt{R_1/3}\right) \leq 2\beta \log\left(\frac{3R_2}{R_1}\right) \\ 2\tilde{h}(R_3/3) - 2\tilde{h}(R_2) &= 2h'(\log \hat{R}) \log\left(\frac{R_3}{3R_2}\right) \leq -\frac{3}{2}\beta \log\left(\frac{R_3}{3R_2}\right), \end{aligned}$$

for some  $R_2 < \hat{R} < R_3/3 < 1$ .

Adding  $\int_{B_{2R_1/3}} |u|^2$  to both sides of the previous inequality and replacing  $\beta$  by  $\beta + 1$  shows that for any sufficiently large  $\beta \in \mathbb{N} + \frac{1}{2}$ , it holds that

$$(4.8) \quad \int_{B_{R_2}} |u|^2 \leq C_7 R_2^{-1/2} \left( e^{\beta E} \int_{B_{R_1}} |u|^2 + e^{-\beta B} \int_{B_{R_3}} |u|^2 \right),$$

where we set  $E = 2 \log\left(\frac{3R_2}{R_1}\right)$  and  $B = \frac{1}{2} \log\left(\frac{R_3}{3R_2}\right)$ . Note that both  $E$  and  $B$  are positive.

Let  $\tilde{\beta}$  denote the smallest  $\beta \in \mathbb{N} + \frac{1}{2}$  for which (4.8) holds. To further simplify the terms on the right hand side of (4.8), we consider two cases. If

$\int_{B_{R_1}} |u|^2 \neq 0$  and

$$e^{\tilde{\beta} E} \int_{B_{R_1}} |u|^2 \leq e^{-\tilde{\beta} B} \int_{B_{R_3}} |u|^2,$$

then we find  $\beta_1 \in \mathbb{N} + \frac{1}{2}$  satisfying  $\beta_1 \geq \tilde{\beta} + 1$  and  $\beta_2 \in (\beta_1 - 1, \beta_1]$  such that

$$(4.9) \quad e^{\beta_2 E} \int_{B_{R_1}} |u|^2 = e^{-\beta_2 B} \int_{B_{R_3}} |u|^2$$

and

$$e^{(\beta_1-1)E} \int_{B_{R_1}} |u|^2 \leq e^{-(\beta_1-1)B} \int_{B_{R_3}} |u|^2.$$

With these choices of  $\beta_1$  and  $\beta_2$ , we derive from (4.8) that

$$\begin{aligned} (4.10) \quad \int_{B_{R_2}} |u|^2 &\leq C_7 R_2^{-1/2} \left( e^{(\beta_1-1)E} \int_{B_{R_1}} |u|^2 + e^{-(\beta_1-1)B} \int_{B_{R_3}} |u|^2 \right) \\ &\leq C_7 R_2^{-1/2} \left( e^{\beta_2 E} \int_{B_{R_1}} |u|^2 + e^{(\beta_2+1-\beta_1)B-\beta_2 B} \int_{B_{R_3}} |u|^2 \right) \\ &\leq C_7 (1 + e^B) R_2^{-1/2} e^{\beta_2 E} \int_{B_{R_1}} |u|^2 \\ &= C_7 (1 + e^B) R_2^{-1/2} \left( \int_{B_{R_1}} |u|^2 \right)^{\frac{B}{B+E}} \left( \int_{B_{R_3}} |u|^2 \right)^{\frac{E}{B+E}}, \end{aligned}$$

where we have used  $0 < \beta_2 + 1 - \beta_1 \leq 1$  and the value of  $\beta_2$  obtained from (4.9). If  $\int_{B_{R_1}} |u|^2 = 0$ , then letting  $\beta \rightarrow \infty$  in (4.8) shows that  $\int_{B_{R_2}} |u|^2 = 0$  as well, and then the three-ball inequality obviously holds.

On the other hand, if

$$e^{-\tilde{\beta}B} \int_{B_{R_3}} |u|^2 \leq e^{\tilde{\beta}E} \int_{B_{R_1}} |u|^2,$$

then since  $R_2 \leq R_3$ , it holds that

$$(4.11) \quad \begin{aligned} \int_{B_{R_2}} |u|^2 &\leq \left( \int_{B_{R_3}} |u|^2 \right)^{\frac{B}{E+B}} \left( \int_{B_{R_1}} |u|^2 \right)^{\frac{E}{E+B}} \\ &\leq e^{\tilde{\beta}B} \left( \int_{B_{R_1}} |u|^2 \right)^{\frac{B}{E+B}} \left( \int_{B_{R_3}} |u|^2 \right)^{\frac{E}{E+B}}. \end{aligned}$$

Putting together (4.10), (4.11), we arrive at

$$(4.12) \quad \int_{B_{R_2}} |u|^2 \leq C_8 \left( \int_{B_{R_1}} |u|^2 \right)^\tau \left( \int_{B_{R_3}} |u|^2 \right)^{1-\tau},$$

where we have set  $C_8 = \max \left\{ C_7 (1 + e^B) R_2^{-1/2}, e^{\tilde{\beta}B} \right\}$  and  $\tau = \frac{B}{E+B}$ . It is readily seen that  $\frac{B}{E+B} \approx (\log(1/R_1))^{-1}$  when  $R_1$  tends to 0. The proof of Theorem 1.1 is complete.

We now turn to the proof of Theorem 1.2. Fix  $R_2, R_3 \leq R_0$  so that Theorem 1.1 is applicable, and assume that  $R_1 \ll R_2, R_3$ . Applying (1.3) to  $u$ , then raising both sides to  $1/\tau$  yields

$$\|u\|_{L^2(B_{R_2})}^{1/\tau} \leq C^{1/\tau} \|u\|_{L^2(B_{R_1})} \|u\|_{L^2(B_{R_3})}^{-1+1/\tau}.$$

Since  $\frac{1}{\tau} = c_1 \log(1/R_1) + c_2$  for some  $c_1 > 0$  and  $c_2 \in \mathbb{R}$  depending on  $R_2, R_3$  and  $R_0$ , then,

$$\|u\|_{L^2(B_{R_1})} \geq C^{-c_2} \|u\|_{L^2(B_{R_2})}^{c_2} \|u\|_{L^2(B_{R_3})}^{1-c_2} R_1^{c_1 \log \left( C \|u\|_{L^2(B_{R_3})} / \|u\|_{L^2(B_{R_2})} \right)}.$$

This completes the proof of Theorem 1.2.

## 5. THE PROOF OF THEOREM 3.1

Here we prove the crucial  $L^p - L^2$  Carleman estimate for  $\bar{\partial}$  that is stated in Theorem 3.1. We rewrite the operator in polar coordinates then use an eigenvalue decomposition to establish our stated bounds. The techniques used here were first implemented in [12] and are very similar to those that appeared in recent papers [8], [9], [6].

We use standard polar coordinates in  $\mathbb{R}^2 \setminus \{0\}$  by setting  $x_1 = |x| \cos \theta$  and  $x_2 = |x| \sin \theta$ , where  $|x| = \sqrt{x_1^2 + x_2^2}$  and  $\theta = \arctan(x_2/x_1)$ . With the new coordinate  $t = \log |x|$ , we see that

$$\partial_1 = e^{-t} \left( \cos \theta \frac{\partial}{\partial t} - \sin \theta \frac{\partial}{\partial \theta} \right), \quad \partial_2 = e^{-t} \left( \sin \theta \frac{\partial}{\partial t} + \cos \theta \frac{\partial}{\partial \theta} \right)$$

so that

$$(5.1) \quad \mathcal{L} := e^{t-i\theta} \bar{\partial} = \partial_t + i\partial_\theta.$$

The eigenvalues of  $\partial_\theta$  are  $ik$ ,  $k \in \mathbb{Z}$ , with corresponding eigenspace  $E_k = \text{span}\{e_k\}$ , where  $e_k = \frac{1}{\sqrt{2\pi}} e^{ik\theta}$  so that  $\|e_k\|_{L^2(S^1)} = 1$ . For any  $v \in L^2(S^1)$ , let  $P_k v = v_k$  denote the projection of  $v$  onto  $E_k$ . We remark that the projection operator,  $P_k$ , acts only on the angular variables. In particular,  $P_k v(t, \omega) = P_k v(t, \cdot)(\omega)$ . We may then rewrite the operator  $\mathcal{L}$  as

$$(5.2) \quad \mathcal{L} = \partial_t - \sum_{k \in \mathbb{Z}} k P_k.$$

Recall that we introduced the radial weight function  $h$  above. With  $|x| = e^t$ , the weight function is given in terms of  $t$  as

$$h(t) = h_\beta(t) = -\beta t + \beta e^{t/4}.$$

Since our result applies to functions that are supported in  $B_{\tilde{R}_0} \setminus \{0\}$ , then in terms of the new coordinate  $t$ , we study the case when  $t$  is sufficiently close to  $-\infty$ .

By a slight modification to the result described by [9, Lemma 2], we get the following lemma.

**Lemma 5.1.** *Let  $M, N \in \mathbb{N}$  and let  $\{c_k\}$  be a sequence of numbers such that  $|c_k| \leq 1$  for all  $k$ . For any  $v \in L^2(S^1)$  and every  $p \in [1, 2]$ , we have that*

$$(5.3) \quad \left\| \sum_{k=N}^M c_k P_k v \right\|_{L^2(S^1)} \leq C \left( \sum_{k=N}^M |c_k|^2 \right)^{\frac{1}{p} - \frac{1}{2}} \|v\|_{L^p(S^1)},$$

where  $C = C(p)$ .

*Proof.* Recall that  $P_k v = v_k$  is the projection of  $v$  onto  $E_k$ . Thus, for every  $k \in \mathbb{Z}$ ,  $P_k v = \langle v, e_k \rangle e_k$ , where we use the notation  $\langle \cdot, \cdot \rangle$  to denote a pairing of elements in dual spaces. By Parseval's identity,

$$\sum_{k=-\infty}^{\infty} |\langle e_k, v \rangle|^2 = \|v\|_{L^2(S^1)}^2.$$

Since  $\|e_k\|_{L^\infty(S^1)} = \frac{1}{\sqrt{2\pi}}$  for all  $k \in \mathbb{Z}$ , then

$$\|P_k v\|_{L^\infty(S^1)}^2 = \|\langle v, e_k \rangle e_k\|_{L^\infty(S^1)}^2 = \frac{1}{2\pi} |\langle e_k, v \rangle|^2 \leq \frac{1}{2\pi} \|v\|_{L^2(S^1)}^2.$$

It follows that for any  $u \in L^2(S^1)$

$$\langle u, P_k v \rangle = \langle P_k u, v \rangle \leq \|P_k u\|_{L^\infty(S^1)} \|v\|_{L^1(S^1)} \leq c \|u\|_{L^2(S^1)} \|v\|_{L^1(S^1)}.$$

By duality, we conclude that

$$(5.4) \quad \|P_k v\|_{L^2(S^1)} \leq c \|v\|_{L^1(S^1)}.$$

It follows from the normalization condition in combination with Parseval's identity that

$$(5.5) \quad \|P_k v\|_{L^2(S^1)} \leq \|v\|_{L^2(S^1)}.$$

Interpolating (5.4) and (5.5) gives that

$$(5.6) \quad \|P_k v\|_{L^2(S^1)} \leq C(p) \|v\|_{L^p(S^1)}$$

for all  $1 \leq p \leq 2$ .

Now we consider a more general setting. Let  $\{c_k\}$  be a sequence of numbers with  $|c_k| \leq 1$ . For all  $N \leq M$ , it follows from orthogonality and Hölder's inequality that

$$\left\| \sum_{k=N}^M c_k P_k v \right\|_{L^2(S^1)}^2 = \sum_{k=N}^M |c_k|^2 \langle P_k v, v \rangle \leq \sum_{k=N}^M |c_k|^2 \|P_k v\|_{L^\infty(S^1)} \|v\|_{L^1(S^1)}.$$

Since each  $e_k$  is normalized in  $L^2(S^1)$ , then  $\sqrt{2\pi} \|e_k\|_{L^\infty(S^1)} = 1 = \|e_k\|_{L^2(S^1)}$  and then  $\|P_k v\|_{L^\infty(S^1)} = \frac{1}{\sqrt{2\pi}} \|P_k v\|_{L^2(S^1)} \leq c \|v\|_{L^1(S^1)}$ , where we have used observation (5.4). It follows that

$$\left\| \sum_{k=N}^M c_k P_k v \right\|_{L^2(S^1)} \leq C \left( \sum_{k=N}^M |c_k|^2 \right)^{\frac{1}{2}} \|v\|_{L^1(S^1)}.$$

Clearly, as long as  $|c_k| \leq 1$ , then

$$\left\| \sum_{k=N}^M c_k P_k v \right\|_{L^2(S^1)} \leq \|v\|_{L^2(S^1)}.$$

As before, we interpolate the last two inequalities to reach (5.3).  $\square$

The following proposition is crucial to the proof of Theorem 3.1.

**Proposition 5.2.** *Let  $1 < p < 2$ . There exists a  $\beta_0 > 0$  such that we have for  $\beta > \beta_0$  and  $z \in C_0^\infty((-\infty, 0) \times S^1)$  that*

$$(5.7) \quad \left\| e^{h_\beta(t)+t/8} z \right\|_{L^2(dtd\theta)} \leq C \beta^{-1+\frac{1}{p}} \left\| e^{h_\beta(t)-t/8} \mathcal{L} z \right\|_{L^p(dtd\theta)},$$

where  $C = C(p)$ .

*Proof.* To prove this lemma, we introduce the conjugated operator  $\mathcal{L}_\beta$  of  $\mathcal{L}$ , defined by

$$\mathcal{L}_\beta z = e^{h_\beta(t)} \mathcal{L} \left( e^{-h_\beta(t)} z \right).$$

Replacing  $z$  with  $e^{-h_\beta(t)}z$  in inequality (5.7) shows that it suffices to prove that

$$(5.8) \quad \left\| e^{t/8} z \right\|_{L^2(dt d\theta)} \leq C \beta^{-1+\frac{1}{p}} \left\| e^{-t/8} \mathcal{L}_\beta z \right\|_{L^p(dt d\theta)}.$$

From (5.1) and (5.2), the operator  $\mathcal{L}_\beta$  takes the form

$$(5.9) \quad \mathcal{L}_\beta = \sum_{k \in \mathbb{Z}} (\partial_t - h'_\beta(t) - k) P_k = \sum_{k \in \mathbb{Z}} \left( \partial_t + \beta - \frac{\beta}{4} e^{t/4} - k \right) P_k.$$

Since  $\sum_{k \in \mathbb{Z}} P_k z = z$ , we split the sum into three parts. Let  $M = \lceil 2\beta \rceil$  and define

$$P_\beta^h = \sum_{k > M} P_k, \quad P_\beta^l = \sum_{k=0}^M P_k, \quad P_\beta^n = \sum_{k < 0} P_k.$$

In order to prove (5.8), it suffices to show that

$$(5.10) \quad \left\| e^{t/8} P_\beta^\square z \right\|_{L^2(dt d\theta)} \leq C \beta^{-1+\frac{1}{p}} \left\| e^{-t/8} \mathcal{L}_\beta z \right\|_{L^p(dt d\theta)}$$

for  $\square = h, l, n$ ,  $z \in C_0^\infty((-\infty, 0) \times S^1)$ , and  $1 < p < 2$ . The sum of all three inequalities will yield (5.8), which implies (5.7).

From (5.9), we have the first order differential equation

$$(5.11) \quad P_k \mathcal{L}_\beta z = (\partial_t - h'_\beta(t) - k) P_k z.$$

For  $z \in C_0^\infty((-\infty, 0) \times S^1)$ , solving the first order differential equation gives that

$$(5.12) \quad \begin{aligned} P_k z(t, \omega) &= - \int_t^\infty e^{k(t-s)+h_\beta(t)-h_\beta(s)} P_k \mathcal{L}_\beta z(s, \omega) ds \\ &= \int_{-\infty}^t e^{k(t-s)+h_\beta(t)-h_\beta(s)} P_k \mathcal{L}_\beta z(s, \omega) ds. \end{aligned}$$

We first establish (5.10) with  $\square = h$ . For  $k > M \geq 2\beta$ , if  $-\infty < t \leq s \leq 0$ , then

$$k(t-s) + h_\beta(t) - h_\beta(s) = -(k-\beta)|t-s| + \beta \left( e^{t/4} - e^{s/4} \right) \leq -\frac{k}{2} |t-s|.$$

Taking the  $L^2(S^1)$ -norm in (5.12) and using this bound gives that

$$\|P_k z(t, \cdot)\|_{L^2(S^1)} \leq \int_{-\infty}^\infty e^{-\frac{1}{2}k|t-s|} \|P_k \mathcal{L}_\beta z(s, \cdot)\|_{L^2(S^1)} ds.$$

With the aid of (5.6), we get

$$\|P_k z(t, \cdot)\|_{L^2(S^1)} \leq C \int_{-\infty}^\infty e^{-\frac{1}{2}k|t-s|} \|\mathcal{L}_\beta z(s, \cdot)\|_{L^p(S^1)} ds$$

for any  $1 \leq p \leq 2$ . Applying Young's inequality for convolution then yields

$$\|P_k z\|_{L^2(dt d\theta)} \leq C \left( \int_{-\infty}^\infty e^{-\frac{\sigma}{2}k|z|} dz \right)^{\frac{1}{\sigma}} \|\mathcal{L}_\beta z\|_{L^p(dt d\theta)} \leq C k^{\frac{1}{p}-\frac{3}{2}} \|\mathcal{L}_\beta z\|_{L^p(dt d\theta)},$$

where  $\frac{1}{\sigma} = \frac{3}{2} - \frac{1}{p}$ . Squaring and summing up  $k > M$  gives that

$$\sum_{k>M} \|P_k z\|_{L^2(dt d\theta)}^2 \leq C \sum_{k>M} k^{-3+\frac{2}{p}} \|\mathcal{L}_\beta z\|_{L^p(dt d\theta)}^2 = C \beta^{-2+\frac{2}{p}} \|\mathcal{L}_\beta z\|_{L^p(dt d\theta)}^2,$$

where we have used that  $p > 1$  to conclude that the series converges. An application of orthogonality shows that

$$\left\| P_\beta^h z \right\|_{L^2(dt d\theta)} \leq C \beta^{-1+\frac{1}{p}} \|\mathcal{L}_\beta z\|_{L^p(dt d\theta)}$$

which implies (5.10) with  $\square = h$ .

Now we prove (5.10) for  $\square = n$ . For  $k < 0$ , if  $-\infty < s \leq t \leq 0$ , then

$$\begin{aligned} k(t-s) + h_\beta(t) - h_\beta(s) &= -(\beta - k)|t-s| + \beta e^{t/4} \left(1 - e^{-|t-s|/4}\right) \\ &\leq -\left(\frac{\beta}{2} - k\right)|t-s|, \end{aligned}$$

where we have performed a Taylor expansion. Repeating the arguments from above shows that for  $k < 0$ ,

$$\|P_k z\|_{L^2(dt d\theta)} \leq C \left(\frac{\beta}{2} - k\right)^{\frac{1}{p}-\frac{3}{2}} \|\mathcal{L}_\beta z\|_{L^p(dt d\theta)}.$$

Squaring and summing up  $k < 0$  gives that

$$\sum_{k<0} \|P_k z\|_{L^2(dt d\theta)}^2 \leq C \beta^{-2+\frac{2}{p}} \|\mathcal{L}_\beta z\|_{L^p(dt d\theta)}^2,$$

where we have again used that  $p > 1$  to conclude that the series converges. As in the previous setting, (5.10) holds with  $\square = n$ .

Fix  $t \in (-\infty, 0)$  and set  $N = \lceil -h'_\beta(t) \rceil$ . Recalling that  $h_\beta(t) = -\beta t + \beta e^{t/4}$ , an application Taylor's theorem shows that for all  $s, t \in (-\infty, 0)$

$$h_\beta(s) - h_\beta(t) = h'_\beta(t)(s-t) + \frac{1}{2} h''_\beta(s_0)(s-t)^2,$$

where  $s_0$  is some number between  $s$  and  $t$ . If  $s > t$ , then

$$\begin{aligned} (5.13) \quad &k(t-s) + h_\beta(t) - h_\beta(s) \\ &= -(k - N - 1)|t-s| - (N+1 + h'_\beta(t))(s-t) - \frac{\beta}{32} e^{s_0/4} (s-t)^2 \\ &\leq -(k - N - 1)|t-s| - (N+1 + h'_\beta(t))|s-t| - \frac{\beta}{32} e^{t/4} (s-t)^2. \end{aligned}$$

Alternatively, if  $s \leq t$ , then

$$\begin{aligned} (5.14) \quad &k(t-s) + h_\beta(t) - h_\beta(s) \\ &= -(N-k)|t-s| - (-h'_\beta(t) - N)(t-s) - \frac{\beta}{32} e^{s_1/4} (s-t)^2 \\ &\leq -(N-k)|t-s| - (-h'_\beta(t) - N)|t-s| - \frac{\beta}{32} e^{s/4} (s-t)^2. \end{aligned}$$

For this reason, we split the sum corresponding to  $\square = l$  and use both representations from (5.12).

First we consider the values  $N + 1 \leq k \leq M$ . Let us denote

$$a(\beta, s, t) = (N + 1 + h'_\beta(t))|s - t| + \frac{\beta}{32}e^{t/4}(s - t)^2.$$

From the first line of (5.12), we sum over  $k$  and use the bound from (5.13) to get

$$\left\| \sum_{k=N+1}^M P_k z(t, \cdot) \right\|_{L^2(S^1)} \leq \int_{-\infty}^{\infty} \left\| \sum_{k=N+1}^M e^{-(k-N-1)|t-s|-a(\beta,s,t)} P_k \mathcal{L}_\beta z(s, \cdot) \right\|_{L^2(S^1)} ds.$$

With  $c_k = e^{-(k-N-1)|t-s|-a(\beta,s,t)}$ , it is clear that  $|c_k| \leq 1$ . Therefore, Lemma 5.1 is applicable, so we may apply estimate (5.3) to obtain

$$(5.15) \quad \begin{aligned} & \left\| \sum_{k=N+1}^M e^{-(k-N-1)|t-s|-a(\beta,s,t)} P_k \mathcal{L}_\beta z(s, \cdot) \right\|_{L^2(S^1)} \\ & \leq C \left( \sum_{k=N}^M e^{-2(k-N-1)|t-s|-a(\beta,s,t)} \right)^{\frac{1}{p}-\frac{1}{2}} \|\mathcal{L}_\beta z(s, \cdot)\|_{L^p(S^1)} \end{aligned}$$

for all  $1 \leq p \leq 2$ . Since

$$\sum_{k=N+1}^M e^{-2(k-N-1)|t-s|-\frac{a(\beta,s,t)}{2}} \leq \sum_{k=0}^{\infty} e^{-2k|t-s|-\frac{a(\beta,s,t)}{2}} \leq C |t-s|^{-1},$$

then

$$\left\| \sum_{k=N+1}^M P_k z(t, \cdot) \right\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} e^{-\frac{\beta\alpha}{16}e^{t/4}(s-t)^2} |t-s|^{-\alpha} \|\mathcal{L}_\beta z(s, \cdot)\|_{L^p(S^1)} ds,$$

where  $\alpha = \frac{2-p}{2p}$ . Given that if  $t \in (-\infty, 0)$

$$e^{\frac{\beta\alpha}{16}e^{t/4}(s-t)^2} \geq \sqrt{1 + \frac{\alpha\beta}{16}e^{t/4}(s-t)^2} \geq Ce^{t/8}(1 + \beta^{1/2}|s-t|),$$

then it follows that

$$(5.16) \quad e^{-\frac{\beta\alpha}{16}e^{t/4}(s-t)^2} \leq Ce^{-t/8}(1 + \beta^{1/2}|s-t|)^{-1}.$$

We see that

$$(5.17) \quad \left\| \sum_{k=N+1}^M P_k z(t, \cdot) \right\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} \frac{e^{-t/8} \|\mathcal{L}_\beta z(s, \cdot)\|_{L^p(S^1)}}{|t-s|^\alpha (1 + \beta^{1/2}|s-t|)} ds.$$

For  $0 \leq k \leq N$ , we use the second line of (5.12), then sum over  $k$  and use the bound from (5.14) to get

$$\begin{aligned} & \left\| \sum_{k=0}^N P_k z(t, \cdot) \right\|_{L^2(S^1)} \\ & \leq \int_{-\infty}^{\infty} \left\| \sum_{k=0}^N e^{-(N-k)|t-s| - (-h'_\beta(t)-N)|t-s| - \frac{\beta}{32} e^{s/4} (s-t)^2} P_k \mathcal{L}_\beta z(s, \cdot) \right\|_{L^2(S^1)} ds. \end{aligned}$$

Arguing as before, we similarly conclude that

$$(5.18) \quad \left\| \sum_{k=0}^N P_k z(t, \cdot) \right\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} \frac{e^{-s/8} \|\mathcal{L}_\beta z(s, \cdot)\|_{L^p(S^1)}}{|t-s|^\alpha (1 + \beta^{1/2}|s-t|)} ds.$$

Combining (5.17) and (5.18) shows that

$$\left\| e^{t/8} P_\beta^l z(t, \cdot) \right\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} \frac{\|e^{-s/8} \mathcal{L}_\beta z(s, \cdot)\|_{L^p(S^1)}}{|t-s|^\alpha (1 + \beta^{1/2}|s-t|)} ds.$$

Applying Young's inequality for convolution, we get

$$\left\| e^{t/8} P_\beta^l z \right\|_{L^2(dt d\theta)} \leq C \left[ \int_{-\infty}^{\infty} \left( \frac{|z|^{-\alpha}}{1 + \beta^{1/2}|z|} \right)^\sigma dz \right]^{\frac{1}{\sigma}} \left\| e^{-t/8} \mathcal{L}_\beta z \right\|_{L^p(dt d\theta)},$$

where  $\frac{1}{\sigma} = \frac{3}{2} - \frac{1}{p}$ . A direct calculation then shows that

$$\left[ \int_{-\infty}^{\infty} \left( \frac{|z|^{-\alpha}}{1 + \beta^{1/2}|z|} \right)^\sigma dz \right]^{\frac{1}{\sigma}} \leq C \beta^{-\frac{1}{2\sigma} + \frac{\alpha}{2}}$$

with  $C = C(p)$  for  $1 < p < 2$ . Since  $-\frac{1}{2\sigma} + \frac{\alpha}{2} = \frac{1}{2p} - \frac{3}{4} + \frac{1}{2p} - \frac{1}{4} = -1 + \frac{1}{p}$ , then we have shown (5.10) with  $\square = l$ , thereby completing the proof of the proposition.  $\square$

We now present the proof of Theorem 3.1.

*Proof of Theorem 3.1.* Note that for any  $\alpha' \ll \beta$ , if we replace  $h_\beta(t)$  with  $\tilde{h}_\beta(t) = -(\beta - \alpha')t + \beta e^{t/4} = h_\beta(t) + \alpha't$ , then the results of Proposition 5.2 still hold with a possibly modified constant. That is,

$$\left\| e^{\tilde{h}_\beta(t)+t/8} z \right\|_{L^2(dt d\theta)} \leq C \beta^{-1+\frac{1}{p}} \left\| e^{\tilde{h}_\beta(t)-t/8} \mathcal{L}z \right\|_{L^p(dt d\theta)}.$$

Since  $e^{2t} dt d\theta = dx$ , then

$$\left\| e^{\tilde{h}_\beta(t)+t/8} z \right\|_{L^2(dt d\theta)} = \left\| e^{h_\beta(t)+(\alpha'-7/8)t} z e^t \right\|_{L^2(dt d\theta)} = \left\| |x|^{\alpha'-7/8} e^h z \right\|_{L^2(dx)}$$

and

$$\begin{aligned} \left\| e^{\tilde{h}_\beta(t)-t/8} \mathcal{L}z \right\|_{L^p(dt d\theta)} &= \left\| e^{h_\beta(t)+(\alpha'+7/8-2/p)t-i\theta} \bar{\partial}z e^{2t/p} \right\|_{L^p(dt d\theta)} \\ &= \left\| |x|^{\alpha'+7/8-2/p} e^h \bar{\partial}z \right\|_{L^p(dx)}, \end{aligned}$$

so we reach the conclusion of the theorem by setting  $\alpha' = \frac{15}{8}$ .  $\square$

## 6. AN INTERIOR ESTIMATE

In this final section, we present the statement and proof of the interior estimate that was used in the proof of our main theorems.

**Lemma 6.1.** *Assume that for some  $s > \frac{4}{3}$ , (1.1) holds. Let  $u \in H_{loc}^1(\Omega)$  be a solution of (1.2). For any  $0 < a_3 < a_1 < a_2 < a_4$  such that  $B_{a_4r} \subset \Omega$  and  $a_4r < 1$  is sufficiently small with respect to  $s$ ,  $M_0$  and  $\delta_0$ , we have*

$$(6.1) \quad \int_{\{a_1r < |x| < a_2r\}} |x|^2 |\nabla u|^2 dx \leq C \int_{\{a_3r < |x| < a_4r\}} |u|^2 dx,$$

where the constant  $C$  is independent of  $r$  and  $u$ .

*Proof.* Let  $\xi \in C_0^\infty(\mathbb{R}^n)$  satisfy  $0 \leq \xi(x) \leq 1$  and

$$(6.2) \quad \xi(x) = \begin{cases} 0, & |x| \leq a_3r, \\ 1, & a_1r < |x| < a_2r, \\ 0, & |x| \geq a_4r. \end{cases}$$

From (1.2), we obtain via integration by parts that

$$\begin{aligned} 0 &= - \int [\operatorname{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \operatorname{div} u)] \cdot \xi^2 u dx \\ &= \int \sum_{i,j,k,l=1}^2 [\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})] \partial_l u_k \partial_j (\xi^2 u_i) dx \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int \xi^2 \left[ \sum_{i,j=1}^2 \lambda \partial_j u_j \partial_i u_i + \sum_{i,j=1}^2 \mu (\partial_i u_j \partial_j u_i + \partial_j u_i \partial_i u_j) \right] dx$$

and

$$I_2 = \int \sum_{i,j,k,l=1}^2 \partial_j (\xi^2) [\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})] \partial_l u_k u_i dx.$$

Another integration by parts shows that

$$\begin{aligned}
& \int \xi^2 (2\mu - \delta_0) \partial_i u_j \partial_j u_i dx \\
&= - \int \partial_j [\xi^2 (2\mu - \delta_0)] \partial_i u_j u_i dx - \int \xi^2 (2\mu - \delta_0) \partial_{ij}^2 u_j u_i dx \\
&= - \int \partial_j [\xi^2 (2\mu - \delta_0)] \partial_i u_j u_i dx \\
&\quad + \int \partial_i [\xi^2 (2\mu - \delta_0)] \partial_j u_j u_i dx + \int \xi^2 (2\mu - \delta_0) \partial_j u_j \partial_i u_i dx.
\end{aligned}$$

Substituting this equality into  $I_1$  shows that

$$\begin{aligned}
I_1 &= \int \xi^2 \left[ \sum_{i,j=1}^2 \lambda \partial_j u_j \partial_i u_i + \sum_{i,j=1}^2 (2\mu - \delta_0) \partial_i u_j \partial_j u_i \right] dx \\
&\quad + \int \xi^2 \sum_{i,j=1}^2 (\mu - \delta_0) (\partial_j u_i \partial_j u_i - \partial_i u_j \partial_j u_i) dx + \delta_0 \int \xi^2 \sum_{i,j=1}^2 \partial_j u_i \partial_j u_i dx \\
&= \int \xi^2 (2\mu + \lambda - \delta_0) \left( \sum_{i=1}^2 \partial_i u_i \right)^2 dx + \delta_0 \int \xi^2 \sum_{i,j=1}^2 (\partial_j u_i)^2 dx \\
&\quad + \int \xi^2 (\mu - \delta_0) \sum_{i,j=1}^2 (\partial_j u_i \partial_j u_i - \partial_i u_j \partial_j u_i) dx - I_3 - I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_3 &= \int (2\mu - \delta_0) \sum_{i,j=1}^2 [\partial_j (\xi^2) \partial_i u_j u_i - \partial_i (\xi^2) \partial_j u_j u_i] dx \\
I_4 &= \int 2\xi^2 \sum_{i,j=1}^2 (\partial_j \mu \partial_i u_j u_i - \partial_i \mu \partial_j u_j u_i) dx.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{i,j=1}^2 (\partial_j u_i \partial_j u_i - \partial_i u_j \partial_j u_i) &= \frac{1}{2} \sum_{i,j=1}^2 (\partial_j u_i \partial_j u_i - 2\partial_i u_j \partial_j u_i + \partial_i u_j \partial_i u_j) \\
&= \frac{1}{2} \sum_{i,j=1}^2 (\partial_j u_i - \partial_i u_j)^2,
\end{aligned}$$

then we conclude from the lower bounds of (1.1) that

$$I_1 \geq \delta_0 \int \xi^2 |\nabla u|^2 dx - I_3 - I_4.$$

As  $I_1 + I_2 = 0$ , then

$$(6.3) \quad \delta_0 \int \xi^2 |\nabla u|^2 dx \leq I_4 + I_5,$$

where with  $I_5 := -I_2 + I_3$ , i.e.,

$$\begin{aligned} I_5 &= \int (\mu - \delta_0) \sum_{i,j=1}^2 \partial_j (\xi^2) \partial_i u_j u_i dx - \int \mu \sum_{i,j=1}^2 \partial_j (\xi^2) \partial_j u_i u_i dx \\ &\quad - \int (\lambda + 2\mu - \delta_0) \sum_{i,j=1}^2 \partial_i (\xi^2) \partial_j u_j u_i dx \\ &\leq C \int \xi |\nabla \xi| |\nabla u| |u| \leq \frac{\delta_0}{4} \int \xi^2 |\nabla u|^2 + \frac{C}{\delta_0} \int |\nabla \xi|^2 |u|^2, \end{aligned}$$

where we have used the upper bounds from (1.1) in combination with a Sobolev embedding. For  $I_4$ , observe that

$$|I_4| \leq C \int \xi^2 |\nabla \mu| |\nabla u| |u| \leq \frac{\delta_0}{4} \int \xi^2 |\nabla u|^2 + \frac{C}{\delta_0} \int \xi^2 |\nabla \mu|^2 |u|^2.$$

Define  $X = B_{a_4 r} \setminus \bar{B}_{a_3 r}$  and  $Y = B_{a_2 r} \setminus \bar{B}_{a_1 r}$  so that  $\text{supp } \xi \subset X$  and  $\xi \equiv 1$  on  $Y$ . Since  $s > 4/3$ , then  $s^* > 4$ , where  $s^*$  denotes the Sobolev conjugate of  $s$ . With  $\alpha = 1 - \frac{2}{2^*} - \frac{2}{s^*}$ , applications of Hölder and Sobolev inequalities show that

$$\begin{aligned} \int \xi^2 |\nabla \mu|^2 |u|^2 &\leq \left( \int_X 1 \right)^{1 - \frac{2}{2^*} - \frac{2}{s^*}} \left( \int_X |\nabla \mu|^{s^*} \right)^{2/s^*} \left( \int_X |\xi u|^{2^*} \right)^{2/2^*} \\ &\leq C (a_4 r)^{2\alpha} \|\mu\|_{W^{2,s}}^2 \int |\nabla (\xi u)|^2 \\ &\leq C (a_4 r)^{2\alpha} \left( \int \xi^2 |\nabla u|^2 + \int |\nabla \xi|^2 |u|^2 \right), \end{aligned}$$

where the last line uses (1.1). Since we can always choose  $2^* \in (2, \infty)$  so that  $\alpha \geq \frac{1}{2}$ , then by substituting these estimates into (6.3) and simplifying, we see that

$$\int \xi^2 |\nabla u|^2 dx \leq \frac{C}{\delta_0^2} \left( a_4 r \int \xi^2 |\nabla u|^2 + \int |\nabla \xi|^2 |u|^2 \right).$$

Assuming that  $a_4 r \leq \frac{\delta_0^2}{2C}$ , we may absorb the first term on the left into the right to obtain

$$\begin{aligned} \int_Y |x|^2 |\nabla u|^2 dx &\leq (a_2 r)^2 \int_Y |\nabla u|^2 dx \leq (a_2 r)^2 \int \xi^2 |\nabla u|^2 dx \\ &\leq C \left( \frac{a_2 r}{\delta_0} \right)^2 \int |\nabla \xi|^2 |u|^2 \leq C \int_X |u|^2, \end{aligned}$$

as required.  $\square$

## REFERENCES

- [1] G. Alessandrini and R. Magnanini, *Elliptic equations in divergence form, geometrical critical points of solutions and Stekloff eigenfunctions*, SIAM J. Math. Anal., **25** (1964), 1259-1268.
- [2] G. Alessandrini and A. Morassi, *Strong unique continuation for the Lamé system of elasticity*, Comm. in PDE., **26**, 1787-1810, 2001.
- [3] L. Bers, F. John, and M. Schechter, *Partial differential equations*. Lectures in Applied Mathematics III. Interscience Publishers, New York, 1964.
- [4] L. Bers and L. Nirenberg, *On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications*, Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, pages 111-140, 1954.
- [5] A. Coffman and Y. Pan, *Smooth counterexamples to strong unique continuation for a Beltrami system in  $\mathbb{C}^2$* , Comm. PDE, **37** (2012), 2228-2244.
- [6] B. Davey, *Quantitative unique continuation for Schrödinger operators*, Preprint, arXiv:1903.04021, 2019.
- [7] B. Davey, C.-L. Lin, and J.-N. Wang, *Liouville-type theorem for the Lamé system with singular coefficients*, Proc AMS, **147** (2019), no. 6, 2619-2624.
- [8] B. Davey and J. Zhu, *Quantitative uniqueness of solutions to second order elliptic equations with singular lower order terms*, Preprint, arXiv:1702.04742, 2017.
- [9] B. Davey and J. Zhu, *Quantitative uniqueness of solutions to second order elliptic equations with singular potentials in two dimensions*, Calc. Var. Partial Differential Equations, **57** (2018), no. 3, Art. 92, 27.
- [10] L. Escauriaza, *Unique continuation for the system of elasticity in the plane*, Proc. Amer. Math. Soc., **134** (2006), 2015-2018.
- [11] L. Escauriaza, F.J. Fernández, and S. Vessella, *Doubling properties of caloric functions*, Appl. Anal., **85** (2006), 205-223.
- [12] D. Jerison, *Carleman inequalities for the Dirac and Laplace operators and unique continuation*, Adv. in Math., **62** (1986), no. 2, 118-134.
- [13] C. Kenig and J.-N. Wang, *Unique continuation for the elasticity system and a counterexample for second order elliptic systems*, Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory, Vol 1:159-178, 2016.
- [14] H. Koch, C.-L. Lin, and J.-N. Wang, *Doubling inequalities for the Lamé system with rough coefficients*, Proc AMS, **144** (2016), no. 4, 5309-5318.
- [15] C.-L. Lin, G. Nakamura, G. Uhlmann, and J.-N. Wang, *Quantitative strong unique continuation for the Lamé system with less regular coefficients*, Methods Appl. Anal., **18** (2011), no. 1, 85-92.

- [16] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in selfadjoint divergence form with Hölder continuous coefficients*, Arch. Rational Mech. Anal., **54** (1974), 105-117.
- [17] C.-L. Lin and J.-N. Wang, *Strong unique continuation for the Lamé system with Lipschitz coefficients*, Math. Ann., **331** (2005), 611-629.
- [18] C.-L. Lin, G. Nakamura and J.-N. Wang *Optimal three-ball inequalities and quantitative uniqueness for the Lamé system with Lipschitz coefficients*, Duke Math Journal., **155** (2010), 189-204.
- [19] A. Plis, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **11** (1963), 95-100.
- [20] F. Schulz, *On the unique continuation property of elliptic divergence form equations in the plane*, Math. Z., **228** (1998), 201-206.

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