QUANTITATIVE UNIQUENESS ESTIMATES FOR THE LAMÉ SYSTEM WITH COEFFICIENTS HAVING ALMOST OPTIMAL REGULARITY

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Abstract. We derive the doubling inequality and the optimal three-ball inequality for the Lamé system in the plane. The main contribution of this work is to derive these quantitative uniqueness estimates when the Lamé coefficients $(\mu, \lambda) \in W^{2,s}(\Omega) \times L^\infty(\Omega)$ for any fixed $s > 1$. Consequently, we establish the strong unique continuation property (SUCP) for the Lamé system in the plane when λ is essentially bounded and μ belongs to some function in $C^{0,\gamma} \cap W^{1,p}$ with $\gamma = 2(s-1)/s$ and $p = 2s/(2-s)$ (note $\gamma \to 0$, $p \to 2$ as $s \to 1$). This result improves the early work [4] where $s > 4/3$.

1. Introduction

In this paper, we are interested in the quantitative strong unique continuation property (SUCP) for the Lamé system in the plane

$$\operatorname{div}\left(\mu\left(\nabla u + (\nabla u)^t\right)\right) + \nabla(\lambda \operatorname{div} u) = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \tag{1.1}$$

where $u = (u_1, u_2)^t$ is the real-valued displacement vector and $(\nabla u)_{jk} = \partial_k u_j$ for $j, k = 1, 2$. The domain $\Omega \subset \mathbb{R}^2$ is an open, connected set that contains the origin. Our aim here is to find the best possible regularity assumption on (μ, λ) such that the SUCP holds for (1.1). This study is motivated by the similar result for the two-dimensional second-order elliptic equation in divergence form where the SUCP holds provided the elliptic coefficients are essentially bounded. We want to remark that in three and higher dimensions, both qualitative and quantitative SUCP for (1.1) have been proved when $(\mu, \lambda) \in C^{0,1} \times L^\infty$, which is optimal in view of the positive and negative results for the second elliptic equation.

Before stating main theorems of the paper, we introduce some notations and assumptions. Let $s > 1$ be fixed. Assume that $\mu \in W^{2,s}(\Omega)$ and

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$\lambda \in L^{\infty}(\Omega)$, where

$$\begin{cases} \mu(x) \geq \delta_0, \quad \lambda(x) + 2\mu(x) \geq \delta_0, \quad \forall\, x \in \Omega, \\ \|\mu\|_{W^{2,s}(\Omega)} + \|\lambda\|_{L^{\infty}(\Omega)} \leq M_0 \end{cases} \tag{1.2}$$

for some positive constants δ_0, M_0. Recall that

$$\|f\|_{W^{2,s}(\Omega)} = \sum_{|\alpha| \leq 2} \|\partial_x^{\alpha} f\|_{L^s(\Omega)}.$$

In the sequel, B_r denotes an open ball of radius $r > 0$ centered at the origin, while $B_r(x_0)$ denotes an open ball of radius r centered at x_0. We now state the main results of the paper.

Theorem 1.1. *Assume that* (1.2) *hold. Then there exists an* $R_0 > 0$*, depending on* s, M_0, δ_0*, such that* $B_{6R_0} \subset \Omega$ *and for any* $u \in H^1_{loc}(B_{6R_0})$ *satisfying* (1.1) *in* B_{6R_0}*, given* $0 < R_1 < R_2 \leq R_0$*, we have*

$$\int_{B_{R_2}} |u|^2 dx \leq C \left(\frac{3R_2}{R_1} \right)^{2\beta_0} \int_{B_{R_1}} |u|^2 dx, \tag{1.3}$$

where

$$\beta_0 \geq \frac{1}{2} \ln \left(\frac{\hat{C} \int_{B_{6R_0}} |u|^2 dx}{\int_{B_{R_0}} |u|^2 dx} \right)$$

with $\hat{C} = \hat{C}(s, M_0, \delta_0)$.

Our next result establishes the vanishing order of solutions to (1.1), thereby proving the strong unique continuation property (SUCP) in our setting.

Theorem 1.2. *Assume that* (1.2) *hold and* R_0 *is defined in Theorem 1.1. There exist positive constants* K *and* m*, depending on* s, M_0, δ_0 *and* u*, such that for all* R *sufficiently small*

$$\int_{B_R} |u|^2 dx \geq K R^m. \tag{1.4}$$

Next result is the optimal three-ball inequality.

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, for any* $u \in H^1_{loc}(B_{R_0})$ *satisfying* (1.1) *in* B_{R_0}*, if* $0 < R_1 < R_2 < R_3/6 < R_3 \leq R_0$*, then*

$$\int_{B_{R_2}} |u|^2 \leq \widetilde{C} \left(\int_{B_{R_1}} |u|^2 \right)^{\tau} \left(\int_{B_{R_3}} |u|^2 \right)^{1-\tau}, \tag{1.5}$$

where $\widetilde{C} = \widetilde{C}(s, M_0, \delta_0, R_3/R_2)$*, and* $\tau = \tau(R_2/R_1, R_2/R_3) \in (0,1)$*. Moreover, for fixed* R_2 *and* R_3*, the exponent* τ *behaves like* $1/(-\log R_1)$ *whenever* R_1 *is sufficiently small.*

We want to point out that $\widetilde{C}$ is independent of R_1 and τ has the asymptotic behavior like $(-\log R_1)^{-1}$. These facts are crucial to the derivation of the vanishing order for nontrivial solutions u to (1.1). Due to the behavior of τ, the three-ball inequality is called optimal [8].

In Theorems 1.1, 1.2 and 1.3, our main interest is in the case where $s \in (1, 2)$. In view of the Sobolev embedding theorem, $\mu \in W^{2,s}$ implies

$$\mu \in C^{0,\gamma} \cap W^{1,p}$$

with $\gamma = 2(s-1)/s$ and $p = 2s/(2-s)$. Note that $\gamma \to 0$ and $p \to 2$ as $s \to 1$. In view of related counterexamples established in [9] where $\mu \in L^\infty$ (i.e., $\gamma = 0$), in this work we establish the quantitative SUCP for (1.1) with coefficients having nearly optimal regularity. On the other hand, in [3], a Liouville-type theorem was proved for the (1.1) in the plane when $\mu \in W^{1,2}$ and $\|\nabla \mu\|_{L^2}$ is small. From this result, we proved a weak unique continuation property and the uniqueness of the Cauchy problem for (1.1) (see Corollaries 1.3, 1.4 in [3]). We now summarize some known qualitative and quantitative results on the SUCP for (1.1):

- $\lambda, \mu \in C^{1,1}$, $n \geq 2$ (quantitative): Alessandrini and Morassi [1].
- $\lambda, \mu \in C^{0,1}$, $n = 2$ (qualitative): Lin and Wang [13].
- $\lambda \in L^\infty, \mu \in C^{0,1}$, $n = 2$ (qualitative): Escauriaza [7].
- $\lambda, \mu \in C^{0,1}$, $n \geq 2$ (quantitative): Lin, Nakamura, and Wang [14].
- $\mu \in C^{0,1}, \lambda \in L^\infty$, $n \geq 2$ (quantitative): Lin, Nakamura, Uhlmann, and Wang [11].
- $\mu \in C^{0,1}, \lambda \in L^\infty$, $n \geq 2$ (doubling inequality): Koch, Lin, and Wang [10].
- $\mu \in W^{2,s}, s > 4/3$, $\lambda \in L^\infty$, $n = 2$ (optimal three-ball inequality): Davey, Lin, and Wang [4].

This work can be seen an improvement of the result in [4] where $s > 4/3$ is needed. The key to the improvement is that we use a system of $\bar{\partial}$ equations here (see (2.2), (2.3)), instead of a combination of $\bar{\partial}$ equation and second-order equation with a divergence term used in [4]. By this observation, we can prove our theorems using an $L^2 - L^q$ Carleman estimate for the $\bar{\partial}$ operator (see Theorem 3.2). On the contrary, in [4], one used an $L^2 - L^q$ Carleman estimate for the second-order equation with a divergence term whose derivation is more involved.

This article is organized as follows. In Section 2, we derive a reduced system (a $\bar{\partial}$-system) from (1.1) and prove a useful interior estimate. The proofs of Theorems 1.1, 1.2, and 1.3, rely on suitable Carleman estimates whose derivations are given in Section 3. Section 4 is devoted to the proofs of the main theorems.

2. Reduced system and interior estimates

In this section, we will derive a reduced system involving the $\bar\partial$ operator from (1.1). Basically, we show that (1.1) can be written as a systems of complex-valued two first-order elliptic equations. These two representations will be instrumental to the Carleman estimate arguments that are used to prove our main theorems.

We now define the operator $\bar\partial = \partial_1 + i\partial_2$ and two auxiliary functions

$$\begin{aligned} v &= \frac{\lambda+2\mu}{\mu}\operatorname{div} u = \frac{\lambda+2\mu}{\mu}(\partial_1 u_1 + \partial_2 u_2), \\ w &= \operatorname{curl} u = \partial_2 u_1 - \partial_1 u_2. \end{aligned} \tag{2.1}$$

It is not hard to show that

$$\bar\partial(u_1 - iu_2) = \frac{\mu}{\lambda+2\mu} v + iw. \tag{2.2}$$

For the other equation, we reall (2.10) and (2.11) in [3]

$$\bar\partial\left(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1\right) = g_3 + g_4. \tag{2.3}$$

where

$$g_3 = -2i\bar\partial\mu w, \quad g_4 = \bar\partial\left(2i\partial_1\mu\right)u_2 - \bar\partial\left(2i\partial_2\mu\right)u_1. \tag{2.4}$$

In the sequel, we will use the system given by (2.2)–(2.4) to prove the main theorems based on the Carleman estimates. Before discussing the Carleman estimates in the next section, we first derive a useful interior estimate related to (2.2)–(2.4).

Lemma 2.1. *Let $1 < s < 2$ and (λ, μ) satisfy (1.2). Assume that $u \in H^1_{loc}(\Omega)$ is a solution of (1.1). Then, there exists $r_0 = r_0(s, M_0, \delta_0)$ such that for any $0 < r < r_0$ and $0 < a_3 < a_1 < a_2 < a_4 < 1$ satisfying $B_{a_4 r_0} \subset \Omega$, we have*

$$\int_{a_1 r < |x| < a_2 r} (|v| + |w|)^2 \le C \int_{a_3 r<|x|<a_4 r} |u|^2 |x|^{-2}\, dx, \tag{2.5}$$

where the constant C is independent of r and u.

Proof. Let $\xi \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \xi(x) \le 1$ and

$$\xi(x) = \begin{cases} 0, & |x| \le a_3 r, \\ 1, & a_1 r < |x| < a_2 r, \\ 0, & |x| \ge a_4 r \end{cases} \tag{2.6}$$

with $B_{a_4 r} \subset \Omega$. Multiplying $-\xi^2(u_1 - iu_2)$ to (2.3) and integrating the new equation over B_1, we have

$$\begin{aligned} &-\Re\int_\Omega \bar\partial\left(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1\right)\xi^2(u_1 - iu_2)\, dx \\ =&-\Re\int_\Omega (g_3 + g_4)\xi^2(u_1 - iu_2)\, dx. \end{aligned} \tag{2.7}$$

We first compute the left hand side of (2.7) and obtain

$$
\begin{aligned}
& -\Re\int_\Omega \bar\partial\left(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1\right)\xi^2(u_1 - iu_2)\,dx\\
=&\Re\int_\Omega \left(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1\right)(u_1 - iu_2)\bar\partial(\xi^2)\,dx\\
& + \Re\int_\Omega \left(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1\right)\xi^2(\frac{\mu}{\lambda+2\mu}v + iw)\,dx\\
=&\Re\int_\Omega \left(\mu v - i\mu w + 2i\partial_1\mu u_2 - 2i\partial_2\mu u_1\right)(u_1 - iu_2)\bar\partial(\xi^2)\,dx\\
& + \Re\int_\Omega \left(2i\partial_1\mu u_2 - 2i\partial_2\mu u_1\right)\xi^2(\frac{\mu}{\lambda+2\mu}v + iw)\,dx\\
& + \Re\int_\Omega \left(\mu v - i\mu w\right)\xi^2(\frac{\mu}{\lambda+2\mu}v + iw)\,dx\\
\geq&\int_\Omega \xi^2(\frac{\mu^2}{\lambda+2\mu}v^2 + \mu w^2)\,dx\\
& - c_1\int_\Omega \xi\left(|v| + |w| + |\nabla\mu||u|\right)|u||\nabla\xi|\,dx\\
& - c_1\int_\Omega \xi^2|\nabla\mu||u|(|v| + |w|)\,dx,
\end{aligned} \tag{2.8}
$$

where c_1 (and $c_2, \cdots, c_5$ below) depends on $\delta_0, M_0,$ and also $a_1, \cdots, a_4$. Moreover, we can see that

$$
\begin{aligned}
&\int_\Omega \xi^2\left(|\frac{\mu}{\lambda+2\mu}v|^2 + |w|^2\right)\,dx\\
=&\int_\Omega (\xi^2|\partial_1u_1 + \partial_2u_2|^2 + \xi^2|\partial_2u_1 - \partial_1u_2|^2)\,dx\\
=&\int_\Omega \xi^2|\nabla u|^2\,dx + 2\int_\Omega \xi^2\partial_1u_1\partial_2u_2\,dx - 2\int_\Omega \xi^2\partial_2u_1\partial_1u_2\,dx\\
=&\int_\Omega \xi^2|\nabla u|^2\,dx - 2\int_\Omega u_1\partial_2u_2\partial_1(\xi^2)\,dx + 2\int_\Omega u_1\partial_1u_2\partial_2(\xi^2)\,dx\\
\geq&\frac{1}{2}\int_\Omega \xi^2|\nabla u|^2\,dx - 16\int_\Omega |\nabla\xi|^2|u|^2\,dx.
\end{aligned} \tag{2.9}
$$

Combining (2.4), (2.7), (2.8), (2.9) and using (1.2) gives

$$
\begin{aligned}
&\int_\Omega \xi^2|\nabla u|^2\,dx + \int_\Omega \xi^2(|v| + |w|)^2\,dx\\
\leq&\frac{1}{2}\int_\Omega \xi^2(|v| + |w|)^2\,dx + c_2\int_\Omega |u|^2|\nabla\xi|^2\,dx\\
& + c_2\int_\Omega \xi^2|\nabla\mu|^2|u|^2\,dx + c_2\int_\Omega \xi^2|D^2\mu||u|^2\,dx,
\end{aligned}
$$

which is reduced to

$$\int_\Omega \xi^2|\nabla u|^2\,dx+\int_\Omega \xi^2(|v|+|w|)^2\,dx\le c_3(I_1+I_2+I_3), \tag{2.10}$$

where

$$I_1=\int_\Omega |u|^2|\nabla\xi|^2\,dx,$$
$$I_2=\int_\Omega |\nabla\mu|^2|\xi u|^2\,dx,$$
$$I_3=\int_\Omega |D^2\mu||\xi u|^2\,dx.$$

Let us now estimate I_j for $j=2,3$. We begin with

$$\begin{aligned}
I_2=&\int_\Omega |\nabla\mu|^2|\xi u|^2\,dx\\
\le&\left(\int_{a_3r<|x|<a_4r}1\,dx\right)^{\frac{s-1}{s}}\left(\int_\Omega|\nabla\mu|^{\frac{2s}{2-s}}\,dx\right)^{\frac{2-s}{s}}\left(\int_\Omega|\xi u|^{\frac{2s}{s-1}}\,dx\right)^{\frac{s-1}{s}}\\
\le& c_4r^{\frac{2(s-1)}{s}}\int_\Omega|\nabla(\xi u)|^2\,dx\quad(\text{recall }\ W_0^{1,2}(\Omega)\subset L^\alpha(\Omega),\ \ \forall\,\alpha\in(1,\infty))\\
\le& 2c_4r^{\frac{2(s-1)}{s}}\int_\Omega\xi^2|\nabla u|^2\,dx+2c_4r^{\frac{2(s-1)}{s}}I_1\\
\le&\frac14\int_\Omega\xi^2|\nabla u|^2\,dx+\frac14 I_1,
\end{aligned}\tag{2.11}$$

where we chose r sufficiently small such that $2c_4r^{\frac{2(s-1)}{s}}<\frac14$. Similarly, we can estimate

$$\begin{aligned}
I_3=&\int_\Omega|D^2\mu||\xi u|^2\,dx\\
\le&\left(\int_{a_3r<|x|<a_4r}1\,dx\right)^{\frac{s-1}{2s}}\left(\int_\Omega|D^2\mu|^s\,dx\right)^{\frac1s}\left(\int_\Omega|\xi u|^{\frac{4s}{s-1}}\,dx\right)^{\frac{s-1}{2s}}\\
\le&\frac14\int_\Omega\xi^2|\nabla u|^2\,dx+\frac14 I_1.
\end{aligned}\tag{2.12}$$

From (2.10) to (2.12), we obtain that

$$\int_{a_1r<|x|<a_2r}(|v|+|w|)^2\le c_5\int_{a_3r<|x|<a_4r}|u|^2|x|^{-2}\,dx, \tag{2.13}$$

which implies (2.5). □

3. Carleman estimates

In this section, we derive the Carleman estimates that will be used in the proofs of our main theorems. Let β be a large number. Define $t=\log|x|$

and the weight function

$$\phi(\log|x|) = \phi(t) = -\beta t.$$

where q is defined in Theorem 1.1 to Theorem 1.3. At times, we will use the notation $\phi = \phi_\beta$ to remind ourselves of the dependence on the constant β.

We first state the following Carleman estimate for first order operators.

Theorem 3.1. *Let $1 < p \le 2$ and $\tilde{R}_0 \in (0,1)$. If $\beta \in \mathbb{N} + \frac{1}{2}$ and $\beta \ge 3$, then for all $z \in C_0^\infty\left(B_{\tilde{R}_0} \setminus \{0\}\right)$, it holds that*

$$\left\| e^{\phi} z \right\|_{L^2(B_{\tilde{R}_0})} \le C \left\| |x|^{2-\frac{2}{p}} e^{\phi}\, \bar{\partial} z \right\|_{L^p(B_{\tilde{R}_0})}, \tag{3.1}$$

where $C = C\left(p, \tilde{R}_0\right)$.

Theorem 3.2. *Let $2 \le q$ and $\tilde{R}_0 \in (0,1)$. If $\beta \in \mathbb{N} + \frac{1}{2}$ and $\beta \ge 3$, then for all $z \in C_0^\infty\left(B_{\tilde{R}_0} \setminus \{0\}\right)$, it holds that*

$$\left\| |x|^{-\frac{2}{q}} e^{\phi} z \right\|_{L^q(B_{\tilde{R}_0})} \le C \left\| e^{\phi}\, \bar{\partial} z \right\|_{L^2(B_{\tilde{R}_0})}, \tag{3.2}$$

where $C = C\left(q, \tilde{R}_0\right)$.

To prove the above two estimates, we use the polar coordinates in $\mathbb{R}^2 \setminus \{0\}$ and write

$$x_1 = |x| \cos\theta = e^t \cos\theta,$$
$$x_2 = |x| \sin\theta = e^t \sin\theta,$$

where $t = \log|x| = \log(\sqrt{x_1^2 + x_2^2})$, $\theta = \arctan(x_2/x_1)$. In the new coordinates, we have

$$\partial_{x_1} = e^{-t}\left(\cos\theta \frac{\partial}{\partial t} - \sin\theta \frac{\partial}{\partial \theta}\right), \quad \partial_{x_2} = e^{-t}\left(\sin\theta \frac{\partial}{\partial t} + \cos\theta \frac{\partial}{\partial \theta}\right)$$

and

$$\mathcal{L} := e^{t - i\theta} \bar{\partial} = \partial_t + i\partial_\theta. \tag{3.3}$$

Recall that the eigenvalues of ∂_θ are ik, $k \in \mathbb{Z}$, with corresponding eigenspace $E_k = \operatorname{span}\{e_k\}$, where $e_k = \frac{1}{\sqrt{2\pi}} e^{ik\theta}$ and $\|e_k\|_{L^2(S^1)} = 1$. Let P_k denote the projection operator from $L^2\left(S^1\right)$ onto E_k. Then we have

(a) $P_k u = \langle u, e_k \rangle_{d\theta} e_k := u_k$,
(b) $\langle P_k u, v \rangle_{d\theta} = \langle u, P_k v \rangle_{d\theta}$,

where $\langle u, v \rangle_{d\theta} := \int_0^{2\pi} u \bar{v} d\theta$ for any two functions $u, v \in L^2\left(S^1\right)$. Since P_k acts only on the angular variables, it follows that $P_k v\,(t, \theta) = P_k v\,(t, \cdot)\,(\theta)$. In other words, $\mathcal{L}$ can be written as

$$\mathcal{L} = \partial_t - \sum_{k \in \mathbb{Z}} k P_k.$$

To establish the Theorem 3.1 and Theorem 3.2, we recall the following lemma from Lemma 5.1 of [4].

Lemma 3.3. *Let $M, N \in \mathbb{N}$ and let $\{c_k\}$ be a sequence of numbers such that $|c_k| \leq 1$ for all k. For any $v \in L^2\left(S^1\right)$, and every $p \in [1,2]$, we have that*

$$\|P_k v\|_{L^2(S^1)} \leq C \|v\|_{L^p(S^1)} \tag{3.4}$$

and

$$\left\| \sum_{k=N}^{M} c_k P_k v \right\|_{L^2(S^1)} \leq C \left(\sum_{k=N}^{M} |c_k|^2 \right)^{\frac{1}{p}-\frac{1}{2}} \|v\|_{L^p(S^1)}, \tag{3.5}$$

where $C = C(p)$.

We now introduce the conjugated operators of $\mathcal{L}$, defined by

$$\mathcal{L}_\beta z = e^{\phi_\beta(t)} \mathcal{L}\left(e^{-\phi_\beta(t)} z\right).$$

Then

$$\mathcal{L}_\beta z = \sum_{k\in\mathbb{Z}} \left(\partial_t + \beta - k\right) P_k z. \tag{3.6}$$

We also define the following two operators related to $\mathcal{L}_\beta$ with respect to t variable by

$$T_\beta z(t,\theta) = \sum_{k\in\mathbb{Z}} \int e^{i\eta t} \left(i\eta + \beta - k\right) P_k \widehat{z}(\eta,\theta) d\eta, \tag{3.7}$$

$$T_\beta^* z(t,\theta) = \sum_{k\in\mathbb{Z}} \int e^{i\eta t} \left(-i\eta + \beta - k\right) P_k \widehat{z}(\eta,\theta) d\eta, \tag{3.8}$$

where $\widehat{z}(\eta,\theta) = \int_{\mathbb{R}} e^{-i\eta t} z(t,\theta) dt$ for fixed θ, $z \in \mathcal{S}(\mathbb{R}\times S^1)$, and $\mathcal{S}(\mathbb{R}\times S^1)$ is the Schwartz class on $\mathbb{R}\times S^1$.

Remark 3.4. *We can see that for $z \in \mathcal{S}\left(\mathbb{R}\times S^1\right)$,*

(a) $T_\beta z = \mathcal{L}_\beta z$.
(b) T_β, $T_\beta^* : \mathcal{S}\left(\mathbb{R}\times S^1\right) \to \mathcal{S}\left(\mathbb{R}\times S^1\right)$ *are invertible and their inverses are*

$$T_\beta^{-1} z(t,\theta) = \sum_{k\in\mathbb{Z}} \int e^{i\eta t} \left(i\eta + \beta - k\right)^{-1} P_k \widehat{z}(\eta,\theta) d\eta,$$

$$\left(T_\beta^*\right)^{-1} z(t,\theta) = \sum_{k\in\mathbb{Z}} \int e^{i\eta t} \left(-i\eta + \beta - k\right)^{-1} P_k \widehat{z}(\eta,\theta) d\eta,$$

where $z \in \mathcal{S}(\mathbb{R}\times S^1)$.

Here we consider functions that are supported in $B_{\tilde{R}_0} \setminus \{0\}$. Therefore in the subsequent parts of this section we consider the functions supported in $(-\infty,0)\times S^1$ in terms of the coordinates (t,θ). The following lemma can be proved by direct computations.

Lemma 3.5. *For $u, v \in C_0^\infty\left((-\infty,0)\times S^1\right)$, we have*

$$\langle T_\beta u, v\rangle = \langle u, T_\beta^* v\rangle, \tag{3.9}$$

$$\langle (T_\beta)^{-1} u, v\rangle = \langle u, \left(T_\beta^*\right)^{-1} v\rangle, \tag{3.10}$$

where $\langle u,v\rangle := \int_{-\infty}^{\infty}\langle u,v\rangle_{d\theta} dt$ for any two functions $u, v \in L^2\left((-\infty,0)\times S^1\right)$.

By a slight modification of the proof of Proposition 5.2 in [4], we prove the next proposition, which is crucial to the proofs of Theorem 3.1 and Theorem 3.2.

Proposition 3.6. *Let $1 < p \le 2$. If $\beta \in \mathbb{N} + \frac{1}{2}$ and $\beta \ge 2$, then for $f \in C_0^\infty\left((-\infty,0)\times S^1\right)$ we have*

$$\left\| T_\beta^{-1} f\right\|_{L^2(dtd\theta)} \le C\left\|f\right\|_{L^p(dtd\theta)}, \tag{3.11}$$

$$\left\| \left(T_\beta^*\right)^{-1} f\right\|_{L^2(dtd\theta)} \le C\left\|f\right\|_{L^p(dtd\theta)}, \tag{3.12}$$

where $C = C(p)$.

Proof. We will only prove (3.11). The proof of (3.12) is similar. Let $f \in C_0^\infty\left((-\infty,0)\times S^1\right)$ and set $z = T_\beta^{-1} f$. Note that $z \in \mathcal{S}\left(\mathbb{R}\times S^1\right)$ and, therefore, z satisfies

$$\mathcal{L}_\beta z = T_\beta z = f. \tag{3.13}$$

Recall from (3.6) that

$$\mathcal{L}_\beta = \sum_{k\in\mathbb{Z}} \left(\partial_t + \beta - k\right) P_k.$$

Applying P_k to (3.13) gives

$$\left(\partial_t + \beta - k\right) P_k z = P_k \mathcal{L}_\beta z = P_k f. \tag{3.14}$$

Since $z \in \mathcal{S}(\mathbb{R}\times S^1)$, solving the above first order differential equations with $\lim_{t\to\pm\infty} P_k z = 0$ gives that

$$\begin{aligned} P_k z(t,\theta) &= -\int_t^\infty e^{(\beta-k)(s-t)} P_k f(s,\theta)\, ds \\ &= \int_{-\infty}^t e^{(\beta-k)(s-t)} P_k f(s,\theta)\, ds. \end{aligned} \tag{3.15}$$

We can split the sum $\sum_{k\in\mathbb{Z}} P_k z = z$ into two parts. One is the sum over k's far from β, the other is the sum over k's near β. Precisely, let $M = \lceil 2\beta\rceil$ and define

$$S_{1,\beta} = \sum_{k>M} P_k + \sum_{k<0} P_k, \qquad S_{2,\beta} = \sum_{k=0}^{M} P_k. \tag{3.16}$$

In order to prove (3.11), it suffices to show that

$$\|S_{\ell,\beta}z\|_{L^2(dtd\theta)} \leq C\,\|T_\beta z\|_{L^p(dtd\theta)} = C\,\|f\|_{L^p(dtd\theta)} \tag{3.17}$$

for $\ell = 1, 2$.

We first establish (3.17) with $\ell = 1$. Note that

$$(\beta - k)(s-t) \leq \begin{cases} -\dfrac{k}{2}\,|s-t|\,, & \text{for } k > M \geq 2\beta,\ t \leq s, \\ -\left(\dfrac{\beta}{2} + |k|\right)|t-s|\,, & k < 0, s \leq t. \end{cases} \tag{3.18}$$

Therefore, taking the $L^2\left(S^1\right)$-norm on the first equality of (3.15), we have that for $k > M$

$$\begin{aligned} \|P_k z(t,\cdot)\|_{L^2(S^1)} &\leq \int_t^\infty e^{-\frac{1}{2}k|s-t|}\,\|P_k f(s,\cdot)\|_{L^2(S^1)}\,ds \\ &\leq \int_{-\infty}^\infty e^{-\frac{1}{2}k|s-t|}\,\|P_k f(s,\cdot)\|_{L^2(S^1)}\,ds. \end{aligned} \tag{3.19}$$

We discuss the case of $k > M$ for $p = 2$ and $1 < p < 2$, separately. The case of $p = 2$ is an easier part. By applying Young's inequality for convolution to (3.19), we obtain

$$\|P_k z\|_{L^2(dtd\theta)} \leq Ck^{-1}\,\|P_k f\|_{L^2(dtd\theta)} \leq C\,\|P_k f\|_{L^2(dtd\theta)}\,. \tag{3.20}$$

Squaring and summing up (3.20) in $k > M$ implies

$$\sum_{k>M} \|P_k z\|^2_{L^2(dtd\theta)} \leq C \sum_{k>M} k^{-2}\,\|P_k f\|^2_{L^2(dtd\theta)} \leq C\beta^{-1}\,\|f\|^2_{L^2(dtd\theta)}\,.$$

For the case of $1 < p < 2$, we apply (3.4) to (3.19) and deduce that

$$\|P_k z(t,\cdot)\|_{L^2(S^1)} \leq C\int_{-\infty}^\infty e^{-\frac{1}{2}k|t-s|}\,\|f(s,\cdot)\|_{L^p(S^1)}\,ds.$$

Applying Young's inequality for convolution again yields

$$\|P_k z\|_{L^2(dtd\theta)} \leq C\left(\int_{-\infty}^\infty e^{-\frac{\sigma}{2}k|z|}dz\right)^{\frac{1}{\sigma}}\|f\|_{L^p(dtd\theta)} \leq Ck^{\frac{1}{p}-\frac{3}{2}}\,\|f\|_{L^p(dtd\theta)}\,,$$

where $\frac{1}{\sigma} = \frac{3}{2} - \frac{1}{p}$, $1 < p < 2$. Once more, squaring and summing up the above estimate in $k > M$ implies

$$\sum_{k>M} \|P_k z\|^2_{L^2(dtd\theta)} \leq C \sum_{k>M} k^{-3+\frac{2}{p}}\,\|f\|^2_{L^p(dtd\theta)} \leq C\beta^{-2+\frac{2}{p}}\,\|f\|^2_{L^p(dtd\theta)}\,.$$

The case of $k < 0$ is similar to the case of $k > M$. By the second formula of (3.18) and repeating the arguments from above, we have

$$\|P_k z\|_{L^2(dtd\theta)} \leq C\left(\frac{\beta}{2} + |k|\right)^{\frac{1}{p}-\frac{3}{2}}\|f\|_{L^p(dtd\theta)}\,, \quad 1 < p < 2,$$

and

$$\|P_k z\|_{L^2(dtd\theta)} \leq C\left(\frac{\beta}{2}+|k|\right)^{-1} \|P_k f\|_{L^p(dtd\theta)}\,, \quad p=2,$$

Squaring and summing up the estimates derived above in $k<0$ yields that for $\beta>1$

$$\sum_{k<0} \|P_k z\|^2_{L^2(dtd\theta)} \leq C\beta^{-2+\frac{2}{p}} \|f\|^2_{L^p(dtd\theta)}\,.$$

An application of orthogonality shows that for $1<p\leq 2$,

$$\|S_{1,\beta} z\|_{L^2(dtd\theta)} \leq C\beta^{-1+\frac{1}{p}} \|f\|_{L^p(dtd\theta)} \leq C\|f\|_{L^p(dtd\theta)}$$

which implies (3.17) with $\ell=1$.

Now let us consider the case of $\ell=2$. Set $N=\lfloor\beta\rfloor$, then $N+1-\beta=\beta-N=\frac{1}{2}$ by the assumption $\beta\in\mathbb{N}+1/2$. For $-\infty<s,t\leq 0$, we have

$$\begin{aligned}(\beta-k)(s-t) &= -(k-N-1)\,|t-s| - (N+1-\beta)(s-t)\\ &\leq -(k-N-1)\,|t-s| - \frac{1}{2}\,|t-s|\,,\ k\geq N+1, t\leq s.\end{aligned} \tag{3.21}$$

Alternatively, if $-\infty<s,t\leq 0$ and $s\leq t$, then

$$\begin{aligned}(\beta-k)(s-t) &= -(N-k)\,|t-s| - (\beta-N)\,|t-s|\\ &\leq -(N-k)\,|t-s| - \frac{1}{2}\,|t-s|\,, \quad 0\leq k\leq N.\end{aligned} \tag{3.22}$$

We now split the sum corresponding to $\ell=2$ into two parts and use both representations from (3.15).

First we consider $N+1\leq k\leq M$. From the first line of (3.15), we sum over k and use the bound from (3.21) to derive that

$$\left\|\sum_{k=N+1}^{M} P_k z(t,\cdot)\right\|_{L^2(S^1)} \leq \int_{-\infty}^{\infty} \left\|\sum_{k=N+1}^{M} e^{-\frac{1}{2}|t-s|} e^{-(k-N-1)|t-s|} P_k f(s,\cdot)\right\|_{L^2(S^1)} ds.$$

With $c_k=e^{-\frac{1}{2}|t-s|}e^{-(k-N-1)|t-s|}$, it is clear that $|c_k|\leq 1$. Therefore, Lemma 3.3 is applicable, so we may apply estimate (3.5) to obtain

$$\begin{aligned}&\left\|\sum_{k=N+1}^{M} e^{-\frac{1}{2}|t-s|} e^{-(k-N-1)|t-s|} P_k f(s,\cdot)\right\|_{L^2(S^1)}\\ \leq & Ce^{-\frac{1}{2}|t-s|}\left(1+\left(\sum_{k=N+2}^{M} e^{-2(k-N-1)|t-s|}\right)^{\frac{1}{p}-\frac{1}{2}}\right)\|f(s,\cdot)\|_{L^p(S^1)}\end{aligned} \tag{3.23}$$

for all $1\leq p\leq 2$. Since

$$\sum_{k=N+2}^{M} e^{-2(k-N-1)|t-s|} \leq \sum_{k=1}^{\infty} e^{-2k|t-s|} \leq C\,|t-s|^{-1}\,,$$

we then have

(3.24)
$$\left\| \sum_{k=N+1}^{M} P_k z(t,\cdot) \right\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} e^{-\frac{1}{2}|t-s|} \left(1 + |t-s|^{-\alpha}\right) \|f(s,\cdot)\|_{L^p(S^1)} \, ds,$$

where $\alpha = \frac{2-p}{2p}$.

On the other hand, for $0 \leq k \leq N$, by the second line of (3.15), summing over k, and using the bound (3.22), we can derive

$$\left\| \sum_{k=0}^{N} P_k z(t,\cdot) \right\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} \left\| \sum_{k=0}^{N} e^{-\frac{1}{2}|t-s|} e^{-(N-k)|t-s|} P_k f(s,\cdot) \right\|_{L^2(S^1)} ds.$$

Similar to the above argument, we then conclude that

(3.25)
$$\left\| \sum_{k=0}^{N} P_k z(t,\cdot) \right\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} e^{-\frac{1}{2}|t-s|} \left(1 + |t-s|^{-\alpha}\right) \|f(s,\cdot)\|_{L^p(S^1)} \, ds.$$

In view of (3.24), (3.25), it follows from Young's inequality for convolution that

$$\|S_{2,\beta} z\|_{L^2(dtd\theta)} \leq C \left[\int_{-\infty}^{\infty} e^{-\frac{\sigma}{2}|z|} \left(1 + |z|^{-\alpha}\right)^{\sigma} dz \right]^{\frac{1}{\sigma}} \|f\|_{L^p(dtd\theta)},$$

where $\frac{1}{\sigma} = \frac{3}{2} - \frac{1}{p}$.

Note that $1 \leq \sigma = \frac{2p}{3p-2} < 2$, if $1 < p \leq 2$. Therefore,

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-\frac{\sigma}{2}|z|} \left(1 + |z|^{-\alpha}\right)^{\sigma} dz \\ &= \int_{|z|\leq 1} e^{-\frac{\sigma}{2}|z|} \left(1 + |z|^{-\alpha}\right)^{\sigma} dz + \int_{|z|>1} e^{-\frac{\sigma}{2}|z|} \left(1 + |z|^{-\alpha}\right)^{\sigma} dz \\ &\leq \int_{|z|\leq 1} \left(1 + |z|^{-\alpha\sigma}\right) dz + \int_{|z|>1} e^{-\frac{\sigma}{2}|z|} dz \end{aligned}$$

Since

$$0 \leq \alpha\sigma = \frac{2-p}{2p} \cdot \frac{2p}{3p-2} = \frac{2-p}{3p-2} < 1, \text{ if } 1 < p \leq 2,$$

we have

$$\int_{-\infty}^{\infty} e^{-\frac{\sigma}{2}|z|} \left(1 + |z|^{-\alpha}\right)^{\sigma} dz \leq C$$

with $C = C(p)$ for $1 < p \leq 2$. In other words, we have shown (3.17) for $\ell = 2$, and therefore, the proof of the proposition is completed. □

Next Proposition 3.6 implies the following corollary.

Corollary 3.7. *Let* $1 < p \leq 2$ *and* $2 \leq q < \infty$*. If* $\beta \in \mathbb{N} + \frac{1}{2}$ *and* $\beta \geq 2$*, then for* $z \in C_0^\infty\left((-\infty, 0) \times S^1\right)$ *that*

$$\|z\|_{L^2(dtd\theta)} \leq C \left\|\mathcal{L}_\beta z\right\|_{L^p(dtd\theta)}, \tag{3.26}$$

$$\|z\|_{L^q(dtd\theta)} \leq C \left\|\mathcal{L}_\beta z\right\|_{L^2(dtd\theta)}, \tag{3.27}$$

where $C = C(p)$ *in* (3.26) *and* $C = C(q)$ *in* (3.27) *.*

Proof. Given any $z \in C_0^\infty\left((-\infty, 0) \times S^1\right)$, we set $f = \mathcal{L}_\beta z$. Then $f \in C_0^\infty\left((-\infty, 0) \times S^1\right)$ and $f = T_\beta z$. Therefore, we can apply Proposition 3.6 and Remark 3.4 to obtain (3.26).

For (3.27), it remains to consider $2 < q < \infty$ and then $1 < q' < 2$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Recall Lemma 3.5: For any $f, u \in C_0^\infty\left((-\infty, 0) \times S^1\right)$, we have

$$\begin{aligned}\langle (T_\beta)^{-1} f, u\rangle = \langle f, \left(T_\beta^*\right)^{-1} u\rangle &\leq \|f\|_{L^2(dtd\theta)} \left\|\left(T_\beta^*\right)^{-1} u\right\|_{L^2(dtd\theta)} \\ &\leq C(q') \|f\|_{L^2(dtd\theta)} \|u\|_{L^{q'}(dtd\theta)},\end{aligned}$$

where we used the conclusion of Proposition 3.6 in the last inequality. This implies

$$\left\|T_\beta^{-1} f\right\|_{L^q(dtd\theta)} \leq C(q) \|f\|_{L^2(dtd\theta)}, \tag{3.28}$$

since $C_0^\infty\left((-\infty, 0) \times S^1\right)$ is dense in $L^{q'}\left((-\infty, 0) \times S^1\right)$. Therefore, using (3.28) together with Remark 3.4 gives (3.27). □

We now present the proof of Theorem 3.1 and Theorem 3.2, which are corollaries of Corollary 3.7.

Proof of Theorem 3.1. If we choose $\beta \geq 3$ and replace β with $\beta_1 = \beta - 1$, then the results of Corollary 3.7 still hold with possibly modified constants. That is, by replacing z with $e^{\phi_{\beta_1}} z = e^{-\beta_1 t} z$ in (3.26), we have

$$\left\|e^{\phi_{\beta_1}(t)} z\right\|_{L^2(dtd\theta)} \leq C \left\|e^{\phi_{\beta_1}(t)} \mathcal{L} z\right\|_{L^p(dtd\theta)}$$

for $1 < p \leq 2$. Since $e^{2t} dtd\theta = dx$, then

$$\left\|e^{\phi_{\beta_1}(t)} z\right\|_{L^2(dtd\theta)} = \left\|e^{-\beta t + t} z\right\|_{L^2(dtd\theta)} = \left\|e^{\phi_\beta} z\right\|_{L^2(dx)}$$

and

$$\begin{aligned}\left\|e^{\phi_{\beta_1}(t)} \mathcal{L} z\right\|_{L^p(dtd\theta)} &= \left\|e^{(-\beta+2)t - i\theta} \bar{\partial} z\right\|_{L^p(dtd\theta)} \\ &= \left\||x|^{2-2/p} e^{\phi_\beta} \bar{\partial} z\right\|_{L^p(dx)}.\end{aligned}$$

□

Proof of Theorem 3.2. As in the proof of Theorem 3.1, replacing z with $e^{\phi_\beta} z = e^{-\beta t} z$ in (3.27) implies

$$\left\| e^{\phi_\beta(t)} z \right\|_{L^q(dtd\theta)} \le C \left\| e^{\phi_\beta(t)} \mathcal{L} z \right\|_{L^2(dtd\theta)}$$

for $2 \le q < \infty$. Note that $e^{2t} dtd\theta = dx$ and thus

$$\left\| e^{\phi_\beta(t)} z \right\|_{L^q(dtd\theta)} = \left\| |x|^{-2/q} e^{\phi_\beta} z \right\|_{L^q(dx)} \tag{3.29}$$

and

$$\left\| e^{\phi_\beta(t)} \mathcal{L} z \right\|_{L^2(dtd\theta)} = \left\| e^{-\beta t + t - i\theta} \bar{\partial} z \right\|_{L^2(dtd\theta)} = \left\| e^{\phi_\beta} \bar{\partial} z \right\|_{L^2(dx)} . \tag{3.30}$$

Combining (3.29) and (3.30) gives (3.2). □

Remark 3.8. *If we replace β in Theorem 3.1 by $\beta + 1$, then we obtain that for $p = 2$*

$$\left\| |x|^{-1} e^{\phi_\beta} z \right\|_{L^2(B_{\tilde{R}_0})} \le C \left\| e^{\phi_\beta} \bar{\partial} z \right\|_{L^2(B_{\tilde{R}_0})}, \tag{3.31}$$

which is the estimate in (3.2) *for $q = 2$.*

4. The proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3

We now have all the tools to prove Theorems 1.1, 1.2 and 1.3. As discussed above, we consider $s \in (1, 2)$. We first choose $\hat{R}_0 \le \tilde{R}_0$ such that $B_{\hat{R}_0} \subset \Omega$. Let $\chi \in C_0^\infty \left(B_{\hat{R}_0} \setminus \{0\} \right)$ be a cutoff function with the property that $\operatorname{supp} \chi \subset A_s \cup A_m \cup A_l$, where $\chi \equiv 1$ on A_m and $\operatorname{supp} \nabla \chi = A_s \cup A_l$. The precise descriptions of these sets will be given later. All constants C and C_j, $j \in \mathbb{N}$, given in this section depend on a priori parameters s, M_0, δ_0.

For μ, λ satisfying (1.2), u_1, u_2 from (1.1), and v, w from (2.1), define the compactly supported function

$$z = \chi \left(\mu v - i \mu w + 2i \partial_1 \mu u_2 - 2i \partial_2 \mu u_1 \right).$$

Applying estimate (3.1) in Theorem 3.1 to z with some $p \in (1, 2)$ to be determined below, and by the equations (2.3), (2.4), we obtain

$$\begin{aligned}
& \left\| e^{\phi} \chi \left(|v| + |w| \right) \right\|_2 - \left\| e^{\phi} \chi \left| \nabla \mu \right| u \right\|_2 \\
\le & C \left\| e^{\phi} \chi \left(\mu v - i \mu w + 2i \partial_1 \mu u_2 - 2i \partial_2 \mu u_1 \right) \right\|_2 \\
\le & C \left\| |x|^{2 - \frac{2}{p}} e^{\phi} \chi \left(g_3 + g_4 \right) \right\|_p \\
+ & C \left\| |x|^{2 - \frac{2}{p}} e^{\phi} \bar{\partial} \chi \left(\mu v - i \mu w + 2i \partial_1 \mu u_2 - 2i \partial_2 \mu u_1 \right) \right\|_p \\
\le & C_1 \left\| |x|^{2 - \frac{2}{p}} e^{\phi} \chi \left| \nabla \mu \right| w \right\|_p + C_1 \left\| |x|^{2 - \frac{2}{p}} e^{\phi} \chi \left| D^2 \mu \right| u \right\|_p \\
+ & C_1 \left\| |x|^{2 - \frac{2}{p}} e^{\phi} \left| \nabla \chi \right| \left(|v| + |w| \right) \right\|_p + C_1 \left\| |x|^{2 - \frac{2}{p}} e^{\phi} \left| \nabla \chi \right| \left| \nabla \mu \right| u \right\|_p .
\end{aligned} \tag{4.1}$$

An application of (3.2) from Theorem 3.2 and (3.31) in Remark 3.8 to the functions $z = \chi(u_1 - iu_2)$, and using (2.2), shows that if $\beta \in \mathbb{N} + \frac{1}{2}$ is large, then

$$\begin{aligned} &\left\| |x|^{-1} e^{\phi} \chi u \right\|_2 + \left\| |x|^{-2/q} e^{\phi} \chi u \right\|_q \\ \le & C_2 \left\| e^{\phi} |\nabla \chi| \, |u| \right\|_2 + C_2 \left\| e^{\phi} \chi \left(|v| + |w| \right) \right\|_2 , \end{aligned} \tag{4.2}$$

where $2 < q < \infty$ that will be specified below. Adding $2C_2 \times$(4.1) to (4.2) yields

$$\begin{aligned} &\left\| |x|^{-2/q} e^{\phi} \chi u \right\|_q + \left\| |x|^{-1} e^{\phi} \chi u \right\|_2 + C_2 \left\| e^{\phi} \chi \left(|v| + |w| \right) \right\|_2 \\ \le & 2C_2 \left\| e^{\phi} \chi \left| \nabla \mu \right| u \right\|_2 + 2C_1 C_2 \left\| |x|^{2-\frac{2}{p}} e^{\phi} \chi \left| \nabla \mu \right| w \right\|_p \\ & + 2C_1 C_2 \left\| |x|^{2-\frac{2}{p}} e^{\phi} \chi \left| D^2 \mu \right| u \right\|_p \\ & + 2C_1 C_2 \left\| |x|^{2-\frac{2}{p}} e^{\phi} \left| \nabla \chi \right| \left(|v| + |w| \right) \right\|_p + 2C_1 C_2 \left\| |x|^{2-\frac{2}{p}} e^{\phi} \left| \nabla \chi \right| \left| \nabla \mu \right| u \right\|_p \\ & + C_2 \left\| e^{\phi} \left| \nabla \chi \right| u \right\|_2 . \end{aligned} \tag{4.3}$$

We now want to choose appropriate indices p and q such that the first three terms on the right of the inequality may be absorbed into the left.

First, we consider the term $\left\| e^{\phi} \chi \left| \nabla \mu \right| u \right\|_2$. From the Hölder inequality, it follows that

$$\left\| e^{\phi} \chi \left| \nabla \mu \right| u \right\|_2 \le C \left\| e^{\phi} \chi u \right\|_q \left\| \nabla \mu \right\|_{\frac{2s}{2-s}}$$

with

$$q \ge \frac{s}{s-1}. \tag{4.4}$$

Next, we consider the term $\left\| |x|^{2-\frac{2}{p}} e^{\phi} \chi \left| \nabla \mu \right| w \right\|_p$. By the Hölder inequality, we have that for $p \le s$

$$\left\| |x|^{2-\frac{2}{p}} e^{\phi} \chi \left| \nabla \mu \right| w \right\|_p \le C \left\| |x|^{2-\frac{2}{p}} e^{\phi} \chi w \right\|_2 \left\| \nabla \mu \right\|_{\frac{2s}{2-s}} .$$

Finally, using similar techniques, we can estimate

$$\left\| |x|^{2-\frac{2}{p}} e^{\phi} \chi \left| D^2 \mu \right| u \right\|_p \le C \left\| |x|^{2-\frac{2}{p}} e^{\phi} \chi u \right\|_q \left\| D^2 \mu \right\|_{\frac{pq}{q-p}} .$$

To handle this term,we need that

$$\frac{pq}{q-p} \le s. \tag{4.5}$$

To satisfy all requirements described above, i.e., (4.4), (4.5), and $1 < p \le s$, we can choose $p = \frac{2s}{1+s}$ and $q = \frac{2rs}{s-1}$ for some $r > \frac{1+s}{2(s-1)}$. Note that if $s \in (1, 2)$, then (4.4) implies $q > 2$. Consequently, by choosing $\tilde{R}_0$

sufficiently small, we can see that the first three terms on the righthand side of (4.3) are absorbed by its lefthand side. Hence, we get

$$
\begin{aligned}
&\left\| |x|^{-2/q} e^{\phi} \chi u \right\|_q + \left\| |x|^{-1} e^{\phi} \chi u \right\|_2 \\
\leq& C_3 \left\| |x|^{2-\frac{2}{p}} e^{\phi} \left|\nabla \chi\right| (|v| + |w|) \right\|_p + C_3 \left\| |x|^{2-\frac{2}{p}} e^{\phi} \left|\nabla \chi\right| \left|\nabla \mu\right| u \right\|_p \\
&+ C_3 \left\| e^{\phi} \left|\nabla \chi\right| u \right\|_2 \\
\leq& C_4 \left\| e^{\phi} |x| \left|\nabla \chi\right| (|v| + |w|) \right\|_2 + C_4 \left\| e^{\phi} \left|\nabla \chi\right| u \right\|_2 ,
\end{aligned} \tag{4.6}
$$

where we have used the Hölder inequality and the structure of $\operatorname{supp} \nabla\chi$ to show that

$$
\begin{aligned}
\left\| |x|^{2-\frac{2}{p}} e^{\phi} \left|\nabla \chi\right| (|v| + |w|) \right\|_p &\leq C \left\| e^{\phi} |x| \left|\nabla \chi\right| (|v| + |w|) \right\|_2 \left\| (\operatorname{supp} \nabla\chi) |x|^{1-\frac{2}{p}} \right\|_{\frac{2p}{2-p}} \\
&\leq C \left\| e^{\phi} |x| \left|\nabla \chi\right| (|v| + |w|) \right\|_2
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| |x|^{2-\frac{2}{p}} e^{\phi} \left|\nabla \chi\right| \left|\nabla \mu\right| u \right\|_p &\leq C \left\| e^{\phi} \left|\nabla \chi\right| u \right\|_2 \left\| \nabla \mu \right\|_{\frac{2s}{2-s}} \\
&\leq C \left\| e^{\phi} \left|\nabla \chi\right| u \right\|_2
\end{aligned}
$$

in the last inequality.

We define the sets $A_s = \{R_1/3 \leq |x| \leq 2R_1/3\}$, $A_m = \{2R_1/3 \leq |x| \leq R_3/3\}$, and $A_l = \{R_3/3 \leq |x| \leq 2R_3/3\}$, where $R_3 \leq \hat{R}_0$. Let χ is a smooth cut-off function described as above. Since $\operatorname{supp} D^\alpha \chi \subset A_s \cup A_l$, then for any multi-index α

$$
\begin{cases}
|D^\alpha \chi| = O\left(R_1^{-|\alpha|}\right) & \text{for all } x \in A_s \\
|D^\alpha \chi| = O\left({R_3}^{-|\alpha|}\right) & \text{for all } x \in A_l.
\end{cases} \tag{4.7}
$$

We also denote $A_s \Subset \tilde{A}_s := \{R_1/4 \leq |x| \leq R_1\}$ and $A_l \Subset \tilde{A}_l := \{R_3/4 \leq |x| \leq R_3\}$. Now it follows from (4.6) and (2.5) that

$$
\begin{aligned}
&\left\| |x|^{-1} e^{\phi} \chi u \right\|_2 \\
\leq& C_5 e^{\tilde{\phi}(R_1/3)} \left\| |v| + |w| \right\|_{L^2(A_s)} + C_5 e^{\tilde{\phi}(R_3/3)} \left\| |v| + |w| \right\|_{L^2(A_l)} + C_5 \left\| e^{\phi} \left|\nabla \chi\right| u \right\|_2 \\
\leq& C_5 e^{\tilde{\phi}(R_1/3)} R_1^{-1} \left\| u \right\|_{L^2(\tilde{A}_s)} + C_5 e^{\tilde{\phi}(R_3/3)} R_3^{-1} \left\| u \right\|_{L^2(\tilde{A}_l)} \\
&+ C_5 \left\| e^{\phi} |x|^{-1} (\operatorname{supp} \nabla\chi) u \right\|_2 \\
\leq& C_6 e^{\tilde{\phi}(R_1/3)} R_1^{-1} \left\| u \right\|_{L^2(\tilde{A}_s)} + C_6 e^{\tilde{\phi}(R_3/3)} R_3^{-1} \left\| u \right\|_{L^2(\tilde{A}_l)} ,
\end{aligned} \tag{4.8}
$$

where we set $\tilde{\phi}(a) = \phi(\ln a)$.

From (4.8), it follows that for $R_2 \le R_3/3$
(4.9)

$$\begin{aligned}
&e^{2\tilde{\phi}(R_2)}R_2^{-2}\int_{\{2R_1/3<|x|<R_2\}}|u|^2dx+e^{2\tilde{\phi}(R_3/6)}\left(\frac{R_3}{6}\right)^{-2}\int_{\{2R_1/3<|x|<R_3/6\}}|u|^2dx\\
\le&2\int|x|^{-2}e^{2\phi}\chi^2|u|^2dx\\
\le&C_7e^{2\tilde{\phi}(R_1/3)}R_1^{-2}\int_{\tilde{A}_s}|u|^2dx+C_7e^{2\tilde{\phi}(R_3/3)}R_3^{-2}\int_{\tilde{A}_l}|u|^2dx.
\end{aligned}$$

Dividing $R_2^{-2}e^{2\tilde{\phi}(R_2)}$ on both sides of (4.9) yields
(4.10)

$$\begin{aligned}
&\int_{\{2R_1/3<|x|<R_2\}}|u|^2dx+\left(\frac{6R_2}{R_3}\right)^{2\beta+2}\int_{\{2R_1/3<|x|<R_3/6\}}|u|^2dx\\
\le&C_7e^{2\tilde{\phi}(R_1/3)-2\tilde{\phi}(R_2)}\left(\frac{R_2}{R_1}\right)^2\int_{\tilde{A}_s}|u|^2dx+C_7e^{2\tilde{\phi}(R_3/3)-2\tilde{\phi}(R_2)}\left(\frac{3R_2}{R_3}\right)^2\int_{\tilde{A}_l}|u|^2dx\\
\le&C_7\left(\frac{3R_2}{R_1}\right)^{2\beta+2}\int_{\tilde{A}_s}|u|^2dx+C_7\left(\frac{3R_2}{R_3}\right)^{2\beta+2}\int_{\tilde{A}_l}|u|^2dx.
\end{aligned}$$

Further choosing $R_2 \le R_3/6$ and adding $(1+(6R_2/R_3)^{2\beta+2})\int_{B_{2R_1/3}}|u|^2$ to both sides of (4.10), we obtain that

$$\begin{aligned}
&\int_{B_{R_2}}|u|^2dx+\left(\frac{6R_2}{R_3}\right)^{2\beta+2}\int_{B_{R_3/6}}|u|^2dx\\
\le&C_8\left(\frac{3R_2}{R_1}\right)^{2\beta+2}\int_{B_{R_1}}|u|^2dx+C_8\left(\frac{3R_2}{R_3}\right)^{2\beta+2}\int_{B_{R_3}}|u|^2dx.
\end{aligned}\tag{4.11}$$

Now, let $R_0 = R_3/6$ be fixed. Then, we choose $\beta + 1 = \beta_0$ large enough such that

$$2C_8\left(\frac{R_2}{2R_0}\right)^{2\beta_0}\int_{B_{6R_0}}|u|^2dx\le\left(\frac{R_2}{R_0}\right)^{2\beta_0}\int_{B_{R_0}}|u|^2dx\tag{4.12}$$

which is equivalent to

$$\beta_0\ge\frac{1}{2}\ln\left(\frac{2C_8\int_{B_{6R_0}}|u|^2dx}{\int_{B_{R_0}}|u|^2dx}\right).$$

From (4.11) and (4.12), we immediately obtain

$$\int_{B_{R_2}}|u|^2dx\le C_9\left(\frac{3R_2}{R_1}\right)^{2\beta_0}\int_{B_{R_1}}|u|^2dx,$$

which is the doubling inequality (1.3). This completes the proof of Theorem 1.1. Furthermore, combining (4.11) and (4.12) also leads to

$$C_8\left(\frac{R_2}{2R_0}\right)^{2\beta_0}\int_{B_{6R_0}}|u|^2dx\le C_8\left(\frac{3R_2}{R_1}\right)^{2\beta_0}\int_{B_{R_1}}|u|^2dx,$$

which implies (1.4) and Theorem 1.2.

Again, in view of (4.11), we have that

$$\int_{B_{R_2}}|u|^2dx\le C_9\left(e^{\beta E}\int_{B_{R_1}}|u|^2dx+e^{-\beta B}\int_{B_{R_3}}|u|^2dx\right), \tag{4.13}$$

where we set $\beta+1\to\beta$, $E=2\log\left(\frac{3R_2}{R_1}\right)$, and $B=2\log\left(\frac{R_3}{3R_2}\right)$. Note that both E and B are positive (recall $R_2\le R_3/6$). Let $\tilde\beta$ denote the smallest $\beta\in\mathbb{N}+\frac{1}{2}$ for which (4.13) holds. We now discuss two cases. Firstly, if $\displaystyle\int_{B_{R_1}}|u|^2\neq 0$ and

$$e^{\tilde\beta E}\int_{B_{R_1}}|u|^2dx\le e^{-\tilde\beta B}\int_{B_{R_3}}|u|^2dx,$$

then we find $\hat\beta\ge\tilde\beta$ such that

$$e^{\hat\beta E}\int_{B_{R_1}}|u|^2dx=e^{-\hat\beta B}\int_{B_{R_3}}|u|^2dx. \tag{4.14}$$

Solving $\hat\beta$ from (4.14) gives

$$\hat\beta=\frac{1}{E+B}\ln\left(\frac{\int_{B_{R_3}}|u|^2dx}{\int_{B_{R_1}}|u|^2dx}\right).$$

Substituting $\hat\beta$ into (4.13) yields

$$\int_{B_{R_2}}|u|^2dx\le 2C_9\left(\int_{B_{R_1}}|u|^2dx\right)^{\frac{B}{B+E}}\left(\int_{B_{R_3}}|u|^2dx\right)^{\frac{E}{B+E}}. \tag{4.15}$$

On the other hand, if

$$e^{-\tilde\beta B}\int_{B_{R_3}}|u|^2\le e^{\tilde\beta E}\int_{B_{R_1}}|u|^2,$$

then since $R_2\le R_3$, it holds that

$$\begin{aligned}\int_{B_{R_2}}|u|^2\le&\left(\int_{B_{R_3}}|u|^2\right)^{\frac{B}{E+B}}\left(\int_{B_{R_3}}|u|^2\right)^{\frac{E}{E+B}}\\ \le& e^{\tilde\beta B}\left(\int_{B_{R_1}}|u|^2\right)^{\frac{B}{E+B}}\left(\int_{B_{R_3}}|u|^2\right)^{\frac{E}{E+B}}.\end{aligned} \tag{4.16}$$

Putting together (4.15), (4.16), we arrive at

$$\int_{B_{R_2}} |u|^2 \le C_{10} \left(\int_{B_{R_1}} |u|^2 \right)^{\tau} \left(\int_{B_{R_3}} |u|^2 \right)^{1-\tau}, \tag{4.17}$$

where we have set $C_{10} = \max\left\{2C_9, e^{\tilde{\beta}B}\right\}$ and $\tau = \frac{B}{E+B}$. The proof of Theorem 1.1 is complete. Finally, we can observe that $\frac{B}{E+B} \approx (\log(1/R_1))^{-1}$ when R_1 is small.

References

[1] G. Alessandrini and A. Morassi, *Strong unique continuation for the Lamé system of elasticity*, Comm. in PDE., **26**, 1787-1810, 2001.

[2] B. Davey, *Quantitative unique continuation for Schrödinger operators*, Preprint, arXiv:1903.04021, 2019.

[3] B. Davey, C.-L. Lin, and J.-N. Wang, *Liouville-type theorem for the Lame system with singular coefficients*, Proc AMS, **147** (2019), no. 6, 2619-2624.

[4] B. Davey, C.-L. Lin, and J.-N. Wang, *Strong unique continuation for the Lamé system with less regular coefficients*, Math. Ann. (2020). https://doi.org/10.1007/s00208-020-02026-0.

[5] B. Davey and J. Zhu, *Quantitative uniqueness of solutions to second order elliptic equations with singular lower order terms*, Preprint, arXiv:1702.04742, 2017.

[6] B. Davey and J. Zhu, *Quantitative uniqueness of solutions to second order elliptic equations with singular potentials in two dimensions*, Calc. Var. Partial Differential Equations, **57** (2018), no. 3, Art. 92, 27.

[7] L. Escauriaza, *Unique continuation for the system of elasticity in the plane*, Proc. Amer. Math. Soc., **134** (2006), 2015-2018.

[8] L. Escauriaza, F.J. Fernández, and S. Vessella, *Doubling properties of caloric functions*, Appl. Anal., **85** (2006), 205-223.

[9] C. Kenig and J.-N. Wang, *Unique continuation for the elasticity system and a counterexample for second order elliptic systems*, Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory, Vol 1:159-178, 2016.

[10] H. Koch, C.-L. Lin, and J.-N. Wang, *Doubling inequalities for the Lame system with rough coefficients*, Proc AMS, **144** (2016), no. 4, 5309-5318.

[11] C.-L. Lin, G. Nakamura, G. Uhlmann, and J.-N. Wang, *Quantitative strong unique continuation for the Lamé system with less regular coefficients*, Methods Appl. Anal., **18** (2011), no. 1, 85-92.

[12] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in selfadjoint divergence form with Hölder continuous coefficients*, Arch. Rational Mech. Anal., **54** (1974), 105-117.

[13] C.-L. Lin and J.-N. Wang, *Strong unique continuation for the Lamé system with Lipschitz coefficients*, Math. Ann., **331** (2005), 611-629.

[14] C.-L. Lin, G. Nakamura and J.-N. Wang *Optimal three-ball inequalities and quantitative uniqueness for the Lamé system with Lipschitz coefficients*, Duke Math Journal., **155** (2010), 189-204.

[15] A. Plis, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **11** (1963), 95-100.

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