OPTIMAL STABILITY ESTIMATE OF THE INVERSE BOUNDARY VALUE PROBLEM BY PARTIAL MEASUREMENTS

HORST HECK AND JENN-NAN WANG

ABSTRACT. In this work we establish log type stability estimates for the inverse potential and conductivity problems with partial Dirichlet-to-Neumann map, where the Dirichlet data is homogeneous on the inaccessible part. This result, to some extent, improves our former result on the partial data problem [HW06] in which log-log type estimates were derived.

1. Introduction

In this paper we study the stability question of the inverse boundary value problem for the Schrödinger equation with a potential and the conductivity equation by partial Cauchy data. This type of inverse problem with full data, i.e., Dirichlet-to-Neumann map, were first proposed by Calderón [Ca80]. For three or higher dimensions, the uniqueness issue was settled by Sylvester and Uhlmann [SU87] and a reconstruction procedure was given by Nachman [Na88]. For two dimensions, Calderón’s problem was solved by Nachman [Na96] for $W^{2,p}$ conductivities and by Astala and Päivärinta [AP06] for $L^\infty$ conductivities. This inverse problem is known to be ill-posed. A log-type stability estimate was derived by Alessandrini [Al88]. On the other hand, it was shown by Mandache [Ma01] that the log-type estimate is optimal.

All results mentioned above are concerned with the full data. Recently, the inverse problem with partial data has received lots of attentions [GU01], [IU04], [BU02], [KSU05], [FKSU07], [Is07]. A log-log type stability estimate for the inverse problem with partial data was derived by the authors in [HW06]. The method in [HW06] was based on [BU02] and a stability estimate for the analytic continuation proved in [Ve99]. We believe that the log type estimate should be the right estimate for the inverse boundary problem, even with partial data. In this paper, motivated by the uniqueness proof in Isakov’s work [Is07], we prove a log type estimate for the inverse boundary value problem.

The second author was supported in part by the National Science Council of Taiwan (NSC 95-2115-M-002-003).
under the same \textit{a priori} assumption on the boundary as given in \cite{Is07}. Precisely, the inaccessible part of the boundary is either a part of a sphere or a plane. Also, one is able to use zero data on the inaccessible part of the boundary. The strategy of the proof in \cite{Is07} follows the framework in \cite{SU87} where complex geometrical optics solutions are key elements. A key observation in \cite{Is07} is that when \( \Gamma_0 \) is a part of a sphere or a plane, we are able to use a reflection argument to guarantee that complex geometrical optics solutions have homogeneous data on \( \Gamma_0 \).

Let \( n \geq 3 \) and \( \Omega \subset \mathbb{R}^n \) be an open domain with smooth boundary \( \partial \Omega \). Given \( q \in L^\infty(\Omega) \), we consider the boundary value problem:

\[
\begin{align*}
(\Delta - q)u &= 0 \quad \text{in } \Omega \\
u &= f \quad \text{on } \partial \Omega,
\end{align*}
\]  

(1.1)

where \( f \in H^{1/2}(\partial \Omega) \). Assume that 0 is not a Dirichlet eigenvalue of \( \Delta - q \) on \( \Omega \). Then (1.1) has a unique solution \( u \in H^1(\Omega) \). The usual definition of the Dirichlet-to-Neumann map is given by

\[
\Lambda_q f = \partial_\nu u|_{\partial \Omega}
\]

where \( \partial_\nu u = \nabla u \cdot \nu \) and \( \nu \) is the unit outer normal of \( \partial \Omega \).

Let \( \Gamma_0 \subset \partial \Omega \) be an open part of the boundary of \( \Omega \). We set \( \Gamma = \partial \Omega \setminus \Gamma_0 \). We further set \( H^{1/2}_0(\Gamma) := \{ f \in H^{1/2}(\partial \Omega) : \text{supp } f \subset \Gamma \} \) and \( H^{-1/2}(\Gamma) \) the dual space of \( H^{1/2}_0(\Gamma) \). Then the partial Dirichlet-to-Neumann map \( \Lambda_{q,\Gamma} \) is defined as

\[
\Lambda_{q,\Gamma} f := \partial_\nu u|_{\Gamma} \in H^{-1/2}(\Gamma)
\]

where \( u \) is the unique weak solution of (1.1) with Dirichlet Data \( f \in H^{1/2}_0(\Gamma) \). In what follows, we denote the operator norm by

\[
\|\Lambda_{q,\Gamma}\|_* := \|\Lambda_{q,\Gamma}\|_{H^{1/2}_0(\Gamma) \to H^{-1/2}(\Gamma)}
\]

We consider two types of domains in this paper: (a) \( \Omega \) is a bounded domain in \( \{ x_n < 0 \} \) and \( \Gamma_0 = \partial \Omega \cap \{ x_n < 0 \} \); (b) \( \Omega \) is a subdomain of \( B(a,R) \) and \( \Gamma_0 = \partial B(a,R) \cap \partial \Omega \) with \( \Gamma_0 \neq \partial B(a,R) \), where \( B(a,R) \) is a ball centered at \( a \) with radius \( R \). Denote by \( \tilde{q} \) the zero extension of the function \( q \) defined on \( \Omega \) to \( \mathbb{R}^n \). The main result of the paper reads as follows:

**Theorem 1.1.** Assume that \( \Omega \) is given as in either (a) or (b). Let \( N > 0, s > \frac{n}{2} \) and \( q_j \in H^s(\Omega) \) such that

\[
\|q_j\|_{H^s(\Omega)} \leq N
\]  

(1.2)
for $j = 1, 2, \text{ and } 0$ is not a Dirichlet eigenvalue of $\Delta - q_j$ for $j = 1, 2$. Then there exist constants $C > 0$ and $\sigma > 0$ such that
\[
\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C \left| \log \|\Lambda_{q_1, \Gamma} - \Lambda_{q_2, \Gamma}\|_* \right|^{-\sigma}
\] (1.3)
where $C$ depends on $\Omega, N, n, s$ and $\sigma$ depends on $n$ and $s$.

Theorem 1.1 can be generalized to the conductivity equation. Let $\gamma \in H^s(\Omega)$ with $s > 3 + \frac{n}{2}$ be a strictly positive function on $\Omega$. The equation for the electrical potential in the interior without sinks or sources is
\[
\text{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega
\]
\[
u \gamma \quad \text{on } \partial \Omega.
\]
As above, we take $f \in H^{1/2}(\Gamma)$. The partial Dirichlet-to-Neumann map defined in this case is
\[
\Lambda_{\gamma, \Gamma} : f \mapsto \gamma \partial_{\nu} u|_{\Gamma}.
\]

**Corollary 1.2.** Let the domain $\Omega$ satisfy (a) or (b). Assume that $\gamma_j \geq N^{-1} > 0$, $s > \frac{n}{2}$, and
\[
\|\gamma_j\|_{H^{s+3}(\Omega)} \leq N
\] (1.4)
for $j = 1, 2, \text{ and } 0 \leq \beta \leq 1$.

Then there exist constants $C > 0$ and $\sigma > 0$ such that
\[
\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C \left| \log \|\Lambda_{\gamma_1, \Gamma} - \Lambda_{\gamma_2, \Gamma}\|_* \right|^{-\sigma}
\] (1.6)
where $C$ depend on $\Omega, N, n, s$ and $\sigma$ depend on $n, s$.

**Remark 1.3.** For the sake of simplicity, we impose the boundary identification condition (1.5) on conductivities. However, using the arguments in [Al88] (also see [HW06]), this condition can be removed. The resulting estimate is still in the form of (1.6) with possible different constant $C$ and $\sigma$.

## 2. Preliminaries

We first prove an estimate of the Riemann-Lebesgue lemma for a certain class of functions. Let us define
\[
g(y) = \|f(\cdot - y) - f(\cdot)\|_{L^1(\mathbb{R}^n)}
\]
for any $f \in L^1(\mathbb{R}^n)$. It is known that $\lim_{|y| \to 0} g(y) = 0$. 
Lemma 2.1. Assume that $f \in L^1(\mathbb{R}^n)$ and there exist $\delta > 0$, $C_0 > 0$, and $\alpha \in (0, 1)$ such that
\begin{equation}
    g(y) \leq C_0|y|^{\alpha}
\end{equation}
whenever $|y| < \delta$. Then there exists a constant $C > 0$ and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the inequality
\begin{equation}
    |\mathcal{F}f(\xi)| \leq C(\exp(-\pi \varepsilon^2 |\xi|^2) + \varepsilon^\alpha)
\end{equation}
holds with $C = C(C_0, \|f\|_{L^1}, n, \delta, \alpha)$.

Proof. Let $G(x) := \exp(-\pi |x|^2)$ and set $G_\varepsilon(x) := \varepsilon^{-n}G(\frac{x}{\varepsilon})$. Then we define $f_\varepsilon := f \ast G_\varepsilon$. Next we write
\begin{equation*}
    |\mathcal{F}f(\xi)| \leq |\mathcal{F}f_\varepsilon(\xi)| + |\mathcal{F}(f_\varepsilon - f)(\xi)|.
\end{equation*}
For the first term on the right hand side we get
\begin{equation*}
    |\mathcal{F}f_\varepsilon(\xi)| \leq |\mathcal{F}f(\xi)| \cdot |\mathcal{F}G_\varepsilon(\xi)|
\end{equation*}
\begin{equation}
    \leq \|f\|_1 \|\varepsilon^{-n}G\|_1 \|\varepsilon^{-n}G_\varepsilon\|_1 \|\xi\|_2 \quad \quad (2.3)
\end{equation}

To estimate the second term, we use the assumption (2.1) and derive
\begin{equation*}
    |\mathcal{F}(f_\varepsilon - f)(\xi)| \leq \|f_\varepsilon - f\|_1
\end{equation*}
\begin{equation*}
    \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y) - f(x)|G_\varepsilon(y) \, dy \, dx
\end{equation*}
\begin{equation*}
    = \int_{|y| < \delta} \int_{\mathbb{R}^n} |f(x - y) - f(x)|G_\varepsilon(y) \, dy \, dx
\end{equation*}
\begin{equation*}
    + \int_{|y| \geq \delta} \int_{\mathbb{R}^n} |f(x - y) - f(x)|G_\varepsilon(y) \, dy \, dx
\end{equation*}
\begin{equation*}
    = I + II.
\end{equation*}
In view of (2.1) we can estimate

\[ I = \int_{|y|<\delta} g(y) G_\varepsilon(y) \, dy \]
\[ \leq C_0 \int_{|y|<\delta} |y|^\alpha G_\varepsilon(y) \, dy \]
\[ = C_0 \int_{S^{n-1}} \int_0^\delta r^n \varepsilon^{-n} \exp\left(-\pi\varepsilon^{-2}r^2\right) r^{n-1} \, dr \, d\psi \]
\[ = C_1 \int_0^\delta \varepsilon^\alpha u^n \varepsilon^{-n} \exp\left(-u^2\right) u^{n-1} \varepsilon \, du \]
\[ = C_2 \varepsilon^\alpha \int_0^\delta u^{n+\alpha-1} \exp(-u^2) \, du = C_3 \varepsilon^\alpha, \]

where \( C_3 = C_3(C_0, n, \delta, \alpha). \)

As for \( II, \) we obtain that for \( \varepsilon \) sufficiently small

\[ II = \int_{|y|\geq\delta} g(y) G_\varepsilon(y) \, dy \]
\[ \leq 2 \| f \|_{L^1} \int_{|y|\geq\delta} G_\varepsilon(y) \, dy \]
\[ \leq C_4 \| f \|_{L^1} \int_0^{\varepsilon^{-1}} \varepsilon^{-n} \exp\left(-\pi\varepsilon^{-2}r^2\right) r^{n-1} \, dr \]
\[ = C_4 \| f \|_{L^1} \int_0^{\varepsilon^{-1}} u^{n-1} \exp\left(-\pi u^2\right) \, du \]
\[ \leq C_4 \| f \|_{L^1} \int_0^{\varepsilon^{-1}} \exp(-\pi u) \, du \]
\[ \leq C_4 \| f \|_{L^1} \frac{1}{\pi} \exp\left(-\pi\varepsilon^{-1}\right) \leq C_5 \varepsilon^\alpha, \]

where \( C_5 = C_5(\| f \|_{L^1}, n, \delta, \alpha). \) Combining the estimates for \( I, II, \) and (2.3), we immediately get (2.2). \( \square \)

We now provide a sufficient condition on \( f, \) defined on \( \Omega, \) such that (2.1) in the previous lemma holds.

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^1 \) boundary. Let \( f \in C^\alpha(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \) and denote by \( \hat{f} \) the zero extension of \( f \) to \( \mathbb{R}^n. \) Then there exists \( \delta > 0 \) and \( C > 0 \) such that

\[ \| \hat{f}(-\cdot - y) - \hat{f}(-\cdot) \|_{L^1(\mathbb{R}^n)} \leq C |y|^\alpha \]

for any \( y \in \mathbb{R}^n \) with \( |y| \leq \delta. \)
Proof. Since \( \Omega \) is bounded and of class \( C^1 \), there exist a finite number of balls, say \( m \in \mathbb{N}, B_i(x_i) \) with center \( x_i \in \partial \Omega, i = 1, \ldots, m \) and associated \( C^1 \)-diffeomorphisms \( \varphi_i : B_i(x_i) \to Q \) where \( Q = \{ x' \in \mathbb{R}^{n-1} : \| x' \| \leq 1 \} \times (-1, 1) \). Set \( d = \text{dist} (\partial \Omega, \partial (\bigcup_{i=1}^m B_i(x_i))) > 0 \) and \( \tilde{Q}_\varepsilon = \bigcup_{x \in \partial \Omega} B(x, \varepsilon) \), where \( B(x, \varepsilon) \) denotes the ball with center \( x \) and radius \( \varepsilon > 0 \). Obviously, for \( \varepsilon < d \), it holds that \( \tilde{Q}_\varepsilon \subset \bigcup_{i=1}^m B_i(x_i) \). Let \( x \in \partial \Omega \) and \( 0 < |y| < \delta \leq d \), then for any \( z_1, z_2 \in B(x, |y|) \cap B_i(x_i) \) we get that

\[
| \varphi_i(z_1) - \varphi_i(z_2) | \leq \| \nabla \varphi_i \|_{L^\infty} |z_1 - z_2| \leq C|y|
\]

for some constant \( C > 0 \). Therefore, \( \varphi_i(\tilde{Q}_\varepsilon \cap B_i(x_i)) \subset \{ x' \in \mathbb{R}^{n-1} : \| x' \| \leq 1 \} \times (-C|y|, C|y|) \). By the transformation formula this yields \( \text{vol}(\tilde{Q}_\varepsilon) \leq C|y| \).

Now let \( |y| < \delta \) we have \( \hat{f}(x - y) - \hat{f}(x) = 0 \) for \( x \not\in \Omega \cup \tilde{Q}_|y| \). Now we write

\[
\| \hat{f}(\cdot - y) - \hat{f} \|_{L^1(\mathbb{R}^n)} = \int_{\Omega \setminus \tilde{Q}_|y|} |\hat{f}(x - y) - \hat{f}(x)| \, dx + \int_{\tilde{Q}_|y|} |\hat{f}(x - y) - \hat{f}(x)| \, dx
\]

\[
\leq C \text{vol}(\Omega)|y|^\alpha + 2\|f\|_{L^\infty} \text{vol}(\tilde{Q}_|y|)
\]

\[
\leq C(|y|^\alpha + |y|) \leq C|y|^\alpha
\]

for \( \delta \leq 1 \). \( \Box \)

Now let \( q_1 \) and \( q_2 \) be two potentials and their corresponding partial Dirichlet-to-Neumann maps are denoted by \( \Lambda_{1,\Gamma} \) and \( \Lambda_{2,\Gamma} \), respectively. The following identity plays a key role in the derivation of the stability estimate.

**Lemma 2.3.** Let \( v_j \) solve \((1.1)\) with \( q = q_j \) for \( j = 1, 2 \). Further assume that \( v_1 = v_2 = 0 \) on \( \Gamma_0 \). Then

\[
\int_{\Omega} (q_1 - q_2) v_1 \overline{v_2} \, dx = \langle (\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma}) v_1, v_2 \rangle
\]

**Proof.** Let \( u_2 \) denote the solution of \((1.1)\) with \( q = q_2 \) and \( u_2 = v_1 \) on \( \partial \Omega \). Therefore

\[
\int_{\Omega} \nabla v_1 \cdot \overline{\nabla v_2} + q_1 v_1 \overline{v_2} \, dx = \langle \partial_\nu v_1, v_2 \rangle
\]

\[
\int_{\Omega} \nabla u_2 \cdot \overline{\nabla v_2} + q_2 u_2 \overline{v_2} \, dx = \langle \partial_\nu u_2, v_2 \rangle.
\]
Setting $v := v_1 - u_2$ and $q_0 = q_1 - q_2$ we get after subtracting these identities

$$
\int_{\Omega} \nabla v \cdot \nabla v_2 + q_2 v_2 v_2 + q_0 v_1 v_2 = \langle (\Lambda_1 - \Lambda_2)v_1, v_2 \rangle.
$$

Since $v_2$ solves $(\Delta - q_2)v_2 = 0$, $v = 0$ on $\partial \Omega$ and $v_2 = 0$ on $\Gamma_0$, we have

$$
\int_{\Omega} \nabla v \cdot \nabla v_2 + q_2 v_2 v_2 = 0,
$$

$$
\langle (\Lambda_1 - \Lambda_2)v_1, v_2 \rangle = \langle (\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma})v_1, v_2 \rangle,
$$

and the assertion follows.

In treating inverse boundary value problems, complex geometrical optics solutions play a very important role. We now describe the complex geometrical optics solutions we are going to use in our proofs. We will follow the idea in [Is07]. Assume that $q_1, q_2 \in L^\infty(\mathbb{R}^n)$ are compactly supported and are even in $x_n$, i.e.

$$
q_1^*(x_1, \ldots, x_{n-1}, x_n) = q_1(x_1, \ldots, x_{n-1}, x_n)
$$

and

$$
q_2^*(x_1, \ldots, x_{n-1}, x_n) = q_2(x_1, \ldots, x_{n-1}, x_n).
$$

Hereafter, we denote

$$
h^*(x_1, \ldots, x_{n-1}, x_n) = h(x_1, \ldots, x_{n-1}, -x_n).
$$

Given $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Let us first introduce new coordinates obtained by rotating the standard Euclidean coordinates around the $x_n$ axis such that the representation of $\xi$ in the new coordinates, denoted by $\tilde{\xi}$, satisfies $\tilde{\xi} = (\tilde{\xi}_1, 0, \ldots, 0, \tilde{\xi}_n)$ with $\tilde{\xi}_1 = \sqrt{\xi_1^2 + \cdots + \xi_{n-1}^2}$ and $\tilde{\xi}_n = \xi_n$. In the following we also denote by $\tilde{x}$ the representation of $x$ in the new coordinates. Then we define for $\tau > 0$

$$
\tilde{\rho}_1 := \left(\frac{\tilde{\xi}_1}{2} - \tau \tilde{\xi}_n, i|\xi| (\frac{1}{4} + \tau^2)^{1/2}, 0, \ldots, 0, 0, 0, 0, \frac{\tilde{\xi}_n}{2} + \tau \tilde{\xi}_1\right),
$$

$$
\tilde{\rho}_2 := \left(\frac{\tilde{\xi}_1}{2} + \tau \tilde{\xi}_n, -i|\xi| (\frac{1}{4} + \tau^2)^{1/2}, 0, \ldots, 0, 0, 0, 0, \frac{\tilde{\xi}_n}{2} - \tau \tilde{\xi}_1\right),
$$

and let $\rho_1$ and $\rho_2$ be representations of $\tilde{\rho}_1$ and $\tilde{\rho}_2$ in the original coordinates. Note that $x_n = \tilde{x}_n$ and $\sum_{i=1}^n x_i y_i = \sum_{i=1}^n \tilde{x}_i \tilde{y}_i$. It is clear that, for $j = 1, 2$, $\rho_j \cdot \rho_j = 0$ as well as $\rho_j^* \cdot \rho_j^* = 0$ hold.

The construction given in [SU87] ensures that there are complex geometrical optics solutions $u_j = e^{i\rho_j \cdot x}(1 + w_j)$ of $(\Delta - q_j)u_j = 0$ in
\[ R^n, \ j = 1, 2, \] and the functions \( w_j \) satisfy \( \| w_j \|_{L^2(K)} \leq C_K \tau^{-1} \) for any compact set \( K \subset R^n \). We then set
\[
\begin{align*}
v_1(x) &= e^{i\rho_1 \cdot x}(1 + w_1) - e^{i\rho_1^* \cdot x}(1 + w_1^*) \\
v_2(x) &= e^{-i\rho_2 \cdot x}(1 + w_2) - e^{-i\rho_2^* \cdot x}(1 + w_2^*). \quad (2.5)
\end{align*}
\]
From this definition it is clear that these functions are solutions of \((\Delta - q_j)v_j = 0\) in \( R^n_+ \) with \( v_j = 0 \) on \( x_n = 0 \).

3. Stability estimate for the potential

Now we are in the position to prove Theorem 1.1. We first consider the case (a) where \( \Gamma_0 \) is a part of a hyperplane. To construct the special solutions described in the previous section, we first perform zero extension of \( q_1 \) and \( q_2 \) to \( R^n_+ \) and then even extension to the whole \( R^n \). As in the last section, we can construct special geometrical optics solutions \( v_j \) of the form \( (2.5) \) to \((\Delta - q_j)v_j = 0\) in \( \Omega \) for \( j = 1, 2 \). Note that \( v_1 = v_2 = 0 \) on \( \Gamma_0 \). We now plug in these solutions into the identity \((2.3)\) and write \( q_0 = q_1 - q_2 \). This gives
\[
\langle (\Lambda_{1, \Gamma} - \Lambda_{2, \Gamma})v_1, v_2 \rangle
\]
\[
= \int_\Omega q_0 v_1 v_2 \, dx \\
= \int_\Omega q_0(x) \left( e^{i(\rho_1 + \rho_2) \cdot x}(1 + w_1)(1 + w_2^*) + e^{i(\rho_1^* + \rho_2^*) \cdot x}(1 + w_1^*)(1 + w_2) \\
- e^{i(\rho_1 + \rho_2^*) \cdot x}(1 + w_1)(1 + w_2^*) - e^{i(\rho_1^* + \rho_2) \cdot x}(1 + w_1^*)(1 + w_2) \right) \, dx \\
= \int_\Omega q_0(x) (e^{i\xi \cdot x} + e^{i\xi^* \cdot x}) \, dx + \int_\Omega q_0(x) f(x, w_1, w_2, w_1^*, w_2^*) \, dx \\
- \int_\Omega q_0(x) (e^{i(\rho_1 + \rho_2^*) \cdot x} + e^{i(\rho_1^* + \rho_2) \cdot x}) \, dx, \quad (3.1)
\]
where
\[
f = e^{i\xi \cdot x}(w_1 + \overline{w_2} + \overline{w_1 w_2}) + e^{i\xi^* \cdot x}(w_1^* + \overline{w_2} + \overline{w_1 w_2^*}) \\
- e^{i(\rho_1 + \rho_2) \cdot x}(w_1^* + \overline{w_2} + \overline{w_1 w_2}) - e^{i(\rho_1^* + \rho_2^*) \cdot x}(w_1 + \overline{w_2} + \overline{w_1 w_2}).
\]
The first term on the right hand side of \((3.1)\) is equal to
\[
\int_{R^n} q_0(x) e^{i\xi \cdot x} \, dx = \mathcal{F}q_0(\xi)
\]
because \( q_0 \) is even in \( x_n \). For the second term, we use the estimate
\[
\|w_1\|_2 + \|w_1^*\|_2 + \|\overline{w_2}\|_2 + \|\overline{w_2^*}\|_2 \leq C\tau^{-1}
\]
to obtain
\[ | \int_{\Omega} q_0f(x, w_1, w_2, w^*_1, w^*_2) \, dx | \leq C \| q_0 \|_{2\tau^{-1}}. \] (3.2)

As for the last term on the right hand side of (3.1), we first observe that
\[(\rho_1 + \rho^*_2) \cdot x = (\tilde{\rho}_1 + \tilde{\rho}^*_2) \cdot \tilde{x} = \tilde{\xi}_1 \tilde{x}_1 + 2\tau \tilde{\xi}_1 \tilde{x}_n = \xi \cdot x' + 2\tau |\xi'| x_n \]
and
\[(\rho^*_1 + \rho_2) \cdot x = (\tilde{\rho}^*_1 + \tilde{\rho}_2) \cdot \tilde{x} = \tilde{\xi}_1 \tilde{x}_1 + 2\tau \tilde{\xi}_1 \tilde{x}_n = \xi \cdot x' - 2\tau |\xi'| x_n, \]
where \(\xi' = (\xi_1, \ldots, \xi_{n-1})\) and \(x' = (x_1, \ldots, x_{n-1})\). Therefore, we can write
\[ \int_{\Omega} q_0(x)e^{i(\rho_1 + \rho^*_2) \cdot x} \, dx = \mathcal{F} q_0(\xi', 2\tau |\xi'|) \]
as well as
\[ \int_{\Omega} q_0(x)e^{i(\rho^*_1 + \rho_2) \cdot x} \, dx = \mathcal{F} q_0(\xi', -2\tau |\xi'|). \]

The Sobolev embedding and the assumptions on \(q_j\) ensure that \(q_0 \in C^\alpha(\Omega)\) for \(\alpha = s - \frac{n}{2}\) and therefore \(q_0\) satisfies the assumption of Lemma 2.2. Applying Lemma 2.1 to \(q_0\) yields that for \(\varepsilon < \varepsilon_0\)
\[ |\mathcal{F} q_0(\xi', 2\tau |\xi'|)| + |\mathcal{F} q_0(\xi', -2\tau |\xi'|)| \leq C(\exp(-\pi \varepsilon^2 (1 + 4\tau^2) |\xi'|^2) + \varepsilon^\alpha). \] (3.3)

Finally, we estimate the boundary integral
\[ \left| \int_{\Gamma} (\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma}) v_1 \cdot v_2 \, d\sigma \right| \]
\[ \leq \|\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma}\|_{*} \|v_1\|_{H^{\frac{1}{2}}(\Gamma)} \|v_2\|_{H^{\frac{1}{2}}(\Gamma)} \]
\[ \leq \|\Lambda_{1,\Gamma} - \Lambda_{2,\Gamma}\|_{*} \|v_1\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} \]
\[ \leq C \exp(|\xi|\tau) \|\Lambda_1 - \Lambda_2\|_{*}. \] (3.4)

Combining (3.1), (3.2), (3.3), and (3.4) leads to the inequality
\[ |\mathcal{F} q_0(\xi)| \leq C\{\exp(|\xi|\tau) \|\Lambda_1 - \Lambda_2\|_{*} + \exp(-\pi \varepsilon^2 (1 + 4\tau^2) |\xi'|^2) + \varepsilon^\alpha + \frac{1}{\tau} \} \] (3.5)
for all \(\xi \in \mathbb{R}^n\) and \(\varepsilon < \varepsilon_0\), where \(C\) only depends on a priori data on the potentials.

Next we would like to estimate the norm of \(q_0\) in \(H^{-1}\). As usual, other estimates of \(q_0\) in more regular norms can be obtained by interpolation. To begin, we set \(Z_R = \{\xi \in \mathbb{R}^n : |\xi_n| < R \text{ and } |\xi'| < R\}\). Note that
B(0, R) \subset Z_R \subset B(0, cR) for some c > 0. Now we use the a priori assumption on potentials and \((3.5)\) and calculate

\[
\|q_0\|_{H^{-1}}^2 \leq \int_{Z_R} |\mathcal{F} q_0(\xi)|^2 (1 + |\xi|^2)^{-1} \, d\xi + \int_{Z_R^c} |\mathcal{F} q_0(\xi)|^2 (1 + |\xi|^2)^{-1} \, d\xi
\]

Finally, we choose a suitable \(R\) here \(\varepsilon = 0.9\). For the second term on the right hand side of \((3.6)\), we choose \(R > \tau\), namely,

\[
\tilde{\tau} = (1 + 4 \pi^2)^{\frac{n}{2}} \tau \leq CR^{\frac{n}{2}} \tau\]

Plugging \((3.7)\) into \((3.6)\) with the choice of \(\varepsilon = (1 + 4 \tau^2)^{-1/4}\) with \(\tau \geq \tau_0 \gg 1\) and integrate

\[
\int_{-R}^{R} \int_{B'(0, R)} \exp(-2\pi \varepsilon^2 (1 + 4 \tau^2) |\xi'|^2) \, d\xi' \, d\xi_n
\]

where \(\tilde{\alpha} = \min\{\alpha, (n - 1)/2\}\). Observing from \((3.3)\), we now choose \(\tau\) such that \(R^n \tau^{-\tilde{\alpha}} = R^{-2}\), namely, \(\tau = R^{(n+2)/\tilde{\alpha}}\). Substituting such \(\tau\) back to \((3.8)\) yields

\[
\|q_0\|_{H^{-1}}^2 \leq C \{R^n \exp(cR\tau) \|\Lambda_1 - \Lambda_2\|^2 + R^n \tau^{-\tilde{\alpha}} + R^{-n/2} + R^{-2}\}
\]

Finally, we choose a suitable \(R\) so that

\[
R^n \exp(cR^{n+2}/\tilde{\alpha}) \|\Lambda_1 - \Lambda_2\|^2 = R^{-2},
\]
i.e., \( R = \left| \log \| \Lambda_1 - \Lambda_2 \| \right|^\gamma \) for some \( 0 < \gamma = \gamma(n, \tilde{\alpha}) \). Thus, we obtain from (3.9) that

\[
\| q_1 - q_2 \|_{H^{-1}(\Omega)} \leq C \left| \log \| \Lambda_1 - \Lambda_2 \| \right|^{-\gamma}.
\] (3.10)

The derivation of (3.10) is legitimate under the assumption that \( \tau \) is large. To make sure that it is true, we need to take \( R \) sufficiently large, i.e. \( R > R_0 \) for some large \( R_0 \). Consequently, there exists \( \tilde{\delta} > 0 \) such that if \( \| \Lambda_1 - \Lambda_2 \| < \tilde{\delta} \) then (3.10) holds. For \( \| \Lambda_1 - \Lambda_2 \| \geq \tilde{\delta} \), (3.10) is automatically true with a suitable constant \( C \) when we take into account the a priori bound (1.2).

The estimate (1.3) is now an easy consequence of the interpolation theorem. Precisely, let \( \epsilon > 0 \) such that \( s = \frac{n}{2} + 2\epsilon \). Using that \( [H^{t_0}(\Omega), H^{t_1}(\Omega)]_\beta = H^{t}(\Omega) \) with \( t = (1 - \beta)t_0 + \beta t_1 \) (see e.g. [Tr95 Theorem 1 in 4.3.1]) and the Sobolev embedding theorem, we get

\[
\| q_1 - q_2 \|_{L^\infty(\Omega)} \leq C \| q_1 - q_2 \|_{H^{t_0}} \| q_1 - q_2 \|_{H^{t_1}}^{(1-\beta)} \| q_1 - q_2 \|_{H^{-1}}^\beta.
\]

Setting \( t_0 = -1 \) and \( t_1 = s \) we end up with

\[
\| q_1 - q_2 \|_{L^\infty(\Omega)} \leq C \| q_1 - q_2 \|_{H^{-1}}^{\frac{n+1-\epsilon}{n+1}}
\]

which yields the desired estimate (1.3) with \( \sigma = \gamma \frac{n+1-\epsilon}{n+1} \).

We now turn to case (b). With a suitable translation and rotation, it suffices to assume \( a = (0, \cdots, 0, R) \) and \( 0 \notin \overline{\Omega} \). As in [Is07], we shall use Kelvin’s transform:

\[
y = \left( \frac{2R}{|x|} \right)^2 x \quad \text{and} \quad x = \left( \frac{2R}{|y|} \right)^2 y. \] (3.11)

Let

\[
\tilde{u}(y) = \left( \frac{2R}{|y|} \right)^n u(x(y)),
\]

then

\[
\left( \frac{|y|}{2R} \right)^{n+2} \Delta_y \tilde{u}(y) = \Delta_x u(x).
\]

Denote by \( \tilde{\Omega} \) the transformed domain of \( \Omega \). In view of this transform, \( \Gamma_0 \) now becomes \( \tilde{\Gamma}_0 \subset \{ y_n = 2R \} \) and \( \Gamma \) is transformed to \( \tilde{\Gamma} \) and \( \tilde{\Gamma} = \partial \tilde{\Omega} \cap \{ y_n > 2R \} \). On the other hand, if \( u(x) \) satisfies \( \Delta u - q(x)u = 0 \) in \( \Omega \), then \( \tilde{u} \) satisfies

\[
\Delta \tilde{u} - \tilde{q} \tilde{u} = 0 \quad \text{in} \quad \tilde{\Omega}, \] (3.12)

where

\[
\tilde{q}(y) = \left( \frac{2R}{|y|} \right)^4 q(x(y)).
\]
Therefore, for (3.12) we can define the partial Dirichlet-to-Neumann map $\tilde{\Lambda}_{\tilde{q},\tilde{r}}$ acting boundary functions with homogeneous data on $\tilde{\Gamma}_0$.

We now want to find the relation between $\Lambda_{q,r}$ and $\tilde{\Lambda}_{\tilde{q},\tilde{r}}$. It is easy to see that for $f, g \in H^{1/2}_0(\Gamma)$

$$\langle \Lambda_{q,r}f, g \rangle = \int_{\Omega} (\nabla u \cdot \nabla \bar{v} + qu\bar{v}) \, dx,$$

where $u$ solves

$$\Delta u - qu = 0 \quad \text{in} \quad \Omega,$$

$$u = f \quad \text{on} \quad \partial \Gamma$$

and $v \in H^1(\Omega)$ with $v|_{\partial \Omega} = g$. Defining

$$\tilde{f} = \left( \frac{2R}{|y|} \right)^{n-2} f \big|_{\partial \tilde{\Omega}}, \quad \tilde{g} = \left( \frac{2R}{|y|} \right)^{n-2} g \big|_{\partial \tilde{\Omega}},$$

and

$$\tilde{v}(y) = \left( \frac{2R}{|y|} \right)^{n-2} v(x(y)).$$

Then we have $\tilde{f}, \tilde{g} \in H^{1/2}_0(\tilde{\Gamma})$ and

$$\langle \Lambda_{q,r}f, g \rangle = \langle \tilde{\Lambda}_{\tilde{q},\tilde{r}}\tilde{f}, \tilde{g} \rangle,$$

in particular,

$$\langle (\Lambda_{q_1,r} - \Lambda_{q_2,r})f, g \rangle = \langle (\tilde{\Lambda}_{\tilde{q}_1,\tilde{r}} - \tilde{\Lambda}_{\tilde{q}_2,\tilde{r}})\tilde{f}, \tilde{g} \rangle. \quad (3.13)$$

With the assumption $0 \notin \overline{\Omega}$, the change of coordinates $x \to y$ by (3.11) is a diffeomorphism from $\overline{\Omega}$ onto $\overline{\tilde{\Omega}}$. Note that $(2R/|y|)^{n-2}$ is a positive smooth function on $\partial \tilde{\Omega}$. Recall a fundamental fact from Functional Analysis:

$$\|\Lambda_{q_1,r} - \Lambda_{q_2,r}\|_* = \sup \left\{ \left| \langle (\Lambda_{q_1,r} - \Lambda_{q_2,r})f, g \rangle \right| : f, g \in H^{1/2}_0(\Gamma) \right\}.$$

The same formula holds for $\|\tilde{\Lambda}_{\tilde{q}_1,\tilde{r}} - \tilde{\Lambda}_{\tilde{q}_2,\tilde{r}}\|_*$. On the other hand, it is not difficult to check that $\|f\|_{H^{1/2}_0(\Gamma)}$ and $\|\tilde{f}\|_{H^{1/2}_0(\tilde{\Gamma})}$, $\|g\|_{H^{1/2}_0(\Gamma)}$ and $\|\tilde{g}\|_{H^{1/2}_0(\tilde{\Gamma})}$ are equivalent, namely, there exists $C$ depending on $\partial \Omega$ such that

$$\frac{1}{C} \|f\|_{H^{1/2}_0(\Gamma)} \leq \|\tilde{f}\|_{H^{1/2}_0(\tilde{\Gamma})} \leq C \|f\|_{H^{1/2}_0(\Gamma)},$$

$$\frac{1}{C} \|g\|_{H^{1/2}_0(\Gamma)} \leq \|\tilde{g}\|_{H^{1/2}_0(\tilde{\Gamma})} \leq C \|g\|_{H^{1/2}_0(\Gamma)}. \quad (3.15)$$
Putting together (3.13), (3.14), and (3.15) leads to
\[\|\tilde{\Lambda}_{\tilde{q}_1, \tilde{r}} - \tilde{\Lambda}_{\tilde{q}_2, \tilde{r}}\|_* \leq C \|\Lambda_{q_1, r} - \Lambda_{q_2, r}\|_*\] (3.16)
with \(C\) only depending on \(\partial\Omega\).

With all the preparations described above, we use case (a) for the domain \(\tilde{\Omega}\) with the partial Dirichlet-to-Neumann map \(\tilde{\Lambda}_{\tilde{q}, \tilde{r}}\). Therefore, we immediately obtain the estimate:
\[\|\tilde{q}_1 - \tilde{q}_2\|_{L^\infty(\tilde{\Omega})} \leq C \left|\log \|\tilde{\Lambda}_{\tilde{q}_1, \tilde{r}} - \tilde{\Lambda}_{\tilde{q}_2, \tilde{r}}\|_*\right|^{-\sigma}.
\]
Finally, rewinding \(\tilde{q}\) and using (3.16) yields the estimate (1.3).

4. Stability estimate for the conductivity

We aim to prove Corollary 1.2 in this section. We recall the following well-known relation: let \(q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}\), then
\[\Lambda_{q, r}(f) = \gamma^{-1/2}|r|\Lambda_{\gamma, r}(\gamma^{-1/2}|r|f) + \frac{1}{2}(\gamma^{-1}\partial_\nu \gamma)|rf|.
\]
In view of the a priori assumption (1.5), we have that
\[(\Lambda_{q_1, r} - \Lambda_{q_2, r})(f) = \gamma^{-1/2}|r|\Lambda_{\gamma, r}(\gamma^{-1/2}|r|f - \frac{1}{2}(\gamma^{-1}\partial_\nu \gamma)|rf|).
\]
where \(\gamma^{-1/2}|r| := \gamma_1^{-1/2}|r| = \gamma_2^{-1/2}|r|\), which implies
\[\|\Lambda_{q_1, r} - \Lambda_{q_2, r}\|_* \leq C\|\Lambda_{\gamma_1, r} - \Lambda_{\gamma_2, r}\|_*\] (4.1)
for some \(C = C(N) > 0\). Hereafter, we set \(q_j = \frac{\Delta \sqrt{\gamma_j}}{\sqrt{\gamma_j}}\), \(j = 1, 2\). The regularity assumption (1.4) and Sobolev’s embedding theorem imply that \(q_1, q_2 \in C^1(\Omega)\). Using this and (1.5), we conclude that \(\tilde{q}_1 - \tilde{q}_2\) satisfies the assumptions of Lemma 2.2 with \(\alpha = 1\). Therefore, Theorem 1.1 and (4.1) imply that
\[\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C \left|\log \|\Lambda_{\gamma_1, r} - \Lambda_{\gamma_2, r}\|_*\right|^{-\sigma_1} \] (4.2)
where \(C\) depend on \(\Omega, N, n, s\) and \(\sigma_1\) depend on \(n, s\). Next, we recall from [Al88, (26) on page 168] that
\[\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\|q_1 - q_2\|^2_{L^\infty(\Omega)}\] (4.3)
for some \(0 < \sigma_2 < 1\), where \(C = C(N, \Omega)\) and \(\sigma_2 = \sigma_2(n, s)\). Finally, putting together (4.2) and (4.3) yields (1.6) with \(\sigma = \sigma_1\sigma_2\) and the proof of Corollary 1.2 is complete.
References


Technische Universität Darmstadt, FB Mathematik, AG 4, Schlossgartenstr. 7, D-64289 Darmstadt, Germany
E-mail address: heck@mathematik.tu-darmstadt.de

Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
E-mail address: jnwang@math.ntu.edu.tw