DOUBLING INEQUALITIES FOR THE LAMÉ SYSTEM

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Abstract. In this paper we derive doubling inequalities for the Lamé system when the Lamé coefficients belong to \( C^{1,1} \). This paper provides a positive answer to the open problem raised by Alessandrini, Morassi, Rosset, and Vessella.

1. Introduction

This work is motivated by a recent article authored by Alessandrini, Morassi, Rosset, and Vessella [3] where they established global doubling inequalities for the Lamé system of elasticity with \( C^{2,1} \) coefficients. In their paper, they posed the proof of similar doubling inequalities with \( C^{1,1} \) coefficients as an open problem. To describe the problem, let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n (n \geq 2) \) with boundary \( \partial \Omega \) whose regularity will be specified later on. Consider the Lamé system of elasticity:

\[
\text{div}(\mu(\nabla u + (\nabla u)^T)) + \nabla(\lambda \text{div} u) = 0 \quad \text{in} \quad \Omega,
\]

where Lamé moduli \( \lambda \) and \( \mu \) belong to \( C^{1,1}(\Omega) \) and satisfy the strong ellipticity condition

\[
\mu \geq \kappa_0 > 0, \quad \lambda + 2\mu \geq \kappa_0 \quad \forall \quad x \in \Omega.
\]

The system (1.1) is a strongly coupled system. However, when \( \lambda, \mu \in C^{1,1} \), one is able to transform (1.1) into an uncoupled system as follows:

\[
-\Delta U + B(\nabla U) + VU = 0 \quad \text{in} \quad \Omega,
\]

where \( U = (u, \text{div} u)^T, B \in L^\infty(\Omega, L(M^{n\times(n+1)}, \mathbb{R}^{n+1})) \) and \( V \in L^\infty(\Omega, L(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \) (see [1]). In view of (1.3), using the frequency function method [4], [5], a doubling inequality of the following form was obtained in [1]

\[
\int_{B_{2r}} |u|^2 + |\text{div} u|^2 \leq K \int_{B_r} |u|^2 + |\text{div} u|^2.
\]

The inequality (1.4) is a quantitative estimate of the strong unique continuation property for (1.1). To apply quantitative estimates of the strong unique continuation property to certain inverse problems for the elasticity, it is desirable to derive a doubling inequality containing \( |u|^2 \) only [2], i.e.,

\[
\int_{B_{2r}} |u|^2 \leq K \int_{B_r} |u|^2.
\]

Indeed, (1.5) for the Lamé system with \( C^{1,1} \) coefficients was proved in [2]. However, as mentioned in [3], the proof given there contains a gap.

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In [3], the authors proved doubling inequalities of the form (1.5) when \( \lambda, \mu \in C^{2,1} \). Moreover, these inequalities depend on global properties of the solution. A key observation in [3] is that for \( \lambda, \mu \in C^{2,1} \), the Lamé system can be transformed into a fourth order system for \( u \) having \( \Delta^2 \) as the leading part and essentially bounded coefficients in the lower orders. For this fourth order system, three-ball inequalities and local doubling inequalities were derived in [8]. Using these inequalities, global doubling inequalities for (1.1) were then obtained.

The aim of this paper is to establish doubling inequalities of the form (1.5) for (1.1) when \( \lambda, \mu \in C^{1,1} \). This result provides a positive answer to the open problem posed in [3]. Since \( \lambda, \mu \in C^{1,1} \), we will work on the reduced system (1.3). Our method is based on Carleman estimates instead of the frequency function method used in [1] and [2]. A key strategy is to choose appropriate polynomial weights for \( u \) and \( \text{div} u \). For other related results about quantitative estimates of the unique continuation property, we refer the reader to papers [7], [8] and references therein.

2. Local doubling inequalities

In this section, we would like to derive local doubling inequalities for (1.1). Without loss of generality, we assume \( 0 \in \Omega \). In addition to (1.2), we let \( \lambda(x) \) and \( \mu(x) \) satisfy
\[
\|\mu(x)\|_{C^{1,1}(\bar{\Omega})} + \|\lambda(x)\|_{C^{1,1}(\bar{\Omega})} \leq M_0.
\]
(2.1)

We aim to prove that

**Theorem 2.1.** Let \( u \in H^1_{loc}(\Omega) \) be a nonzero solution to (1.1). There exist a constant \( 0 < R_1 < 1 \) depending on \( n, \kappa_0, M_0 \) and a constant \( m_1 \) depending on \( n, \kappa_0, M_0, \|u\|_{L^2(|x| \leq 2R_1)} \) satisfying
\[
\int_{|x| \leq r} |u|^2 dx \geq Cr^{2m_1+n} \quad \text{for all small } r,
\]
(2.2)

where \( C \) depends on \( n, \kappa_0, M_0 \) and \( u \).

Theorem 2.1 provides an upper bound on the vanishing order of a nontrivial solution to (1.1). The following doubling inequality is another quantitative estimate of the strong unique continuation for (1.1).

**Theorem 2.2.** Let \( u \in H^1_{loc}(\Omega) \) be a nonzero solution to (1.1). Then there exist positive constants \( R_2 \) depending on \( n, \kappa_0, M_0 \), and \( \tilde{C} \) depending on \( n, \kappa_0, M_0, m_1 \) such that if \( 0 < r \leq R_2 \), then
\[
\int_{|x| \leq 2r} |u|^2 dx \leq \tilde{C} \int_{|x| \leq r} |u|^2 dx,
\]
(2.3)

where \( m_1 \) is the constant obtained in Theorem 2.1.

Theorem 2.1 and 2.2 will be proved together.

**Proof.** We begin with two interior estimates. By repeating the proof of Corollary 17.1.4 in [6], we can show that for \( u \) satisfying (1.1)
\[
\int_{a_1 r < |x| < a_2 r} |x|^{\alpha} |D^\alpha u|^2 dx \leq C_0 \int_{a_3 r < |x| < a_4 r} |u|^2 dx, \quad |\alpha| \leq 2
\]
(2.4)

and for \( U \) satisfying (1.3)
\[
\int_{a_1 r < |x| < a_2 r} |x|^{\alpha} |D^\alpha U|^2 dx \leq C_0 \int_{a_3 r < |x| < a_4 r} |U|^2 dx, \quad |\alpha| \leq 2
\]
(2.5)
for all $0 < a_3 < a_1 < a_2 < a_4$ such that $B_{a_4} \subset \Omega$ with all sufficiently small $r$, where the constant $C_0$ depends on $n$, $\kappa_0$, and $M_0$. From (2.5) and $U = (u, \text{div} u)^T = (u, v)^T$, we immediately obtain that

$$\int_{a_4 r < |x| < a_2 r} |x|^{\alpha}|D^\alpha v|^2 \, dx \leq C_0 \int_{a_3 r < |x| < a_4 r} (|u|^2 + |\nabla u|^2) \, dx, \quad |\alpha| \leq 2. \tag{2.6}$$

Now we recall the following Carleman estimate derived in [10, Lemma 2.1]: for any $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any $m \in \{j + \frac{1}{2}, j \in \mathbb{N}\}$ we have

$$\sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{|x| \leq 5R/4} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 \, dx \leq C_1 \int |x|^{-2m+4-n} |\Delta u|^2 \, dx, \tag{2.7}$$

where $C_1$ only depends on the dimension $n$ (see also [9, Theorem 2] for a similar estimate). We remark that using the cut-off function and regularization, estimate (2.7) remains valid for any fixed $m$ if $u \in H^m_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ with compact support.

In view of this remark, we can apply (2.7) to the function $\chi u$ with $\chi(x) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Therefore, we define $\chi(x) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$\chi(x) = \begin{cases} \ 0 & \text{if } |x| \leq \delta/3, \\ \ 1 & \text{in } \delta/2 \leq |x| \leq 5R/4, \\ \ 0 & \text{if } |x| \geq 3R/2, \end{cases}$$

where $\delta \leq R/4$, $R > 0$ is a small number which will be chosen later. Using the estimate (2.7) to $\chi u$ with parameter $m$ and to $\chi v$ with parameter $m-1$ respectively, we can derive that

$$\sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{|x| \leq 5R/4} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 \, dx$$

$$+ k \sum_{|\alpha| \leq 2} (m-1)^{2-2|\alpha|} \int_{|x| \leq 5R/4} |x|^{2-2m+2|\alpha|-n} |D^\alpha v|^2 \, dx$$

$$\leq C_1 \int |x|^{-2m+4-n} |\Delta u|^2 \, dx + C_1 k \int |x|^{-2m+4-n} |\Delta v|^2 \, dx$$

$$\leq C_1 \int_{|x| \leq 5R/4} |x|^{-2m+4-n} |\Delta u|^2 \, dx + C_1 k \int_{|x| \leq 5R/4} |x|^{-2m+6-n} |\Delta v|^2 \, dx$$

$$+ \sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{|x| \leq 5R/4} |x|^{-2m+4-n} |D^\alpha \chi u|^2 \, dx$$

$$+ \sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{|x| \leq 5R/4} |x|^{-2m+4-n} |D^\alpha \chi v|^2 \, dx$$

$$+ \sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{|x| \leq 5R/4} |x|^{-2m+4-n} |\Delta \chi u|^2 \, dx$$

$$+ \sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{|x| \leq 5R/4} |x|^{-2m+4-n} |\Delta \chi v|^2 \, dx,$$

(2.8)
where the constant $k > 0$ will be determined later. By (1.3) and (2.1), we have that
\[
C_1 \int_{\delta/2 \leq |x| \leq 5R/4} |x|^{-2m+4-n} |\Delta u|^2 dx + C_1 k \int_{\delta/2 \leq |x| \leq 5R/4} |x|^{-2m+6-n} |\Delta v|^2 dx
\leq C_1 M_0 \int_{\delta/2 \leq |x| \leq 5R/4} |x|^{-2m+4-n} (|u|^2 + |\nabla u|^2 + |v|^2 + |\nabla v|^2) dx
\]
\[
+ C_1 k M_0 \int_{\delta/2 \leq |x| \leq 5R/4} |x|^{-2m+6-n} (|u|^2 + |\nabla u|^2 + |v|^2 + |\nabla v|^2) dx. \tag{2.9}
\]
Observing the left hand side of (2.8) and the right hand side of (2.9), we pay attention to $k \int_{\delta/2 \leq |x| \leq 5R/4} |x|^{-2m+4-n} |\nabla v|^2 dx$ and $C_1 M_0 \int_{\delta/2 \leq |x| \leq 5R/4} |x|^{-2m+4-n} |\nabla v|^2 dx$. To absorb the latter by the former, we choose $k = 2C_1 M_0$ and fix it. With this choice of $k$, we can find a small constant $R_1 < 1$ depending on $n$ and $M_0$ such that if $R \leq R_1$ then the right hand side of (2.9) can be absorbed by the left hand side of (2.8).

In view of the definition of $\chi$, it is easy to see that for any multiindex $\alpha$,
\[
\begin{cases}
|D^{\alpha} \chi| = O(\delta^{-|\alpha|}) & \text{for all } \delta/3 < |x| < \delta/2, \\
|D^{\alpha} \chi| = O(R^{-|\alpha|}) & \text{for all } 5R/4 < |x| < 3R/2.
\end{cases} \tag{2.10}
\]
Therefore, using (2.10) and the discussion above, (2.8) implies
\[
m^2 (2\delta)^{-2m-n} \int_{\delta/2 \leq |x| \leq \delta} |u|^2 dx + m^2 R^{-2m-n} \int_{\delta \leq |x| \leq R} |u|^2 dx
\leq \sum_{|\alpha| \leq 2} m^2 \delta^{-2|\alpha|} \int_{\delta/2 \leq |x| \leq 5R/4} |x|^{-2m+2|\alpha|-n} |D^{\alpha} u|^2 dx
\]
\[
\leq C_4 \sum_{|\alpha| \leq 2} \delta^{-4+2|\alpha|} \int_{\delta/3 \leq |x| \leq \delta/2} |x|^{-2m+4-n} |D^{\alpha} u|^2 dx
\]
\[
+ C_4 \sum_{|\alpha| \leq 2} \delta^{-4+2|\alpha|} \int_{\delta/3 \leq |x| \leq \delta/2} |x|^{-2m+6-n} |D^{\alpha} v|^2 dx
\]
\[
+ C_4 \sum_{|\alpha| \leq 2} R^{-4+2|\alpha|} \int_{5R/4 \leq |x| \leq 3R/2} |x|^{-2m+4-n} |D^{\alpha} u|^2 dx
\]
\[
+ C_4 \sum_{|\alpha| \leq 2} R^{-4+2|\alpha|} \int_{5R/4 \leq |x| \leq 3R/2} |x|^{-2m+6-n} |D^{\alpha} v|^2 dx
\]
\[
\leq C_4 \sum_{|\alpha| \leq 2} (\delta/3)^{-2m-n} \int_{\delta/3 \leq |x| \leq \delta/2} |x|^{2|\alpha|} |D^{\alpha} u|^2 dx
\]
\[
+ C_4 \sum_{|\alpha| \leq 2} (\delta/3)^{-2m-n+2} \int_{\delta/3 \leq |x| \leq \delta/2} |x|^{2|\alpha|} |D^{\alpha} v|^2 dx
\]
\[
+ C_4 \sum_{|\alpha| \leq 2} (5R/4)^{-2m-n} \int_{5R/4 \leq |x| \leq 3R/2} |x|^{2|\alpha|} |D^{\alpha} u|^2 dx
\]
\[
+ C_4 \sum_{|\alpha| \leq 2} (5R/4)^{-2m-n+2} \int_{5R/4 \leq |x| \leq 3R/2} |x|^{2|\alpha|} |D^{\alpha} v|^2 dx \tag{2.11}
\]
for all $R \leq R_1$, where $C_2$ and $C_3$ depend on $n$ and $M_0$. To further simplify the estimate, we first use (2.6) and then (2.4) on the right side of (2.11). Hence, we get that

$$m^2(2\delta)^{-2m-n} \int_{\delta/2 < |x| \leq 2\delta} |u|^2 dx + m^2 R^{-2m-n} \int_{2\delta < |x| \leq R} |u|^2 dx \leq C_4 \sum_{|\alpha| \leq 2} (\delta/3)^{-2m-n} \int_{\delta/3 < |x| < \delta/2} |x|^{2|\alpha|} |D^\alpha u|^2 dx + C_4 \sum_{|\alpha| \leq 1} (\delta/3)^{-2m-n+2} \int_{\delta/4 < |x| < \delta/3} |D^\alpha u|^2 dx + C_3 \sum_{|\alpha| \leq 2} (5R/4)^{-2m-n} \int_{5R/4 < |x| < 3R/2} |x|^{2|\alpha|} |D^\alpha u|^2 dx + C_4 \sum_{|\alpha| \leq 1} (5R/4)^{-2m-n+2} \int_{R < |x| < 11R/6} |D^\alpha u|^2 dx \leq C_5 \sum_{|\alpha| \leq 2} (\delta/3)^{-2m-n} \int_{\delta/4 < |x| < 2\delta/3} |x|^{2|\alpha|} |D^\alpha u|^2 dx + C_5 \sum_{|\alpha| \leq 2} (5R/4)^{-2m-n} \int_{R < |x| < 11R/6} |x|^{2|\alpha|} |D^\alpha u|^2 dx \leq C_6 (\delta/3)^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx + C_6 (5R/4)^{-2m-n} \int_{|x| \leq 2R} |u|^2 dx, \quad (2.12)$$

where $C_4, C_5, C_6$ depend on $n$, $\kappa_0$, and $M_0$.

Now we add $m^2(2\delta)^{-2m-n} \int_{|x| \leq \delta/2} |u|^2 dx$ to both sides of (2.12) and obtain

$$\frac{1}{2} m^2(2\delta)^{-2m-n} \int_{|x| \leq \delta/2} |u|^2 dx + m^2 R^{-2m-n} \int_{|x| \leq R} |u|^2 dx = \frac{1}{2} m^2(2\delta)^{-2m-n} \int_{|x| \leq \delta/2} |u|^2 dx + m^2 R^{-2m-n} \int_{|x| \leq \delta/2} |u|^2 dx + m^2 R^{-2m-n} \int_{2\delta < |x| \leq R} |u|^2 dx \leq (C_0 + m^2)(\delta/3)^{-2m-n} \int_{|x| \leq \delta} |u|^2 dx + C_6 (5R/4)^{-2m-n} \int_{|x| \leq 2R} |u|^2 dx. \quad (2.13)$$

We note that the estimate (2.13) is valid for $R \leq R_1$. So from now on, we fix $R = R_1$ in (2.13). Now choosing $m_1$ from $\{ j + 1/2 | j \in \mathbb{N} \}$ satisfying

$$\log \left( 2C_0 \int_{|x| \leq 2R_1} |u|^2 dx / \int_{|x| \leq R_1} |u|^2 dx \right) \leq (2m_1 + n) \log(5/4) + 2 \log m_1,$$
that is, we have
\[
C_6(5R_1/4)^{−2m_1−n} \int_{|x| \leq 2R_1} |u|^2 dx \leq \frac{1}{2} m_1^2 (5R_1)^{−2m_1−n} \int_{|x| \leq R_1} |u|^2 dx. \tag{2.14}
\]
Using (2.14) in (2.13), we get that
\[
\frac{1}{2} m_1^2 (2\delta)^{−2m_1−n} \int_{|x| \leq 2\delta} |u|^2 dx + \frac{1}{2} m_1^2 R_1^{−2m_1−n} \int_{|x| \leq R_1} |u|^2 dx 
\leq (C_6 + m_1^2)(\delta/3)^{−2m_1−n} \int_{|x| \leq \delta} |u|^2 dx. \tag{2.15}
\]
From (2.15), we immediately obtain that
\[
\left( \frac{m_1^2}{2C_6 + 2m_1^2}(3R_1) \right)^{−2m_1−n} \int_{|x| \leq R_1} |u|^2 dx \leq \int_{|x| \leq \delta} |u|^2 dx,
\]
which implies (2.2), and
\[
\frac{1}{2} m_1^2 (2\delta)^{−2m_1−n} \int_{|x| \leq 2\delta} |u|^2 dx \leq (C_6 + m_1^2)(\delta/3)^{−2m_1−n} \int_{|x| \leq \delta} |u|^2 dx
\]
which implies (2.3) with
\[
\bar{C} = \frac{2C_6 + 2m_1^2}{m_1^2} 6^{2m_1+n}.
\]
Here it suffices to take \( \delta \leq R_2 = R_1/4 \). The proof of Theorem 2.1 and 2.2 is now complete. \( \Box \)

3. Global doubling inequalities

In this section we derive global doubling inequalities along the lines in [3]. For brevity, we will not give detailed arguments here. We refer to [3] for detailed proofs.

To begin, we give the definition of Lipschitz boundary.

**Definition 3.1.** We say that the boundary \( \partial \Omega \) is of Lipschitz class with constants \( r_0 \) and \( L_0 \), if, for any \( x_0 \in \partial \Omega \), there exists a rigid transformation of coordinates under which \( x_0 = 0 \) and
\[
\Omega \cap B_{r_0}(0) = \{ x \in B_{r_0}(0) : x_n > \psi(x') \},
\]
where \( x = (x', x_n) \) with \( x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R} \) and \( \psi \) is a Lipschitz continuous function on \( B_{r_0}(0) \subset \mathbb{R}^{n-1} \) satisfying \( \psi(0) = 0 \) and
\[
\| \psi \|_{C^{0,1}(B_{r_0}(0))} \leq L_0 r_0.
\]
To proceed, we will need the following three-ball inequalities.

**Theorem 3.1.** Let \( B_R \subset \Omega \) and \( \lambda, \mu \) satisfy (1.2), (2.1). Then there exist \( 0 < \theta < 1, 0 < \tau < 1 \), depending on \( n, \kappa_0, M_0 \) only, such that for any solution \( u \in H^1(B_R, \mathbb{R}^n) \) to (1.1) and for every \( r_1, r_2, r_3 \), \( 0 < r_1 < r_2 < r_3 \leq \theta R \), we have
\[
\int_{B_{r_2}} |u|^2 dx \leq C \left( \int_{B_{r_1}} |u|^2 dx \right)^{\tau} \left( \int_{B_{r_3}} |u|^2 dx \right)^{1-\tau},
\]
where \( C > 0 \) depends only on \( n, \kappa_0, M_0, r_1/r_3, \) and \( r_2/r_3 \).
Theorem 3.1 was proved in [1] using the frequency function method. We reproduce a proof in the appendix based on the Carleman method.

Let us denote $\Omega_d = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > d \}$. Using three-ball inequalities, one can prove the following theorem (see [2], [3]).

**Theorem 3.2.** [3, Theorem 3.2] Let $\partial \Omega$ be of Lipschitz class with constants $r_0$, $L_0$, and $\lambda$, $\mu$ satisfy (1.2), (2.1). Then for every $\rho > 0$ and for every $x \in \Omega_{\rho}$, we have

$$\int_{B_{\rho}(x)} |u|^2 dx \geq C_\rho \int_{\Omega} |u|^2 dx,$$

where $\theta$ is defined in Theorem 3.1 and $C_\rho$ depends on $n$, $\kappa_0$, $M_0$, $r_0$, $L_0$, $|\Omega|$, $\|u\|_{H^{1,2}(\Omega)}$, $\|u\|_{L^2(\Omega)}$, and $\rho$.

Recall the local doubling inequality (2.3) and its proof: if $u \in H^1_{loc}(\Omega)$ be a non-trivial solution to (1.1), then there exist positive constants $0 < R_1 < 1$, depending on $n, \kappa_0, M_0$, and $R_2 = R_1/4$ such that for all $0 < r \leq R_2$, we have

$$\int_{|x| \leq 2r} |u|^2 dx \leq C \int_{|x| \leq r} |u|^2 dx,$$

(3.1)

where $C$ depending on $n, \kappa_0, M_0$, and $\|u\|_{L^2(|x| \leq 2R_1)}$ such that for all $0 < r \leq R_2$, we have

Using (3.1), we then derive global doubling inequalities [3].

To describe the theorem, we introduce more notations. Instead of the strong convexity, we say that Lamé coefficients $\lambda$, $\mu$ satisfy the strong convexity condition if

$$\mu(x) \geq \kappa_0 > 0, \quad 2\mu(x) + n\lambda(x) \geq \tilde{\kappa}_0 \quad \forall \ x \in \Omega.$$  

(3.2)

It is known that the strong convexity implies the strong ellipticity. Let $\varphi \in L^2(\partial \Omega, \mathbb{R}^n)$ be a vector field satisfying the compatibility condition

$$\int_{\partial \Omega} \varphi \cdot r ds = 0$$

for every infinitesimal rigid displacement $r$, that is, $r = c + Wx$, where $c$ is a constant vector and $W$ is a skew $n \times n$ matrix. Consider the boundary value problem:

$$\begin{align*}
\text{div}(\mu(\nabla u + (\nabla u)^T)) + \nabla(\lambda \text{div} u) &= 0 \quad \text{in} \quad \Omega, \\
(\mu(\nabla u + (\nabla u)^T) + (\lambda \text{div} u) I_n) \nu &= \varphi \quad \text{on} \quad \partial \Omega,
\end{align*}$$

(3.3)

where $I_n$ is the $n \times n$ identity matrix, $\nu$ is the unit outer normal to $\partial \Omega$, and $\varphi$ satisfies the compatibility condition. In order to ensure the uniqueness of the solution to (3.3), we assume the following normalization conditions:

$$\int_{\Omega} u dx = 0, \quad \int_{\Omega} (\nabla u - (\nabla u)^T) dx = 0.$$  

(3.4)

**Theorem 3.3.** [3, Theorem 3.7] Let $\partial \Omega$ be of Lipschitz class with constants $r_0$, $L_0$, and $\lambda$, $\mu$ satisfy (3.2), (2.1). If $u \in H^1(\Omega, \mathbb{R}^n)$ is the weak solution to (3.3) satisfying the normalization condition (3.4). Then there exists a constant $0 < \vartheta < 1$, only depending on $n, \tilde{\kappa}_0, M_0$, such that for every $\tilde{r} > 0$ and for every $x_0 \in \Omega_{\tilde{r}}$, we have

$$\int_{B_{r}(x_0)} |u|^2 dx \leq C \int_{B_{\vartheta r}(x_0)} |u|^2 dx$$

for every $r$ with $0 < r \leq \frac{\vartheta}{2} \tilde{r}$, where $C$ depends on $n, \tilde{\kappa}_0, r_0, L_0, |\Omega|, \tilde{r}$, and $\|\varphi\|_{H^{-1/2}(\partial \Omega)}$, $\|\varphi\|_{H^{-1}(\partial \Omega)}$. 
Appendix A. Proof of Theorem 3.1

We first consider the case where $0 < r_1 < r_2 < \tilde{R} < 1/2$ and $B_{\tilde{R}} \subset \Omega$. The constant $\tilde{R}$ will be determined later. We follow the same argument as in the proof of Theorem 2.1. Here, by replacing $\delta$ by $r_1$ and $R$ by $r_2$, going over the same steps, we can reach estimate (2.13), i.e.,

$$m^2(r_2)^{-2m-n} \int_{|x| \leq r_2} |u|^2 \, dx$$

$$\leq (C_0 + m^2(r_1/3)^{-2m-n}) \int_{|x| \leq r_1} |u|^2 \, dx + C_0(5r_2/4)^{-2m-n} \int_{|x| \leq 1} |u|^2 \, dx.$$  \hspace{1cm} (A.1)

Dividing $m^2(r_2)^{-2m-n}$ on the both sides of (A.1) implies

$$\int_{|x| < r_2} |u|^2 \, dx$$

$$\leq C_7 \{ (3r_2/r_1)^{2m+n} \int_{|x| < r_1} |u|^2 \, dx + (4/5)^{2m+n} \int_{|x| < 1} |u|^2 \, dx \}. \hspace{1cm} (A.2)$$

We emphasize that (A.2) holds when $m \geq m_0$ for some $m_0 \in \{ j + 1/2 | j \in \mathbb{N} \}$. For simplicity, by denoting

$$A = A(r_1, r_2) = (2m + n)^{-1} \log[(3r_2/r_1)^{2m+n}] = \log 3r_2 - \log r_1 > 0,$$

$$B = -(2m + n)^{-1} \log[(4/5)^{2m+n}] = \log 5 - \log 4 > 0,$$

(A.2) becomes

$$\int_{|x| < r_2} |u|^2 \, dx$$

$$\leq C_7 \{ \exp((2m + n)A) \int_{|x| < r_1} |u|^2 \, dx + \exp(-B(2m + n)) \int_{|x| < 1} |u|^2 \, dx \}. \hspace{1cm} (A.3)$$

To further simplify the terms on the right hand side of (A.3), we consider two cases. If

$$\exp((2m_0 + n)A) \int_{|x| < r_1} |u|^2 \, dx < \exp(-B(2m_0 + n)) \int_{|x| < 1} |u|^2 \, dx,$$

then we can pick a $m$ from $\{ j + 1/2 | j \in \mathbb{N} \}$ satisfying $m \geq m_0 + 1$ such that

$$\exp((2m + n)A) \int_{|x| < r_1} |u|^2 \, dx \geq \exp(-B(2m + n)) \int_{|x| < 1} |u|^2 \, dx$$

and

$$\exp((2m - 2 + n)A) \int_{|x| < r_1} |u|^2 \, dx < \exp(-B(2m - 2 + n)) \int_{|x| < 1} |u|^2 \, dx.$$
Using such \( m \), we obtain from (A.3) that

\[
\int_{|x|<r_2} |u|^2\,dx \\
\leq 2C_7 \exp((2m + n)A) \int_{|x|<r_1} |u|^2\,dx \\
= 2C_7 (3r_2/r_1)^2 \exp(A(2m - 2 + n)) \int_{|x|<r_1} |u|^2\,dx \\
\leq 2C_8 \left( \int_{|x|<r_1} |u|^2\,dx \right)^{\frac{n}{n+m}} \left( \int_{|x|<1} |u|^2\,dx \right)^{\frac{A}{n+m}}. \tag{A.4}
\]

On the other hand, if

\[
\exp(-B(2m_0 + n)) \int_{|x|<1} |u|^2\,dx \leq \exp((2m_0 + n)A) \int_{|x|<r_1} |u|^2\,dx,
\]

then we have

\[
\int_{|x|<r_2} |u|^2\,dx \\
\leq \left( \int_{|x|<1} |u|^2\,dx \right)^{\frac{n}{n+m}} \left( \int_{|x|<1} |u|^2\,dx \right)^{\frac{A}{n+m}} \\
\leq \exp((2m_0 + n)B) \left( \int_{|x|<r_1} |u|^2\,dx \right)^{\frac{n}{n+m}} \left( \int_{|x|<1} |u|^2\,dx \right)^{\frac{A}{n+m}}. \tag{A.5}
\]

Putting together (A.4), (A.5), we arrive at

\[
\int_{|x|<r_2} |u|^2\,dx \leq C_9 \left( \int_{|x|<r_1} |u|^2\,dx \right)^{\frac{n}{n+m}} \left( \int_{|x|<1} |u|^2\,dx \right)^{\frac{A}{n+m}}. \tag{A.6}
\]

Now for the general case, we take \( \tilde{R} \leq 1/4 \) and consider \( 0 < r_1 < r_2 < r_3 \) with \( r_1/r_3 < r_2/r_3 \leq 1/4 \). By scaling, i.e. defining \( \tilde{u}(y) := u(r_3y) \), \( \tilde{\mu}(y) := \mu(r_3y) \) and \( \tilde{\lambda}(y) := \lambda(r_3y) \), we derive from (A.6) that

\[
\int_{|y|<r_2/r_3} |\tilde{u}|^2\,dy \leq C_9 \int_{|y|<r_1/r_3} |\tilde{u}|^2\,dy \left( \int_{|y|<1} |\tilde{u}|^2\,dy \right)^{1-\tau}, \tag{A.7}
\]

where \( \tau = B/(A + B) \) with

\[
A = A(r_1/r_3, r_2/r_3) = \log(3r_2/r_1), \quad B = \log(5/4).
\]

Restoring the variable \( x = r_3y \) in (A.7) gives

\[
\int_{|x|<r_2} |u|^2\,dx \leq C \left( \int_{|x|<r_1} |u|^2\,dx \right)^\tau \left( \int_{|x|<r_3} |u|^2\,dx \right)^{1-\tau}.
\]

The proof now is complete. \( \square \)
References


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