Doubling inequalities for anisotropic plate equations and size estimates of inclusions *

M. Di Cristo† C-L. Lin‡ A. Morassi§
E. Rosset¶ S. Vessella‖ J-N. Wang**

Abstract

We prove upper and lower estimates of the area of an unknown elastic inclusion in a thin plate by one boundary measurement. The plate is made of non-homogeneous linearly elastic material belonging to a general class of anisotropy and the domain of the inclusion is a measurable subset of the plate. The size estimates are expressed in terms of the work exerted by a couple field applied at the boundary and of the induced transversal displacement and its normal derivative taken at the boundary of the plate. Main new mathematical tool is a doubling inequality for solutions to fourth order elliptic equations whose principal part $P(x, D)$ is the product of two second order elliptic operators $P_1(x, D), P_2(x, D)$ such that $P_1(0, D) = P_2(0, D)$. The proof of the doubling inequality is based on Carleman method, a sharp three spheres inequality and a bootstrapping argument.

1 Introduction

This paper deals with the inverse problem of detecting, inside a thin elastic plate, an unknown inclusion made of different elastic material, in terms of measurements taken at the boundary of the plate. Concerning the basic issue of uniqueness, let us recall that, even for the analogous inverse problem in conductivity, which involves a second-order elliptic equation instead of the fourth-order elliptic equation governing the static equilibrium of a plate, a general result under a finite number of boundary measurements has not yet been proved. In

*The first, the second, the fifth and the sixth authors are partially supported by CNR-NSC bilateral action between Italy and Taiwan. The third author is partially supported by the Carlos III University of Madrid-Banco de Santander Chairs of Excellence Programme for the 2013-2014 Academic Year. The fourth author is supported by Università degli Studi di Trieste FRA 2012 “Problemi Inversi”.

†Politecnico di Milano, Dipartimento di Matematica, 20133 Milano, Italy. Email: michele.dicristo@polimi.it

‡Department of Mathematics, NCTS, National Cheng Kung University, Tainan 701, Taiwan. Email: cllin2@mail.ncku.edu.tw

§Università degli Studi di Udine, Dipartimento di Ingegneria Civile e Architettura, 33100 Udine, Italy. Email: antonino.morassi@uniud.it

¶Università degli Studi di Trieste, Dipartimento di Matematica e Geoscienze, 34127 Trieste, Italy. Email: rossedi@univ.trieste.it

‖Università degli Studi di Firenze, Italy. Email: sergio.vessella@unifi.it

**Institute of Applied Mathematical Sciences, National Taiwan University, Taiwan. Email: jnwang@math.ntu.edu.tw

the electrostatic context, Isakov [Is] proved that the inclusion is uniquely determined when all possible measurements are at disposal, but, unfortunately, the inverse problem shows a weak stability (logarithmic) (see Alessandrini and Di Cristo [A-DiC] and Di Cristo and Rondi [DiC-Ron]), which represents a strong obstruction for reconstruction techniques. For these reasons, it is of interest for applications to find constructive stable estimates of some relevant geometrical parameters of the unknown inclusion, such as its measure.

Following a line of research on "Size Estimates" initiated in [Ka-S-S], [Al-Ro], [Al-Ro-Se] in the electrostatic context and developed in [Ik] and [Al-Mo-Ro02] in linear elasticity (see also the review paper [Al-Mo-Ro03]), in this note we derive constructive upper and lower bounds for the area of the inclusion in terms of the difference between the work exerted in deforming the defective plate and a reference plate (i.e., a plate without inclusion) by applying the same couple field at the boundary.

Size estimates of this type for general inclusions, that is measurable subsets of the plate, have been proved in [Mo-Ro-Ve09] when the material of the reference plate is isotropic. Analogous results for the case of inclusions in shells have been recently obtained in [DiC-Li-Ve-Wa], [DiC-Li-Wa]. The assumption of isotropic material for the reference plate is rather restrictive, since in an increasing number of practical applications the use of materials with various degree of anisotropy is required to achieve better structural performances. In [Mo-Ro-Ve13], the isotropy condition on the reference plate was removed and size estimates were obtained for a large class of anisotropic background materials satisfying an algebraic condition including (2.17), under the so-called a-priori *fatness condition* on the unknown inclusion E, namely that, for a given $h_1 > 0$, $\text{area}\left(\{x \in E | \text{ dist}\{x, \partial E\} > h_1\}\right) \geq \frac{1}{2}\text{area}(E)$. In the present paper (see Theorem 2.2, Section 2) we remove this geometrical assumption and we prove size estimates for an inclusion which is a measurable subset of the plate, assuming condition (2.17) on the background material.

We refer to [Mo-Ro-Ve11, Remark 3.3] for concrete examples of anisotropic materials satisfying condition (2.17) which are significant for engineering applications. Finally, for the sake of completeness, we mention the interesting alternative approach to size estimates recently developed in [Ka-Mi].

Under various perspectives, a single unifying theme has been used in order to deal with the class of inverse boundary value problems posed by the 'Size Estimates' approach, namely quantitative estimates of unique continuation. In this paper we study the so called doubling inequality property of solutions. A connection between this property and the strong unique continuation principle for elliptic partial differential equations has been originally investigated in [Ga-Li86, Ga-Li87]. Subsequently, the doubling inequality property has been widely used in inverse boundary value problems to obtain bounds on the measure of unknown cavities and inclusions.

More generally, we consider a class of fourth order differential equations, which includes the plate equation under condition (2.17). Precisely, let u be a solution of the differential equation

$$P(x, D)u = Qu \qquad \text{in } B_1 := B_1(0), \tag{1.1}$$

where for $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x) = \{y \in \mathbb{R}^n \,:\, |x - y| < r\}$ and

$$P(x, D)u = (P_1(x, D)P_2(x, D))u, \tag{1.2}$$

with

$$P_k(x, D)u = g_k^{ij}(x)D_{ij}^2 u, \qquad k = 1, 2, \tag{1.3}$$

being g_k^{ij} a tensor such that

$$g_1^{ij}(0) = g_2^{ij}(0), \tag{1.4}$$

$$\lambda|\xi|^2 \leq g_k^{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^2, \forall\, x \in \overline{B}_1 \tag{1.5}$$

and

$$\sum_{|\alpha|\leq 2} \sum_{i,j=1}^{2} \|D^\alpha g_k^{ij}\|_{L^\infty(B_1)} \leq C_0, \tag{1.6}$$

for given positive constants $C_0 > 0$ and $\lambda \in (0, 1)$. Here Q is a third order differential operator such that there exists a positive constant M so that

$$|Qu| \leq M \sum_{|\alpha|\leq 3} |D^\alpha u|\,, \qquad \forall\, u \in H^4(B_1). \tag{1.7}$$

We want to study the doubling property of solutions u of (1.1) which says that for any compact subset G of B_1 and any concentric balls $B_r, B_{2r} \subset G$, the following inequality holds

$$\int_{B_{2r}} u^2 \leq K \int_{B_r} u^2, \tag{1.8}$$

where K depends on G, the ellipticity, the regularity bounds but also, necessarily, on the solution u.

Under the above assumptions, we prove the following theorem.

Theorem 1.1. *Let $P(x, D)$ be defined as* (1.2) *satisfying* (1.3)–(1.6). *Let $u \in H^4(B_1)$ be such that*

$$|P(x, D)u| \leq M \sum_{|\alpha|\leq 3} |D^\alpha u|, \qquad \text{in } \overline{B}_1.$$

Then there exist constants $R \in (0, 1)$, $\theta \in (0, 1/2)$ and K such that

$$\int_{B_{2r}} |u|^2 \leq K \int_{B_r} |u|^2, \quad \forall\, r,\, 0 < r < \theta, \tag{1.9}$$

where R depends on λ, M and C_0 only and θ and K on λ, M, C_0 and

$$\mathcal{F}_{loc} = \frac{\|u\|_{L^2(B_R)}}{\|u\|_{L^2(B_{\lambda^2 R^2})}} \tag{1.10}$$

only.

In order to apply the doubling inequality (1.9) to our inverse problem, it is crucial to estimate the constant K in terms of the available boundary data instead of the interior values of the solution u, which may be not known.

The paper is organized as follows. In next Section 2 we derive doubling inequalities for a class of anisotropic plate equations and we apply them to the size estimate problem. In Section 3 we provide a detailed proof of Theorem 1.1.

2 Doubling Inequalities and Size Estimates of Inclusions in Plates

Let us premise some notation and definitions.

For $P = (x_1(P), x_2(P))$ a point in $\mathbb{R}^2$, we denote by $R_{a,b}(P)$ the rectangle of center P and sides parallel to the coordinate axes of length $2a$ and $2b$, namely $R_{a,b}(P) = \{x = (x_1, x_2) : |x_1 - x_1(P)| < a, |x_2 - x_2(P)| < b\}$. We set also $R_{a,b}(0) = R_{a,b}$.

Definition 2.1 ($C^{k,\alpha}$ regularity)**.** *Let Ω be a bounded domain in $\mathbb{R}^2$. Given k, α with $k \in \mathbb{N}$, $0 < \alpha \leq 1$, we say that a portion S of $\partial\Omega$ is of class $C^{k,\alpha}$ with constants $\rho_0, M_0 > 0$ if for any $P \in S$ there exists a rigid transformation of coordinates under which we have $P = 0$ and*

$$\Omega \cap R_{\frac{\rho_0}{M_0},\rho_0} = \{x = (x_1, x_2) \in R_{\frac{\rho_0}{M_0},\rho_0} : x_2 > \psi(x_1)\},$$

where ψ is a $C^{k,\alpha}$ function on $\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)$ satisfying

$$\psi(0) = 0, \qquad \psi^{(k)}(0) = 0, \text{ when } k \geq 1,$$
$$\|\psi\|_{C^{k,\alpha}(-\rho_0/M_0, \rho_0/M_0)} \leq M_0\rho_0.$$

When $k = 0$, $\alpha = 1$ we say that S is of Lipschitz class with constants ρ_0, M_0.

Hereafter, we shall consider a bounded domain Ω satisfying

$$|\Omega| \leq M_1\rho_0^2. \tag{2.11}$$

Working in the framework of the Kirchhoff-Love theory, the transversal displacement u of the middle plane Ω of the plate $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ satisfies the following fourth-order equation

$$\operatorname{div}(\operatorname{div}(\mathbb{P}D^2u)) = 0, \quad \text{in } \Omega, \tag{2.12}$$

with

$$\mathbb{P} = \frac{h^3}{12}\mathbb{C}, \tag{2.13}$$

where h is the uniform thickness of the plate and $\mathbb{C} = \{C_{ijkl}\}$, $\mathbb{C} \in L^\infty(\Omega)$, is the elastic tensor of the material. On $\mathbb{C}$ we shall assume

$$C_{ijkl} = C_{klij} = C_{klji}, \qquad i,j,k,l = 1,2, \quad \text{a.e. in } \Omega, \tag{2.14}$$

$$\gamma|A|^2 \leq \mathbb{C}A \cdot A \leq \gamma^{-1}|A|^2, \quad \text{a.e. in } \Omega, \tag{2.15}$$

$$\sum_{i,j,k,l=1}^{2} \sum_{|\alpha| \leq 2} \rho_0^{|\alpha|}\|D^\alpha C_{ijkl}\|_{L^\infty(\Omega)} \leq M_2, \tag{2.16}$$

where $\gamma \in (0,1)$ and $M_2 > 0$ are given constants. Let us notice that condition (2.14) implies that instead of 16 coefficients we actually deal with 6 coefficients.

Denoting by $a_0 = C_{1111}$, $a_1 = 4C_{1112}$, $a_2 = 2C_{1122} + 4C_{1212}$, $a_3 = 4C_{2212}$, $a_4 = C_{2222}$ and by $S(x)$ the following 7×7 matrix

$$S(x) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \end{pmatrix},$$

we define

$$\mathcal{D}(x) = \frac{1}{a_0}|\det S(x)|$$

and we assume that

$$\mathcal{D}(x) = 0, \qquad \forall\, x \in \mathbb{R}^2. \tag{2.17}$$

Under the above conditions on the plate tensor $\mathbb{P}$, we can state the following doubling inequality.

Proposition 2.1 (Doubling inequality for the plate equation)**.** *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of Lipschitz class with constants ρ_0, M_0, and let $u \in H^4_{loc}(\Omega)$ be a non trivial solution of the equation (2.12). Then, there exists a constant $\theta \in (0,1)$, depending on γ and M_2 only, such that for every $\overline{r} > 0$ and $x_0 \in \Omega_{\overline{r}\rho_0}$ we have*

$$\int_{B_{2r}(x_0)} u^2 \leq K \int_{B_r(x_0)} u^2, \qquad \forall\; r,\; 0 < r \leq \frac{\theta}{2}\overline{r}\rho_0, \tag{2.18}$$

where K only depends on $\gamma, M_0, M_1, M_2, \overline{r}$ and $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$.

Proof. Without loss of generality, we consider the case $\rho_0 = 1$. By [Mo-Ro-Ve11, Section 6], there exists a matrix $\{g^{ij}((x)\}_{i,j=1}^2$ such that

$$\lambda|\xi|^2 \leq g^{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \qquad \forall\; x \in \Omega, \forall\; \xi \in \mathbb{R}^2,$$

$$\sum_{i,j=1}^{2} \sum_{|\alpha|\leq 2} \|D^\alpha g^{ij}\|_{L^\infty(\Omega)} \leq C,$$

where λ, $0 < \lambda \leq 1$, C depend on γ and M_2, and

$$\mathrm{div}(\mathrm{div}(\mathbb{P}D^2\cdot)) = P(x, D\cdot) + Q(\cdot),$$

with $P(x,D) = (g^{ij}(x)D^2_{ij})(g^{lk}(x)D^2_{lk})$, and Q is a third order operator such that

$$|Q(v)| \leq cM_2 \sum_{2\leq|\alpha|\leq 3} |D^\alpha v|, \qquad \forall\, v \in H^4(\Omega),$$

where c is an absolute constant. Now, since u is solution to (2.12), u satisfies the doubling inequality (1.9) of Theorem 1.1. In order to prove the proposition, it suffices to estimate the local frequency (1.10).

To this aim, let us notice that under our assumptions the solution u satisfies the following *Lipschitz propagation of smallness property*

$$\int_{B_\rho(x)} u^2 \geq C_\rho \int_\Omega u^2, \qquad \forall\ \rho > 0 \text{ and } \forall\ x \in \Omega_{s\rho}, \tag{2.19}$$

where $s > 1$ only depends on γ, M_2 and $C_\rho > 0$ only depends on ρ_0, M_0, M_1, γ, M_2, $\frac{\|u\|_{H^{1/2}(\Omega)}}{\|u\|_{L^2(\Omega)}}$ and ρ. The proof of (2.19) is essentially based on a three-spheres inequality for the solution u, which has been derived in [Mo-Ro-Ve11, Section 6], see for details the arguments in [Al-Mo-Ro-Ve, Theorem 3.2]. □

Next, we consider the problem of the detection of an unknown inclusion E in Ω. Let E be a measurable, possible disconnected subset of Ω satisfying

$$dist(E, \partial\Omega) \geq d_0\rho_0, \tag{2.20}$$

for some positive constant d_0. Let us assume that the plate tensor $\widetilde{\mathbb{P}} = \frac{h^3}{12}\widetilde{\mathbb{C}}$ in the inclusion belongs to $L^\infty(\Omega)$, where $\widetilde{\mathbb{C}}$ satisfies symmetry conditions analogous to (2.14). Moreover, we assume the following jump conditions on the elastic tensors $\mathbb{P}$ and $\widetilde{\mathbb{P}}$: either there exist $\eta > 0$ and $\delta > 1$ such that

$$\eta\mathbb{P} \leq \widetilde{\mathbb{P}} - \mathbb{P} \leq (\delta - 1)\mathbb{P}, \quad \text{a.e. in } \Omega \text{ (hard inclusion)}, \tag{2.21}$$

or there exist $\eta > 0$ and $0 < \delta < 1$ such that

$$-(1 - \delta)\mathbb{P} \leq \widetilde{\mathbb{P}} - \mathbb{P} \leq -\eta\mathbb{P}, \quad \text{a.e. in } \Omega \text{ (soft inclusion)}. \tag{2.22}$$

Let us apply a couple field $\widehat{M} = \widehat{M}_2 e_1 + \widehat{M}_1 e_2$ on the boundary of Ω such that:

$$\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2), \tag{2.23}$$

$$supp(\widehat{M}) \subset \Gamma, \tag{2.24}$$

where Γ is an open subarc of $\partial\Omega$ such that

$$|\Gamma| \leq (1 - \delta_0)|\partial\Omega|, \tag{2.25}$$

for some positive constant δ_0. Moreover, the couple field $\widehat{M}$ is assumed to satisfy the obvious *compatibility conditions*

$$\int_{\partial\Omega} \widehat{M}_\alpha = 0, \quad \alpha = 1, 2. \tag{2.26}$$

When the inclusion E is absent, the transversal displacement $w_0 \in H^2(\Omega)$, normalized by $\int_\Omega w_0 = 0$ and $\int_\Omega \nabla w_0 = 0$, satisfies the Neumann boundary value problem

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}D^2 w_0)) = 0, & \text{in } \Omega, \\ (\mathbb{P}D^2 w_0)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \operatorname{div}(\mathbb{P}D^2 w_0) \cdot n + ((\mathbb{P}D^2 w_0)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s} & \text{on } \partial\Omega. \end{cases} \tag{2.27}$$

In the above equations, $\widehat{M}_\tau = \widehat{M} \cdot n$ is the twisting moment and $\widehat{M}_n = \widehat{M} \cdot \tau$ is the bending moment, where τ and n are respectively the tangent and the normal vector to the boundary $\partial\Omega$, with $n \times \tau = e_3$. Moreover, $(\cdot)_{,s}$ stands for the derivative with respect to the arc length s.

The equilibrium problem for the defective plate is governed by the boundary value problem

$$
\begin{cases}
\operatorname{div}(\operatorname{div}((\chi_{\Omega\setminus E}\mathbb{P} + \chi_E\widetilde{\mathbb{P}})\nabla^2 w)) = 0, & \text{in } \Omega, \\
(\mathbb{P}D^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\
\operatorname{div}(\mathbb{P}D^2 w) \cdot n + ((\mathbb{P}D^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s} & \text{on } \partial\Omega,
\end{cases}
\tag{2.28}
$$

where w is normalized by $\int_\Omega w = 0$ and $\int_\Omega \nabla w = 0$.

Our size estimates of $|E|$ are given in terms of works W, W_0 exerted by the boundary couple field $\widehat{M}$ when E is present or absent, respectively, namely

$$
W = -\int_{\partial\Omega} \left(\widehat{M}_{\tau,s} w + \widehat{M}_n w_{,n} \right), \tag{2.29}
$$

$$
W_0 = -\int_{\partial\Omega} \left(\widehat{M}_{\tau,s} w_0 + \widehat{M}_n w_{0,n} \right). \tag{2.30}
$$

Theorem 2.2 (Size estimates of $|E|$)**.** *Under the above hypotheses, and assuming in addition that $\partial\Omega$ is of class $C^{4,1}$ with constants ρ_0, M_0, if* (2.21) *holds then*

$$
\frac{1}{\delta - 1} C_1^+ \rho_0^2 \frac{W_0 - W}{W_0} \le |E| \le \left(\frac{\delta}{\eta}\right)^{\frac{1}{p}} C_2^+ \rho_0^2 \left(\frac{W_0 - W}{W_0}\right)^{\frac{1}{p}}. \tag{2.31}
$$

Conversely, if (2.22) *holds then*

$$
\frac{\delta}{1 - \delta} C_1^- \rho_0^2 \frac{W - W_0}{W_0} \le |E| \le \left(\frac{1}{\eta}\right)^{\frac{1}{p}} C_2^- \rho_0^2 \left(\frac{W - W_0}{W_0}\right)^{\frac{1}{p}}, \tag{2.32}
$$

where C_1^+, C_1^- depend only on M_0, M_1, M_2, d_0, γ, whereas C_2^+, C_2^-, $p > 1$ only depend on the same quantities and also on δ_0 and on

$$
\frac{\|\widehat{M}\|_{L^2(\partial\Omega,\mathbb{R}^2)}}{\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}. \tag{2.33}
$$

Sketch of the proof. As a first step, the variational formulation for the reference and defective plate problems, both for hard and soft inclusions, leads to the following double inequality

$$
C_l \int_E |D^2 w_0|^2 \le |W - W_0| \le C_u \int_E |D^2 w_0|^2, \tag{2.34}
$$

where C_l, C_u are positive constants only depending on η, δ, h and γ. The above inequalities mean that the strain energy of the reference plate, stored in the set E, is comparable with the work gap $|W - W_0|$.

The lower bounds in (2.31), (2.32) follow from the right hand side of (2.34) and from regularity estimates for the solution to the unperturbed plate problem (2.27).

The derivation of the upper bounds for $|E|$ requires a lower estimate of $\int_E |D^2 w_0|^2$ in (2.34). It is exactly at this point that the doubling inequality (2.18) plays a crucial role. In fact, arguing similarly to [Mo-Ro-Ve09, Section 4], one can derive from (2.18) that there exists a constant $\widetilde{\theta} \in (0,1)$, depending on γ and M_2 only, such that for every $\overline{r} > 0$ and $x_0 \in \Omega_{\overline{r}\rho_0}$ we have

$$\int_{B_{2r}(x_0)} |D^2 w_0|^2 \leq K \int_{B_r(x_0)} |D^2 w_0|^2, \qquad \forall\, r,\ 0 < r \leq \frac{\widetilde{\theta}}{2}\overline{r}\rho_0, \tag{2.35}$$

where K only depends on M_0, M_1, δ_0, M_2, γ, $\overline{r}$ and $\|\widehat{M}\|_{L^2(\partial\Omega)}/\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}$.

By the general theory of Garofalo and Lin [Ga-Li86], [Ga-Li87], inequality (2.35) ensures that $|D^2 w_0|^2$ is an A_p-weight for some $p > 1$ only depending on M_0, M_1, δ_0, M_2, γ and $\|\widehat{M}\|_{L^2(\partial\Omega)}/\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}$, that is $|D^2 w_0|^{-\frac{2}{p-1}}$ is locally integrable. Such an integrability property enables, through Hölder's inequality, to get the upper bounds in (2.31), (2.32), see for details [Mo-Ro-Ve09]. $\square$

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we will first prove a doubling type inequality (Proposition 3.1) where (1.4) will be replaced by the slightly stronger assumption

$$g_1^{ij}(0) = g_2^{ij}(0) = \delta^{ij}. \tag{3.36}$$

Afterwards, we derive Theorem 1.1 by a suitable change of variables.

Proposition 3.1. *Let $P(x,D)$ be as in* (1.2) *satisfying* (1.3), (1.5), (1.6) *and* (3.36). *Let $u \in H^4(B_1)$ be such that*

$$|P(x,D)u| \leq M \sum_{|\alpha|\leq 3} |D^\alpha u| \qquad \text{in } \overline{B}_1, \tag{3.37}$$

where M is a given positive constant. There exist constants $R_1 \in (0,1)$, $\theta_1 \in (0,1/2)$ and $K_1 > 0$ such that

$$\int_{B_{2r}} |u|^2 \leq K_1 \int_{B_r} |u|^2 \qquad \forall\, r, 0 < r < \theta_1, \tag{3.38}$$

where R_1 depends on λ, M, C_0 only and θ_1 and K_1 depend on λ, M, C_0 and

$$\mathcal{F}^{(1)}_{\mathrm{loc}} = \frac{\|u\|_{L^2(B_{R_1^2})}}{\|u\|_{L^2(B_{R_1^4})}} \tag{3.39}$$

only.

We shall prove Proposition 3.1 by a bootstrapping argument based on three spheres inequalities developed in [Li-Nak-Wa] and [Li-Nag-Wa]. It may be possible to derive a doubling inequality like (3.38) using a Carleman estimate with more sophisticated weight functions, which has been obtained in [Co-Ko]. However, we would like to emphasize again that our ultimate goal is to apply doubling inequalities to inverse problems described above. Therefore, it is crucial to know precisely how the constant K_1 of (3.38) depends on u. The quantity $\mathcal{F}^{(1)}_{\mathrm{loc}}$ of (3.39) is important in the investigation of inverse problems. In the bootstrapping argument, we use a simple Carleman estimate (see (3.42)) and the derivation of doubling inequalities is rather elementary.

We begin by stating two results that will be used later. The first ones are two Caccioppoli type inequalities which are simple consequences of [Ho, Theorem 17.1.13]. Let $u \in H^4(B_1)$ such that

$$|P(x,D)u| \leq L \sum_{|\alpha|\leq 3} |D^\alpha u|, \qquad \text{in } B_1,$$

where $P(x,\partial)$ is defined in (1.2)–(1.4), then

$$\sum_{|\alpha|\leq 4} \int_{a_1 r<|x|<a_2 r} ||x|^\alpha D^\alpha u|^2 \leq C' \int_{a_3 r<|x|<a_4 r} |u|^2, \tag{3.40}$$

with $0 < a_3 < a_1 < a_2 < a_4 < 1$, $r < 1$, where $C' > 1$ depends on L, λ, C_0, $a_1 - a_3$ and $a_4 - a_2$ only. Let us stress here that the smaller are the differences $a_1 - a_3$ and $a_4 - a_2$, the larger is the constant C';

$$\sum_{|\alpha|\leq 3} \int_{B_{r/2}} |D^\alpha u|^2 \leq C'' \left(\frac{2}{r}\right)^{2|\alpha|} \int_{B_r} u^2, \tag{3.41}$$

where $C'' > 1$ depends on L, λ and C_0 only.

Moreover we recall the following Carlemann type estimate derived in [LeB] and [Li-Nag-Wa]. For any $v \in C_0^\infty(B_1 \setminus \{0\})$ and $m = j + 1/2 \geq j_* + 1/2 =: m_*$, $j \in \mathbb{N}$, there exits a constant $C_1 > 1$ such that

$$\sum_{|\alpha|\leq 4} m^{4-2|\alpha|} \int |x|^{-2m+2|\alpha|-n} |D^\alpha v|^2 \leq C_1 \int |x|^{-2m+8-n} |\Delta^2 v|^2. \tag{3.42}$$

Let R_0 and $\bar{R}$, $R_0, \bar{R} \in (0,1)$, be numbers that will be chosen later and assume $0 < \bar{R} \leq R_0$. Setting $r_4 = \frac{R_0(R_0+1)}{4}$ (which implies $r_4 < R_0/2$) and picking an arbitrary δ such that

$$0 < \delta \leq \frac{1}{4} R_0^2 \bar{R}, \tag{3.43}$$

we define a function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned}
&0 \leq \chi \leq 1, && \chi = 1 \text{ on } B_{r_4\bar{R}} \setminus B_{\delta/2},\\
&\chi = 0, && \text{for } |x| \leq \tfrac{\delta}{3} \text{ and } |x| \geq 2r_4\bar{R},\\
&|D^\alpha \chi(x)| \leq \tfrac{C_3}{\delta^{|\alpha|}}, && \text{for } \tfrac{\delta}{3} \leq |x| \leq \tfrac{\delta}{2},\\
&|D^\alpha \chi(x)| \leq \tfrac{C_3}{(r_4\bar{R})^{|\alpha|}}, && \text{for } r_4\bar{R} \leq |x| \leq 2r_4\bar{R},
\end{aligned}$$

where C_3 is an absolute constant. After a possible regularization, we insert in (3.42) the function $v = u\chi$, where $u \in H^4(B_1)$ satisfies (1.1). We have

(3.44)
$$\sum_{|\alpha|\leq 4} m^{4-2|\alpha|} \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 \leq C_1 \int |x|^{-2m+8-n} |\Delta^2(\chi u)|^2,$$

for every $m \geq m_*$. Now splitting the integral on the right hand side

$$\int |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 = \int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2$$

$$+ \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2 u|^2 + \int_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2$$

and using the following chain of inequalities

$$\begin{aligned} & |\Delta^2 u| \leq |P(0,D)u - P(x,D)u| + |P(x,D)u| \\ \leq \quad & cC_0^2|x| \sum_{|\alpha|=4} |D^\alpha u| + (cC_0^2 + M) \sum_{|\alpha|\leq 3} |D^\alpha u| \\ \leq \quad & cC_0^2 r_4 \sum_{|\alpha|=4} |D^\alpha u| + (cC_0^2 + M) \sum_{|\alpha|\leq 3} |D^\alpha u|, \end{aligned}$$

which follows, for $|x| \leq r_4\bar{R}$, by Lipschitz continuity of coefficients and by (1.7), we obtain

$$\int |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 \leq \int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2$$

$$+C_4 \left[(r_4\tilde{R})^2 \int\limits_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|=4} |D^\alpha u|^2 + \int\limits_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|\leq 3} |D^\alpha u|^2 \right]$$

$$+ \int_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2,$$

where $C_4 = c(C_0^2 + M^2)$. Now by properties of χ we have

$$\int\limits_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 \leq cC_3^2 \sum_{|\alpha|\leq 4} \delta^{2(|\alpha|-4)} \int\limits_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|D^\alpha u|^2 := I^{(\delta)}$$

and

$$\begin{aligned} & \int\limits_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 \\ \leq \quad & cC_3 \sum_{|\alpha|\leq 4} (r_4\bar{R})^{2(|\alpha|-4)} \int\limits_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|D^\alpha u|^2 =: I^{(\bar{R})}. \end{aligned}$$

Inserting everything in (3.44) we get

$$
\begin{aligned}
(3.45)\qquad & \sum_{|\alpha|\le 4} m^{4-2|\alpha|} \int\limits_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+2|\alpha|-n}|D^\alpha u|^2 \\
\le\; & I^{(\delta)} + I^{(\bar{R})} + C_4(r_4\bar{R})^2 \int\limits_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|=4} |D^\alpha u|^2 \\
& + C_4 \int\limits_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|\le 3} |D^\alpha u|^2, \qquad \forall\, m \ge m_*,
\end{aligned}
$$

that we write in the form

$$
S_1 \le I^{(\delta)} + I^{(R)}, \qquad \forall\, m \ge m_*, \tag{3.46}
$$

where by S_1 we denote the left hand side integral of (3.45) minus the two remaining integrals of right hand side. Our aim now is to estimate the five orders of derivatives of of S_1, starting from the fourth order one (i.e. $\alpha = 4$), which decays to zero faster than the others, up to the zero order integral ($\alpha = 0$), which likewise the first order term ($\alpha = 1$), can be estimate easily taking $\bar{R}$ small enough and m large enough.

Let $\mu \ge 1$ (to be chosen later) and set $\bar{R} = \frac{1}{\mu m^2}$, we have

$$
\begin{aligned}
& \sum_{|\alpha|=4} \left(\frac{1}{m^4} - C_4(r_4\bar{R})^2 \right) \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n}|D^\alpha u|^2 \\
= \; & \sum_{|\alpha|=4} \frac{1}{m^4} \left(1 - \frac{C_4 r_4^2}{\mu^2} \right) \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n}|D^\alpha u|^2.
\end{aligned}
$$

In order to have this quantity greater or equal to zero it suffices taking

$$
\mu \ge \sqrt{C_4}\, r_4. \tag{3.47}
$$

Let us consider now the third order term. By (3.42) and (3.47), we have by the choice of $\bar{R}$,

$$
\begin{aligned}
& \sum_{|\alpha|=3} \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} \left(\frac{1}{m^2} - C_4|x|^2 \right) |x|^{-2m+6-n}|D^\alpha u|^2 \\
\ge \; & \sum_{|\alpha|=3} \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} \left(\frac{1}{m^2} - C_4(r_4\bar{R})^2 \right) |x|^{-2m+6-n}|D^\alpha u|^2 \ge 0.
\end{aligned}
$$

To estimate the second order term we can proceed as follows.

$$
\begin{aligned}
& \sum_{|\alpha|=2} \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} (1 - C_4|x|^4)|x|^{-2m+4-n}|D^\alpha u|^2 \\
\ge \; & \sum_{|\alpha|=2} \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} \left(1 - \frac{C_4}{m^8} \right) |x|^{-2m+4-n}|D^\alpha u|^2.
\end{aligned}
$$

To have this term positive, we take $m \geq C_4^{1/8}$. First order term can be estimated similarly, whereas for the zero order term we have

$$\int_{\frac{\delta}{2}<|x|<r_4\bar{R}} (m^4 - C_4|x|^8)|x|^{-2m-n}|u|^2 \geq \frac{m^4}{2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m-n}|u|^2,$$

as long as $m \geq (2C_4)^{1/4}$. Summarizing, picking $m_1 := \max\{(2C_4)^{n/2}+1, m_*\}$, if $\mu \geq \sqrt{C_4}r_4$,

$$S_1 \geq \frac{m^4}{2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m-n}|u|^2,$$

which recalling (3.46) leads to

$$\frac{m^4}{2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m-n}|u|^2 \leq I^{(\delta)} + I^{(\bar{R})}, \qquad \forall\, m \geq m_1, \tag{3.48}$$

with

$$\bar{R} = \frac{1}{\mu m^2} \qquad \text{and} \qquad \mu = \sqrt{C_4}. \tag{3.49}$$

We now use (3.40) to get rid of terms of zero derivatives in $I^{(\delta)}$ and $I^{(\bar{R})}$ appearing in the inequality (3.48). Let us consider $I^{(\delta)}$ first, we have

$$\begin{aligned} I^{(\delta)} &= cC_3^2 \sum_{|\alpha|\leq 4} \delta^{2(|\alpha|-4)} \int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n-2|\alpha|} \left||x|^{|\alpha|}|D^\alpha u|\right|^2 \\ &\leq c'C_3^2C_2\left(\frac{\delta}{3}\right)^{-2m-n}\int_{\frac{\delta}{6}<|x|<\delta} |u|^2. \end{aligned} \tag{3.50}$$

Similarly, choosing in (3.40) $a_1 = 1$, $a_2 = 2$, $a_3 = 1/2$ and $a_4 = \frac{4}{R_0+1}$ with the further restriction that $R_0 \leq 1/2$, entailing $a_4 - a_2 \geq 2/3$,

$$I^{(\bar{R})} \leq c''C_2C_3^2(r_4\bar{R})^{-2m-n}\int_{\frac{r_4\bar{R}}{2}<|x|<\bar{R}R_0} |u|^2. \tag{3.51}$$

Inserting (3.50) and (3.51) into (3.48) we get

$$\begin{aligned} & m^4 \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m-n}|u|^2 \\ \leq\ & C_5\left(\frac{\delta}{3}\right)^{-2m-n}\int_{|x|<\delta} |u|^2 + C_5(r_4\bar{R})^{-2m-n}\int_{|x|<R_0\bar{R}} |u|^2, \end{aligned} \tag{3.52}$$

for every $m \geq m_1$, $\bar{R}$ and μ satisfying (3.49) and $R_0 \leq 1/2$, where $C_5 = \max\{C_2, c'C_3^2, c''C_2, C_3^2\}$. Now, observing that for $R_0 \leq 1/3$ we get $R_0^2 \leq \frac{R_0(R_0+1)}{4} = r_4$, we have that the left hand side of (3.52) can be trivially bounded from below by an integral over the set $\{\frac{\delta}{2} < |x| < R_0^2\bar{R}\}$, which yields to the following inequality.

$$\begin{aligned} & m^4 \int_{\frac{\delta}{2}<|x|<2\delta} |x|^{-2m-n}|u|^2 + m^4 \int_{2\delta<|x|<R_0\bar{R}} R_0^2|x|^{-2m-n}|u|^2 \\ \leq\ & C_5\left(\frac{\delta}{3}\right)^{-2m-n}\int_{|x|<\delta} |u|^2 + C_5(r_4\bar{R})^{-2m-n}\int_{|x|<R_0\bar{R}} |u|^2, \end{aligned}$$

for every $m \geq m_1$. Let us consider now the left hand side of the above inequality. Observing that $|x| \leq 2\delta$ in the first integral and $|x| \leq R_0^2\bar{R}$ in the second one and adding to both sides the term $m^4(2\delta)^{-2m-n}\int_{|x|<\delta/2}|u|^2$, we have

$$\begin{aligned}(3.53)\qquad & m^4(2\delta)^{-2m-n}\int_{|x|<2\delta}|u|^2 + m^4(R_0^2\bar{R})^{-2m-n}\int_{2\delta<|x|<R_0^2\bar{R}}|u|^2\\ \leq & \left[\left(C_5\frac{\delta}{3}\right)^{-2m-n} + m^4(2\delta)^{-2m-n}\right]\int_{|x|<\delta}|u|^2 + C_5(r_4\bar{R})^{-2m-n}\int_{|x|<R_0\bar{R}}|u|^2,\end{aligned}$$

for every $m \geq m_1$, with $R_0 \leq 1/3$, and R, μ as in (3.49). Now by (3.43) we have $(2\delta)^{-2m-n} \geq \left(\frac{R_0^2\bar{R}}{2}\right)^{-2m-n}$, which allows us to estimate from below the left hand side (LHS) of (3.53) by

$$\mathrm{LHS} \geq m^4\frac{1}{2}(2\delta)^{-2m-n}\int_{|x|<2\delta}|u|^2 + m^4(R_0^2\bar{R})^{-2m-n}\int_{|x|<R_0^2\bar{R}}|u|^2,$$

whereas to estimate from above the second integral of the right hand side let us notice that

$$C_5 m^{-4}\left(\frac{R_0^2}{r_4}\right)^{2m+n} \leq (4R_0)^{2m} \leq \mathrm{e}^{-2m},$$

with $R_0 \leq \frac{1}{4\mathrm{e}}$ and $m \geq m_2 := \max\{m_1, \left[\sqrt[4]{C_5}\right]+1\}$. Now inserting all previous inequalities in (3.53) we get

$$\begin{aligned}& \frac{1}{2}m^4(2\delta)^{-2m-n}\int_{|x|<2\delta}|u|^2 + m^4(R_0^2\bar{R})^{-2m-n}\int_{|x|<R_0^2\bar{R}}|u|^2\\ \leq \quad & \left(C_5\left(\frac{\delta}{3}\right)^{-2m-n} + m^4(2\delta^{-2m-n})\right)\int_{|x|<\delta}|u|^2\\ (3.54)\qquad & +m^4(R_0^2\bar{R})^{-2m-n}\mathrm{e}^{-2m}\int_{|x|<R_0\bar{R}}|u|^2.\end{aligned}$$

We use now (5.36) of [Mo-Ro-Ve11, Theorem 5.3]. Let us point out that, according to their notations, by (1.4) we have $\nu_* = \nu^* = \mu_* = \mu^* = 1$, which means that for every $\beta > 0$, there exists $s_1 \in (0,1)$ and $C \geq 1$, depending on λ, C_0, M and β such that for every $\rho_1 \in (0, s_1)$ and r, ρ such that $r < \rho < \rho_1\lambda^2/2$ we have

$$\begin{aligned}(3.55)\sum_{|\alpha|\leq 3}\rho^{2|\alpha|}\int_{B_\rho}|D^\alpha u|^2 \leq & C\max\{1, \rho^{-(5\beta-2)}\}\mathrm{e}^{C\left((\lambda^{-1}\rho)^{-\beta}-\left(\frac{\rho_1\lambda}{2}\right)^{-\beta}\right)}\\ \times & \left(r^{5\beta-2}\sum_{|\alpha|\leq 3}r^{2|\alpha|}\int_{B_r}|D^\alpha u|^2\right)^{\theta_0}\left(\rho_1^{5\beta-2}\sum_{|\alpha|\leq 3}\rho_1^{2|\alpha|}\int_{B_{\rho_1}}|D^\alpha u|^2\right)^{1-\theta_0},\end{aligned}$$

where

$$\theta_0 = \frac{(\lambda^{-1}\rho)^{-\beta} - \left(\frac{\lambda\rho_1}{2}\right)^{-\beta}}{\left(\frac{\lambda r}{2}\right)^{-\beta} - \left(\frac{\lambda\rho_1}{2}\right)^{-\beta}}.$$

Now, assuming $r < \rho < \frac{\rho_1\lambda^2}{4}$, using the Caccioppoli inequality (3.41) and taking $\beta \leq 2/5$, we can write (3.55) as follows. There exists a constant C_6 depending on λ, C_0, M and β only such that if $r_1 < r_2 < \frac{r_3}{C_6}$, with $r_3 \leq \frac{1}{C_6}$, then

$$\int_{B_{r_2}} u^2 \leq \mathrm{e}^{C_6 r_2^{-\beta}} \left(\int_{B_{r_1}} u^2\right)^{\theta} \left(\int_{B_{r_3}} u^2\right)^{1-\theta} \tag{3.56}$$

where

$$\theta = \frac{(\lambda^{-1}r_2)^{-\beta} - \left(\frac{\lambda r_3}{4}\right)^{-\beta}}{\left(\frac{\lambda r_1}{4}\right)^{-\beta} - \left(\frac{\lambda r_3}{4}\right)^{-\beta}}. \tag{3.57}$$

Proposition 3.2. *Let* $R_j = \frac{1}{\mu(j+1/2)^2}$. $j \in \mathbb{N}$. *We have*

$$R_{j+1} < R_j < 2R_{j+1}, \qquad \textit{for } j \geq 3. \tag{3.58}$$

Furthermore, for every $R_{j+1} < R < R_j$, $j \geq 3$, *we have*

$$\mathrm{e}^{-\frac{2}{\sqrt{\mu R_j}}} \int_{|x|<R_0 R_j} u^2 \leq \mathrm{e}^{-\frac{\sqrt{2}}{\sqrt{\mu R}}} \int_{|x|<R} u^2. \tag{3.59}$$

Proof. Inequality (3.58) is easy to check. Let us prove (3.59). We first observe that, since $R_0 \leq \frac{1}{4\mathrm{e}}$ we have

$$R_0 R_j \leq \frac{1}{4\mathrm{e}} R_j < \frac{1}{2\mathrm{e}} R_{j+1},$$

thus if $R_{j+1} < R < R_j$, we get the thesis. □

We can write (3.54) with $\bar{R} = R_j$ and, recalling that $m = \frac{1}{\sqrt{\mu R_j}}$, by replacing the term e^{-2m} with $\mathrm{e}^{-\frac{2}{\sqrt{\mu R_j}}}$. By (3.59) for every $R \in (R_{j+1}, R_j]$, with $j \geq j_3$, where $j_3 = \max\{j_2, 3\}$, we have

$$\begin{aligned}
&\frac{1}{2} m^4 (2\delta)^{-2m-n} \int_{|x|<2\delta} u^2 + m^4 (R_0^2 R_j)^{-2m-n} \int_{|x|<R_0^2 R} u^2 \\
\leq\ & \left(C_5 \left(\frac{\delta}{3}\right)^{-2m-n} + m^4 (2\delta)^{-2m-n} \right) \int_{|x|<\delta} u^2 \\
&+ m^4 (R_0^2 R_j)^{-2m-n} \mathrm{e}^{-\frac{\sqrt{2}}{\sqrt{\mu R}}} \int_{|x|<R} u^2,
\end{aligned} \tag{3.60}$$

for every $m \geq m_3 := j_3 + 1/2$. Now by (3.60), if there exist $s \in \mathbb{N}$ and $\hat{j} \geq j_3$ such that

$$R_{\hat{j}+1} \leq R_0^{2s} \leq R_{\hat{j}}, \tag{3.61}$$

then, setting $\hat{m} = \hat{j} + \frac{1}{2}$, we have

$$\begin{aligned}
&\frac{1}{2} \hat{m}^4 (2\delta)^{-2\hat{m}-n} \int_{|x|<2\delta} u^2 + \hat{m}^4 (R_0^2 R_{\hat{j}})^{-2\hat{m}-n} \int_{|x|<R_0^{2s+2}} u^2 \\
\leq\ & \left(C_5 \left(\frac{\delta}{3}\right)^{-2\hat{m}-n} + \hat{m}^4 (2\delta)^{-2\hat{m}-n} \right) \int_{|x|<\delta} u^2 \\
&+ \hat{m}^4 (R_0^2 R_{\hat{j}})^{-2\hat{m}-n} \mathrm{e}^{-\frac{\sqrt{2}}{\sqrt{\mu R_0^{2s}}}} \int_{|x|<R_0^{2s}} u^2.
\end{aligned} \tag{3.62}$$

Proposition 3.3. *Let* $\bar{R}_0 = \frac{\lambda^8}{2^{18} C_6^5 (2\mu)^{5/2}}$ *and*

$$s_0 = 1 + \left\lfloor \max\{\log[(\lambda/2)^{4/5}\sqrt{2\mu}\log(\mathrm{e}N)], 2\} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ *stands for the integer part, then*

$$\mathrm{e}^{-\frac{\sqrt{2}}{(\mu R_0^{2k})^{1/2}}} \left(\frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2} \right) \leq 1, \tag{3.63}$$

for every $k \geq s_0$ *and* $R_0 \leq \bar{R}_0$.

Proof. We begin by using three sphere inequality (3.56) with $r_1 = R_0^{2k+2}$, $r_2 = R_0^{2k}$, $r_3 = R_0^{2k-2}$, $k \in \mathbb{N}$, $k > 1$. Thus we require $R_0 \leq \min\left\{\frac{1}{4\mathrm{e}}, \frac{1}{\sqrt{C_6}}\right\}$. Our goal is to estimate the exponent θ. By (3.57) we have

$$\frac{1-\theta}{\theta} = \frac{(R_0^4)^{-\beta} - (4\lambda^{-2}R_0^2)^{-\beta}}{(4\lambda^{-2}R_0^2)^{-\beta} - 1}.$$

Setting

$$\alpha_{k+1} = \frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2},$$

by (3.56) we have

$$\alpha_{k+1} \leq \left(\mathrm{e}^{C_6 (R_0^{2k})^{-\beta}}\right)^{\frac{1}{\theta}} \alpha_k^{\frac{1-\theta}{\theta}}. \tag{3.64}$$

Now if $R_0 \leq \left[\frac{1}{4\lambda^{-2}} \left(\frac{1}{2}\right)^{1/\beta}\right]^{1/2}$, we have

$$\frac{1-\theta}{\theta} \leq 2 \left(\frac{\lambda^2 R_0^2}{4}\right)^{-\beta} =: \omega.$$

Setting $E_k = \mathrm{e}^{C_6 (R_0^{2k})^{-\beta}}$, (3.64) can be written

$$\alpha_{k+1} \leq E_k^{\omega} \alpha_k^{\omega}, \qquad k \geq 2,$$

that, iterating, leads to

$$\alpha_{k+1} \leq G_k \alpha_2^{\omega^{k-1}}, \quad k \geq 2, \tag{3.65}$$

where

$$G_k := \exp\left(C_6 2^{k-1} k \left(\frac{R_0 \lambda}{2}\right)^{-2\beta(k+1)}\right) \alpha_2^{\omega^{k-1}}, \quad k \geq 2.$$

Defining

$$N := N(R_0^4) := \frac{\int_{|x|<R_0^2} u^2}{\int_{|x|<R_0^4} u^2},$$

inequality (3.65) can be written as

$$\frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2} \leq G_k N^{\omega^{k-1}}, \quad k \geq 2.$$

Now by the previous inequality we have

$$\mathrm{e}^{-\frac{\sqrt{2}}{\sqrt{\mu R_0^{2k}}}}\left(\frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2}\right) \le G_k \mathrm{e}^{-\frac{1}{\sqrt{2\mu R_0^{2k}}}} N^{\omega^{k-1}} \mathrm{e}^{-\frac{1}{\sqrt{2\mu R_0^{2k}}}}. \tag{3.66}$$

Now taking $\beta = \frac{2}{5} < \frac{1}{2}$ and requiring $4^{4/5}\lambda R_0 \le 1$ and $4^{4/5}\lambda R_0 \le \frac{1}{2}(2\mu)^{-1/2}C_6^{-1}$, it is simple to check that

$$G_k \mathrm{e}^{-(2\mu R_0^{2k})^{-1/2}} \le 1, \qquad \forall\, k \ge 2.$$

Let us consider now the other part of the right hand side of (3.66). Recalling that, by our choice of $\tilde{R}_0$

$$\frac{2R_0^{1/2}}{\left(\frac{\lambda}{2}\right)^{4/5}} \le \frac{1}{\mathrm{e}},$$

taking

$$k \ge \max\left\{\log\left[\left(\frac{\lambda}{2}\right)^{4/5}\sqrt{2\mu}\,\log(\mathrm{e}N)\right], 2\right\},$$

we easily get

$$\mathrm{e}^{-\frac{1}{(2\mu R_0^{2k})^{1/2}}} N^{\omega^{k-1}} \le 1,$$

which gives us the thesis. □

To complete the proof of the theorem, we have to check (3.61), that is we have to determine $\hat{j} \ge j_3$ and $s \ge s_0$, where s_0 has been defined in Proposition 3.3, such that

$$R_{\hat{j}+1} < \bar{R}_0^{2s} \le R_{\hat{j}}. \tag{3.67}$$

Now, let

$$s_1 = 2 + \left\lfloor \max\left\{s_0, \frac{\log\sqrt{\mu}(j_3 + 1/2)}{|\log \bar{R}_0|}\right\}\right\rfloor,$$

we have

$$\bar{R}_0^{2s_1} \le \frac{1}{\mu(j_3+1/2)^2}.$$

Let

$$J = \left\{ j \in \mathbb{N},\, j \ge j_3 \,:\, \bar{R}_0^{2s_1} \le \frac{1}{\mu(j+1/2)^2}\right\}.$$

Clearly $J \neq \emptyset$. Setting $j_4 = \max J$, (3.67) holds for $s = s_1$ and $\hat{j} = j_4$. We can now conclude the proof of Theorem 3.1. Namely, defining $\hat{m}_4 = j_4 + 1/2$ and $k = s_1$, by (3.61), (3.62) and (3.63) with $R_0 = \bar{R}_0$ we obtain

$$\frac{1}{2}\hat{m}_4^4(2\delta)^{-2\hat{m}_4-n}\int_{|x|<2\delta} u^2 \le \left[C_5\left(\frac{\delta}{3}\right)^{-2\hat{m}_4-n} + \hat{m}_4^4(2\delta)^{-2\hat{m}_4-n}\right]\int_{|x|<\delta} u^2,$$

for every δ such that $0 < \delta \le \frac{1}{4}\bar{R}_0^2 R_{j_4}$. Now (3.38) follows dividing the previous inequality by $\frac{1}{2}\hat{m}_4^4(2\delta)^{-2\hat{m}_4-n}$, with

$$R_1 = \bar{R}_0, \quad K_1 = \frac{6^n(2C_5+1)6^{2\hat{m}_4}}{\hat{m}_4^4} \quad \text{and} \quad \theta_1 = \frac{1}{4}\bar{R}_0^2 R_{j_4}.$$

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $J = \sqrt{g_1^{-1}(0)}$, where

$$g_1^{-1}(0) = \{g_1^{ij}(0)\}_{i,j=1}^n = \{g_2^{ij}(0)\}_{i,j=1}^n = g_2^{-1}(0)$$

and

$$\psi : \mathbb{R}^n \to \mathbb{R}^n, \qquad \psi(x) = Jx.$$

Setting $\tilde{g}_k^{-1}$ such that $\tilde{g}_k^{-1}(\psi(x)) = Jg_k^{-1}(x)J^T$, $k = 1, 2$, we have $\tilde{g}_1^{-1}(0) = \tilde{g}_2^{-1}(0) = Id$. Let $u \in H^4(B_1)$ be a solution to (3.37). We define

$$\begin{aligned}
&U(y) := u(\psi^{-1}(y)),\\
&\tilde{P}(y, D) := \tilde{P}_2(y, D)\tilde{P}_1(y, D),\\
&\tilde{P}_k(y, D) := \textstyle\sum_{i,j=1}^n \tilde{g}_k^{ij}(y)D_{ij}^2, \qquad k = 1, 2
\end{aligned}$$

and for any $r > 0$,

$$\mathcal{E}_r := \{x \in \mathbb{R}^n : g_1^{-1}(0)x \cdot x < r^2\}.$$

We have that $\mathcal{E}_r = \psi^{-1}(B_r)$ (see [Al-Ro-Ro-Ve, page 16]),

$$B_{\sqrt{\lambda}r} \subset \mathcal{E}_r \subset B_{r/\sqrt{\lambda}} \qquad \text{for every } r > 0 \tag{3.68}$$

and (see [Mo-Ro-Ve11, page 1523])

$$|\tilde{P}(y, D)U(y)| \leq cM \sum_{|\alpha|\leq 3} |D^\alpha U|, \qquad \text{in } \mathcal{E}_1 \supset B_{\sqrt{\lambda}},$$

where c depends on λ only. By Theorem 3.1, performing the variable change $y \to \sqrt{\lambda}y$, we have that there exists constants $\tilde{R}_1$ and $\tilde{\theta}_1$, $\tilde{R}_1 \in (0, 1)$, $\tilde{\theta}_1 \in (0, 1/2)$, such that

$$\int_{B_{2^l r}} |U(y)|^2 dy \leq \tilde{K}_1^l \int_{B_r} |U(y)|^2 dy, \qquad \forall\, r,\ 0 < r < \tilde{\theta}_1/2^l, \tag{3.69}$$

where l will be chosen later on, $\tilde{R}_1$ depends on λ, M and C_0 only and $\tilde{\theta}_1$ and $\tilde{K}_1$ depend on λ, M, C_0 and

$$\tilde{\mathcal{F}}_{loc}^{(1)} = \frac{\|U\|_{L^2(B_{\tilde{R}_1})}}{\|U\|_{L^2(B_{\tilde{R}_1^2})}} \tag{3.70}$$

only. By (3.68) and (3.69) we have

$$\begin{aligned}
\int_{B_{2^l r}} |u|^2 dx &= \int_{B_{\frac{2^l r}{\sqrt{\lambda}}}} |u(\psi^{-1}(y))|^2 \left|\det \frac{\partial \psi^{-1}}{\partial y}(y)\right| dy \\
&\leq \lambda^{-1} \int_{B_{2^l r/\sqrt{\lambda}}} |U(y)|^2 dy \leq \lambda^{-1}\tilde{K}_1^l \int_{B_{r/\sqrt{\lambda}}} |U(y)|^2 dy \\
&= \lambda^{-1}\tilde{K}_1^l \int_{\mathcal{E}_{r/\sqrt{\lambda}}} |U(\psi(x))|^2 \left|\det \frac{\partial \psi}{\partial x}(x)\right| dx \leq \lambda^{-2}\tilde{K}_1^l \int_{B_{r/\lambda}} |u(x)|^2 dx.
\end{aligned} \tag{3.71}$$

Let $l = [\log_2 \lambda^{-1}] + 2$, which implies $2^l r \geq 2r\lambda^{-1}$ and by (3.71)

$$\int_{B_{2r/\lambda}} |u|^2 dx \leq \lambda^{-2} (\tilde{K}_1)^{[\log_2 \lambda^{-1}]+2} \int_{B_{r/\lambda}} |u|^2 dx,$$

that can be written as

$$\int_{B_{2s}} |u|^2 dx \leq \lambda^{-2} (\tilde{K}_1)^{[\log_2 \lambda^{-1}]+2} \int_{B_s} |u|^2 dx, \qquad \text{for } 0 < s < \frac{\tilde{\theta}_1}{\sqrt{\lambda} 2^{[\log_2 \lambda^{-1}]+2}}.$$

Now by the inequalities

$$\int_{B_{\tilde{R}_1}} |U(y)|^2 dy \leq \lambda^{-1} \int_{B_{\tilde{R}_1/\sqrt{\lambda}}} |u(x)|^2 dx,$$

$$\int_{B_{\tilde{R}_1^2}} |U(y)|^2 dy \geq \lambda \int_{B_{(\tilde{R}_1\sqrt{\lambda})^2}} |u(x)|^2 dx,$$

we get the thesis with $R = \tilde{R}_1/\sqrt{\lambda}$. $\square$

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