Quantitative uniqueness estimates for the general second order elliptic equations

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Abstract

In this paper we study quantitative uniqueness estimates of solutions to general second order elliptic equations with magnetic and electric potentials. We derive lower bounds of decay rate at infinity for any nontrivial solution under some general assumptions. The lower bounds depend on asymptotic behaviors of magnetic and electric potentials. The proof is carried out by the Carleman method and the bootstrapping arguments.

1 Introduction

In this paper we study the asymptotic behaviors of solutions to the general second order elliptic equation

\begin{equation}
PV + W(x) \cdot \nabla v + V(x)v + q(x)v = 0 \quad \text{in} \quad \Omega := \mathbb{R}^n \setminus B,
\end{equation}

where $B$ is a bounded set in $\Omega$. Here $P(x, D) = \sum_{jk} a_{jk}(x) \partial_j \partial_k$ is uniformly elliptic, i.e., for some $\lambda_0 > 0$

\begin{equation}
\lambda_0 |\xi|^2 \leq \sum_{jk} a_{jk}(x) \xi_j \xi_k \leq \lambda_0^{-1} |\xi|^2 \quad \forall \ x \in \Omega, \ \xi \in \mathbb{R}^n.
\end{equation}
and $a_{jk}(x)$ is Lipschitz continuous. We are interested in deriving lower bounds of the decay rate for any nontrivial solution $v$ to (1.1) under certain a priori assumptions. This kind of problem was originally posed by Landis in the 60’s [10]. He conjectured that if $v$ is a bounded solution of

$$\Delta v + q(x)v = 0 \quad \text{in} \quad \mathbb{R}^n$$

with $\|q\|_{L^\infty} \leq 1$ and $|v(x)| \leq C \exp(-C|x|^{1+})$ for some constant $C$, then $v$ is identically zero. This conjecture was disproved by Meshkov [13] who constructed a $q(x)$ and a nontrivial $v(x)$ with $|v(x)| \leq C \exp(-k|x|^{4/3})$ satisfying (1.3). He also proved that if $|v(x)| \leq C_k \exp(-k|x|^{4/3})$ for all $k > 0$ then $v \equiv 0$. Note that $q(x)$ and $v(x)$ constructed by Meshkov are complex valued.

In 2005, Bourgain and Kenig [2] derived a quantitative version of Meshkov’s result in their resolution of Anderson localization for the Bernoulli model. Precisely, they showed that if $v$ is a bounded solution of $\Delta v + q(x)v = 0$ in $\mathbb{R}^n$ satisfying $\|q\|_{L^\infty} \leq 1$ and $v(0) = 1$, then

$$\inf_{|x_0| = R} \sup_{B(x_0,1)} |v(x)| \geq C \exp(-R^{4/3} \log R).$$

In view of Meshkov’s example, the exponent $4/3$ is optimal.

Recently, Davey [4] derived similar quantitative asymptotic estimates for (1.1) with $P = -\Delta$ and $q(x) = -E \in \mathbb{C}$, i.e.,

$$-\Delta v + W(x) \cdot \nabla v + V(x)v = Ev.$$ (1.4)

To describe her result, we define

$$I(x_0) = \int_{|y-x_0|<1} |v(y)|^2 \, dy$$

and

$$M(t) = \inf_{|x_0|=t} I(x_0).$$

Assume that $|V(x)| \lesssim \langle x \rangle^{-N}$ and $|W(x)| \lesssim \langle x \rangle^{-P}$, where $\langle x \rangle = \sqrt{1 + |x|^2}$. Then it was shown that for any nontrivial bounded solution $v$ of (1.4) with $v(0) = 1$, we have

$$M(t) \gtrsim \exp(-Ct^{\beta_0}(\log t)^{b(t)}),$$ (1.5)

where

$$\beta_0 = \max\{2 - 2P, \frac{4 - 2N}{3}, 1\}$$

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and \( b(t) \) is either a constant \( C \) or \( C \log \log t \). Moreover, in [4], some Meshkov’s type examples were constructed to ensure the optimality of (1.5). There are also some related qualitative results in [3], [6], [7], and [8]. Especially, in [3] and [8], the authors studied the Schrödinger equation with potential \(-\Delta v + V(x)v = Eu\), where \(|V(x)| \lesssim \langle x \rangle^{-N}\) with \(0 < N < 1/2\) (in [3]) and \(N \leq 0, N > 1/2\) (in [8]). In addition to qualitative results, they also showed the optimality of \( \beta_0 \) (here \( \beta_0 = \max\{\frac{4-2N}{3}, 1\} \)). For the case of \( N = 1/2 \), the qualitative result was proved in [7].

In this work, we extend Davey’s results to more general cases. Precisely, we consider the second order elliptic operator \( P \) with more general assumptions on the asymptotic behaviors of \( W, V, \) and \( q \). The main theorem is stated as follows.

**Theorem 1.1.** Let \( v \in H^1_{\text{loc}}(\Omega) \) be a nontrivial solution of (1.1) satisfying
\[
|v(x)| \leq \lambda |x|^\alpha \quad \text{with} \quad \alpha \geq 0
\]
for some \( \lambda > 0 \). Assume that the ellipticity condition (1.2) holds and for \( \epsilon > 0, \kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathbb{R} \),
\[
\begin{aligned}
&|W(x)| \leq \lambda |x|^{-\kappa_1}, |V(x)| \leq \lambda |x|^{-\kappa_2}, \\
&|\nabla a_{ij}(x)| \leq \lambda |x|^{-1-\epsilon}, \\
&|q(x)| \leq \lambda |x|^{\kappa_3}, |\nabla q(x)| \leq \lambda |x|^{-\kappa_4}.
\end{aligned}
\]
Denote \( \kappa_0 = \max\{2 - 2\kappa_1, \frac{4-2\kappa_2}{3}, \frac{2+\kappa_3}{2}, \frac{3-\kappa_4}{2}\} \), \( \kappa = \max\{\kappa_0, 1\} \). Then we have that

- For \( \kappa > 1 \) (i.e., \( \kappa_0 > 1 \)), there exist \( t_0 \) depending on \( \lambda_0, \lambda, \epsilon \) and positive constants \( C, C' \) such that
  \[
  M(t) \geq \exp \left(-C t^{\kappa_0}(\log t)^{\gamma(t)}\right) \quad \text{for} \quad t \geq t_0
  \]
  with
  \[
  \gamma(t) = \frac{C'(\log t)(\log \log \log t)}{(\log \log t)^2},
  \]
  where \( C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \alpha) \) and \( C' = C'(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \alpha) \).

- For \( \kappa = 1 \) (i.e., \( \kappa_0 \leq 1 \)), there exists a positive constant \( C'' \) such that
  \[
  M(t) \geq \exp \left(-C'' t(\log t)^{\gamma(t)}\right) \quad \text{for} \quad t \geq t_0,
  \]
where $C''$ depends on $\alpha, \lambda_0, \lambda, \kappa_1, \kappa_2, \kappa_3, \epsilon$,

$$\left| \log \left( \min\left\{ \inf_{t_0 \leq |x| < t_0} \int_{|y-x| < 1} |v(y)|^2 dy, 1 \right\} \right| \right|.$$ 

**Remark 1.2.**

1. The condition on the decay rate of $\nabla a_{ij}$ was also used by T Nguyen [14] in his proof of qualitative and quantitative Landis-Oleinik conjecture, the parabolic counterpart of Landis conjecture.

2. We have made general assumptions on the asymptotic behaviors of $W$, $V$, and $q$. They may grow in $|x|$. Our theorem provides quantitative uniqueness estimates for solutions of $-\Delta v + V(x)v = Ev$ with $|V| \lesssim |x|^m$, $m > 0$. Moreover, our method works for any $\kappa_0 \in \mathbb{R}$ including the case $\kappa_0 = 1$ that is missing in [4].

Similar to the arguments in [2] and [4], a key ingredient to prove Theorem 1.1 is the Carleman type estimate. Before applying the Carleman estimate, $v$ is shifted and rescaled appropriately. We will modify our ideas in [12]. Basically, we use the Carleman estimate for the shifted and rescaled solution on a ball of radius depending on $|x_0|$ (see the definition of $\Omega_t$ in Section 3). Note that in order to use the behaviors of coefficients in (1.7), the radius of the ball is sufficiently small. In fact, after undoing sifting and rescaling, any point in $\Omega_t$ is at least $|x_0|^{1-\delta(x_0)}$ distance from the origin, where $\delta(x_0) = (\log \log |x_0|)^2 / \log |x_0|$ (see Section 3). We then apply the Carleman estimate to derive three-ball inequalities in which we can estimate the $L^2$ bound of the solution in a unit ball centered at $x_0/|x_0|^{\delta(x_0)}$ by the $L^2$ bound of the solution in a unit ball centered at $x_0$ up to certain power (see (3.21)). To obtain the desired estimates, we want to apply bootstrapping arguments based on a chain of balls similar to what we did in [12]. An bootstrapping step was also used in [4] to prove estimate (1.5). However, our method here is simpler than that of [4].

We now discuss the optimality of (1.8) and (1.9), at least, for some simple cases. It is readily seen that if $v(x) = \exp(-|x|^{1+\epsilon})$ with $0 < \epsilon \leq 1/2$, then $\Delta v + q(x)v = 0$ in $\{|x| < 1\}$ with $|q(x)| \sim |x|^{2\epsilon}$ and $|\nabla q(x)| \sim |x|^{2\epsilon-1}$. In this case, we can see that $\kappa = 1 + \epsilon$. So the exponent $\kappa$ of (1.8) is optimal. This example also shows that if the first derivatives of the potential possess certain decaying property, we can break the 4/3 barrier in the case of bounded potentials. On the other hand, for $\epsilon = 0$, we obtain that $v = \exp(-|x|)$
satisfies $\Delta v + q(x)v = 0$ in $\{|x| < 1\}^c$ with $q(x) = -1 + (n - 1)|x|^{-1}$. Since we can write

$$(\log t)^2(t) = t^{\frac{C\log \log \log t}{\log \log \log t}},$$

(1.9) is equivalent to

$$M(t) \geq \exp (-Ct^{1+o(1)}).$$

Thus, (1.9) is almost optimal.

This paper is organized as follows. In Section 2, we prove a Carleman estimate for the operator $P + q$, which plays an essential role in our proof. In Section 3, we begin to prove the main theorem, Theorem 1.1, by deriving three-ball inequalities for solutions of (1.1). In Section 4, we give detailed arguments of bootstrapping and complete the proof of Theorem 1.1.

2 Carleman estimate

In this section, we would like to derive a Carleman estimate for $P + q$ with $q$ is $C^1$. Similar Carleman estimate for such operator with $P$ being the Laplace-Beltrami operator was also derived in [1] using Donnelly and Fefferman’s approach [5]. Since we are working in the Euclidean space, we give a more elementary proof motivated by the ideas in [15]. To begin, we introduce polar coordinates in $\mathbb{R}^n\setminus\{0\}$ by setting $x = r\omega$, with $r = |x|$, $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. Furthermore, using new coordinate $t = \log r$, we can see that

$$\frac{\partial}{\partial x_j} = e^{-t}(\omega_j \partial_t + \Omega_j), \quad 1 \leq j \leq n,$$

where $\Omega_j$ is a vector field in $S^{n-1}$. We could check that the vector fields $\Omega_j$ satisfy

$$\sum_j \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_j \Omega_j \omega_j = n - 1.$$

Since $r \to 0$ iff $t \to -\infty$, we are mainly interested in values of $t$ near $-\infty$.

It is easy to see that

$$\frac{\partial^2}{\partial x_j \partial x_\ell} = e^{-2t}(\omega_j \partial_\ell - \omega_\ell \partial_j + \Omega_j)(\omega_\ell \partial_t + \Omega_\ell), \quad 1 \leq j, \ell \leq n.$$

and, therefore, the Laplacian becomes

$$e^{2t} \Delta = \partial_t^2 + (n - 2)\partial_t + \Delta_\omega,$$

(2.1)
where $\Delta_\omega = \sum_j \Omega_j^2$ denotes the Laplace-Beltrami operator on $S^{n-1}$. We recall that the eigenvalues of $-\Delta_\omega$ are $k(k + n - 2), k \in \mathbb{N}$, and the corresponding eigenspaces are $E_k$, where $E_k$ is the space of spherical harmonics of degree $k$. It follows that

$$\iint |\Delta_\omega v|^2 dtd\omega = \sum_{k \geq 0} k^2(k + n - 2)^2 \iint |v_k|^2 dtd\omega \quad (2.2)$$

and

$$\sum_j \iint |\Omega_j v|^2 dtd\omega = \sum_{k \geq 0} k(k + n - 2) \iint |v_k|^2 dtd\omega, \quad (2.3)$$

where $v_k$ is the projection of $v$ onto $E_k$.

Our aim is to derive Carleman-type estimates with weights $\varphi_\beta = \varphi_\beta(x) = \exp(-\beta \tilde{\psi}(x))$, where $\beta > 0$ and $\tilde{\psi}(x) = \log |x| + \log((\log |x|)^2)$. For simplicity, we denote $\psi(t) = t + \log t^2$, i.e., $\tilde{\psi}(x) = \psi(\log |x|)$.

**Lemma 2.1.** Assume that $P = \sum_{jk} a_{jk}(x)\partial_j\partial_k$ is a second order elliptic operator satisfying

$$\lambda_0 |\xi|^2 \leq \sum_{jk} a_{jk}(x)\xi_j\xi_k \leq \lambda_0^{-1} |\xi|^2 \quad \forall \ x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n,$$

and

$$\|\nabla a_{jk}\|_{L^\infty(\mathbb{R}^n)} \leq L.$$

Then there exist a sufficiently small $r_1 = r_1(\lambda_0, L) > 0$ such that for all $u \in U_{r_1}$ and

$$\beta \geq (||q||_{L^\infty(\mathbb{R}^n)} + ||\nabla q||_{L^\infty(\mathbb{R}^n)})^{1/2},$$

we have

$$\beta \int \varphi_\beta^2(\log |x|)^{-2}|x|^{-n}(|x|^2|\nabla u|^2 + |u|^2)dx$$

$$\leq C_0 \int \varphi_\beta^2|x|^{-n}|x|^4|Pu + q(x)u|^2dx, \quad (2.4)$$

where $U_{r_1} = \{u \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) : \text{supp}(u) \subset B_{r_1}\}$ and $C_0 = C_0(\lambda_0, L) > 0$. 

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Proof. If we set \( u = e^{\beta \psi(t)}v \) and \( P_{\beta}v = e^{-\beta \psi(t)}P(e^{\beta \psi(t)}v) \), then

\[
e^{2t}P_{\beta}v + e^{2t}qv = \partial_t^2 v + b\partial_t v + av + \Delta_{\omega}v + e^{2t}qv + \sum_{j+|\alpha| \leq 2} C_{ja}(t, \omega) \partial_t^j \Omega^\alpha v \\
+ \sum_{j+|\alpha| \leq 1} C_{ja}(t, \omega) \beta \psi' \partial_t^j \Omega^\alpha v + C_{20}(t, \omega)(\beta^2 \psi'^2 + \beta \psi'')v
= \partial_t^2 v + b\partial_t v + av + \Delta_{\omega}v + e^{2t}qv + S + h,
\]

where

\[
\left\{
\begin{aligned}
a &= \beta \psi'' + \beta^2 (\psi')^2 + (n-2)\beta \psi' \\
&= (1 + 2t^{-1})^2 \beta^2 + (n-2)\beta + 2(n-2)t^{-1} \beta - 2t^{-2} \beta, \\
b &= 2\beta \psi' + n - 2 \\
&= 2\beta + 4\beta t^{-1} + n - 2, \\
S(v) &= \sum_{j+|\alpha| \leq 2} C_{ja}(t, \omega) \partial_t^j \Omega^\alpha v + C_{20}(t, \omega)(\beta^2 \psi'^2 + \beta \psi'')v, \\
h(v) &= \sum_{j+|\alpha| \leq 1} C_{ja}(t, \omega) \beta \psi' \partial_t^j \Omega^\alpha v, \\
C_{ja} &= O(e^t), \partial_t C_{ja} = O(e^t), \Omega^\alpha C_{ja} = O(e^t).
\end{aligned}
\right.
\]

It is clear that (2.4) holds if for \( t \) near \(-\infty\) we have

\[
\iint |e^{2t}P_{\beta}v + e^{2t}qv|^2 dt d\omega \\
\geq C \left\{ \iint \beta t^{-2} |\partial_t v|^2 dt d\omega + \beta \sum_{j} \iint t^{-2} |\Omega_j v|^2 dt d\omega + \beta^3 \iint t^{-2} |v|^2 dt d\omega \right\}.
\]

(2.6)

We obtain from (2.5) that

\[
|e^{2t}P_{\beta}v + e^{2t}qv|^2 = |L(v)|^2 + 2b\partial_t v L(v) + 2h(v) L(v) + |b\partial_t v + h(v)|^2,
\]

(2.7)
where \( L(v) := \partial_t^2 v + av + \Delta_\omega v + e^{2t}qv + S(v) \). Now we write
\[
\int\int 2b\partial_t vL(v)dt\omega = \int\int 2b\partial_t v\partial_t^2 vdt\omega + \int\int 2abv\partial_t vdtd\omega + \int\int 2be^{2t}q\partial_t vvdt\omega + \int\int 2b\partial_t v\Delta_\omega vdt\omega + \int\int 2b\partial_t vS(v)dt\omega.
\]
(2.8)

By straightforward computations, we can get that
\[
\int\int 2b\partial_t v\partial_t^2 vdt\omega = -\int\int (\partial_t b)|\partial_t v|^2 dt\omega, \tag{2.9}
\]
\[
\int\int 2abv\partial_t vdtd\omega = -\int\int \partial_t(ab)|v|^2 dt\omega, \tag{2.10}
\]
\[
\int\int 2b\partial_t v\Delta_\omega vdt\omega = \sum_j \int\int (\partial_t b)|\Omega_j v|^2 dt\omega, \tag{2.11}
\]
and
\[
\int\int 2be^{2t}q\partial_t vdtd\omega = -\int\int (\partial_t b)e^{2t}q|v|^2 dt\omega - \int\int \partial_t(e^{2t}q)b|v|^2 dt\omega. \tag{2.12}
\]

Combining (2.7) to (2.12) yields
\[
\int\int |e^{2t}P_\beta v + e^{2t}qv|^2 dt\omega = \int\int |L(v)|^2 dt\omega + 4\beta \int\int t^{-2} |\partial_t v|^2 dt\omega + 12\beta^3 \int\int t^{-2}(1 + O(t^{-1}))|v|^2 dt\omega - 4\beta \sum_j \int\int t^{-2} |\Omega_j v|^2 dt\omega + 4\beta \int\int t^{-2} e^{2t}q|v|^2 dt\omega - \int\int \partial_t(e^{2t}q)b|v|^2 dt\omega + \int\int 2b\partial_t vS(v)dt\omega + \int\int |b\partial_t v + h(v)|^2 dt\omega. \tag{2.13}
\]

Likewise, we write
\[
|L(v)|^2 = |L(v) + 3\beta t^{-2}v|^2 - 6\beta t^{-2} vL(v) - (3\beta t^{-2})^2 |v|^2. \tag{2.14}
\]
It is easy to check that

\[-6\beta \int \int t^{-2}v L(v) dt \, d\omega\]

\[= -6\beta \int \int t^{-2}(\partial_t^2 v + \alpha v + \Delta v + e^{2t} q v + S(v)) dt \, d\omega\]

\[= -6\beta^3 \int \int t^{-2}(1 + O(t^{-1})|v|^2 dt \, d\omega - 6\beta \int \int e^{2t} q t^{-2}|v|^2 dt \, d\omega\]

\[+ 6\beta \int \int t^{-2}|\partial_t v|^2 dt \, d\omega - 12\beta \int \int t^{-3} v \partial_t v dt \, d\omega + 6\beta \sum_j \int \int t^{-2}|\Omega_j v|^2 dt \, d\omega\]

\[-6\beta \int \int t^{-2}v S(v) dt \, d\omega. \tag{2.15}\]

From (2.13)-(2.15), we have that for \(t \leq \tau\) (\(\tau\) depends on \(\lambda_0, L\))

\[\int \int |e^{2t} P_\beta v + e^{2t} q v|^2 dt \, d\omega\]

\[\geq \beta \int \int t^{-2}|\partial_t v|^2 dt \, d\omega + 2\beta \sum_j \int \int t^{-2}|\Omega_j v|^2 dt \, d\omega + 3\beta^3 \int \int t^{-2}|v|^2 dt \, d\omega\]

\[-2\beta \int \int (q + \partial_t q) t^{-2}|v|^2 dt \, d\omega + \int \int 2\partial_t v S(v) dt \, d\omega\]

\[+ \int \int 2h(v) L(v) dt \, d\omega - 6\beta \int \int t^{-2}v S(v) dt \, d\omega. \tag{2.16}\]

Using integration by parts and choosing an even smaller \(\tau\), if necessary, we can see that \(\int \int 2\partial_t v S(v) dt \, d\omega, \int \int 2h(v) L(v) dt \, d\omega,\) and \(6\beta \int \int t^{-2}v S(v) dt \, d\omega\) are bounded by the first three terms on the right side of (2.16). Therefore, taking

\[\beta \geq \sqrt{\|q\|_{L^\infty} + \|\nabla q\|_{L^\infty}},\]

(2.6) follows from (2.16).

\[\square\]

3 Proof of Theorem 1.1 – Part I: three-ball inequalities

We begin to prove Theorem 1.1 in this section. As in [2], [4], and [12], the solution of (1.1) is shifted and rescaled properly. Fixing \(x_0\) with \(|x_0| = t >>\)

1, we define

\[
\begin{aligned}
    w(x) &= u(atx + x_0), \quad \tilde{a}_{jk}(x) = a_{jk}(atx + x_0), \quad \tilde{P}(x, D) = \sum_{jk} \tilde{a}_{jk}(x) \partial_j \partial_k, \\
    \tilde{W}(x) &= (at)W(atx + x_0), \quad \tilde{V}(x) = (at)^2V(atx + x_0), \\
    \tilde{q}(x) &= (at)^2q(atx + x_0),
\end{aligned}
\]

where \( a \geq 1/r_1 \) will be determined later in the proof. Here \( r_1 \) is the constant appeared in Lemma 2.1. We now denote

\[
\Omega_t := B_{\frac{1}{2} - \frac{1}{20(1+\epsilon)t}}(0) = \{ x : |x| < \frac{1}{a} - \frac{1}{20at^\delta} \}.
\]

It follows from (1.1) that

\[
Pw + \tilde{W}(x) \cdot \nabla w + \tilde{V}(x)w + \tilde{q}(x)w = 0 \quad \text{in} \quad \Omega_t. \tag{3.1}
\]

It is clear that \( \tilde{a}_{jk}(x) \) satisfies (1.2) in \( \Omega_t \) with same constant \( \lambda_0 \). Furthermore, in view of (1.7), we have that

\[
\begin{aligned}
    |\tilde{W}(x)| &\leq 20\lambda at^{1-\kappa_1+\kappa_1 \delta}, \quad |\tilde{V}(x)| \leq 20\lambda a^2t^{2-\kappa_2+\kappa_2 \delta}, \\
    |\nabla \tilde{a}_{ij}(x)| &\leq 40\lambda at^{(1+\epsilon)\delta-\epsilon}, \\
    |\tilde{q}(x)| &\leq 2\lambda a^2t^{2+\kappa_3}, \quad |\nabla \tilde{q}(x)| \leq 20\lambda a^3t^{3-\kappa_4+\kappa_4 \delta}. \tag{3.2}
\end{aligned}
\]

Unlike in [12], where \( \delta \) is a fixed constant, here we take \( \delta = \delta(t) = \frac{\log \log t}{\log t} \).

We now choose \( t_0 \) such that \( \log t_0 \geq 1/r_1 \) and

\[
-\epsilon_0 := (1 + \epsilon)\delta(t_0) - \epsilon < 0. \tag{3.3}
\]

By setting \( a = t_0^{\epsilon_0} \), one can see that \( at^{(1+\epsilon)\delta(t)-\epsilon} \leq 1 \) for all \( t \geq t_0 \). Let \( r_1 \) and \( C_0 \) be constants in Lemma 2.1 determined by \( \lambda_0 \) and \( L = 40\lambda \). Then the Carleman estimate (2.4) can be applied to \( w \) in \( \Omega_t \) for all \( t \geq t_0 \) with same \( r_1 \) and \( C_0 \). For simplicity, in this section, \( C \) denotes a general constant whose value may vary from line to line. Furthermore, it depends on \( \lambda_0 \), \( \lambda \), and \( \epsilon \) unless indicated otherwise.

Besides of the Carleman estimate (2.4), we also need the following interior estimate for solutions of (3.1) in our proof.
Lemma 3.1. For any $0 < a_1 < a_2$ such that $B_{a_2} \subset \Omega_t$ for $t > 1$ and a large enough, let $X = B_{a_2} \setminus B_{a_1}$ and $d(x)$ be the distant from $x \in X$ to $\mathbb{R}^n \setminus X$. Then we have

\[
(1 + ||\tilde{W}||_{L^\infty(X)}) \int_X d(x)^2 |\nabla w|^2 \, dx \\
\leq C \left(||\tilde{W}||_{L^\infty(X)}^4 + ||\tilde{V}||_{L^\infty(X)}^2 + ||\tilde{q}||_{L^\infty(X)}^2\right) \int_X |w|^2 \, dx \tag{3.4}
\]

with $C = C(\lambda_0, L)$.

The lemma can be proved using similar arguments in [11]. We omit the details here.

Now we are ready to apply (2.4) to $w$ solving (3.1). Before doing so, we need to introduce a suitable cut-off function. Let $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \chi(x) \leq 1$ and

\[
\chi(x) = \begin{cases} 
0, & |x| \leq \frac{1}{8at}, \\
1, & \frac{1}{4at} < |x| < \frac{1}{a} - \frac{3}{20at^2}, \\
0, & |x| \geq \frac{1}{a} - \frac{2}{20at^2}.
\end{cases}
\]

It is easy to see that for any multiindex $\alpha$

\[
\begin{cases} 
|D^\alpha \chi| = O((at)^{|\alpha|}) & \text{if } \frac{1}{8at} \leq |x| \leq \frac{1}{4at}, \\
|D^\alpha \chi| = O((at^\delta)^{|\alpha|}) & \text{if } \frac{1}{a} - \frac{3}{20at^2} \leq |x| \leq \frac{1}{a} - \frac{2}{20at^2}.
\end{cases} \tag{3.5}
\]

To use (2.4), it suffices to take $\beta \geq \beta_1 = \sqrt{20/\lambda_0 t^8t|\kappa|^{4/2}}$. Thus, we have

\[
\int (\log |x|)^{-2} \varphi_\beta^2 |x|^{-n}(\beta |x|^2|\nabla (\chi w)|^2 + \beta^3 |\chi w|^2) \, dx \\
\leq C_0 \int \varphi_\beta^2 |x|^{-n} |x|^4 |\tilde{P}(x)(\chi w)| + \tilde{q}(\chi w)^2 \, dx. \tag{3.6}
\]
Using equation (3.1), we obtain that
\[
\int_T (\log |x|)^{-2} \varphi_{\beta}^2 |x|^{-n} (\beta |x|^2 |\nabla w|^2 + \beta^3 |w|^2) \, dx \\
\leq \int \varphi_{\beta}^2 (\log |x|)^{-2} |x|^{-n} (\beta |x|^2 \nabla (\chi w))^2 + \beta^3 |\chi w|^2) \, dx \\
\leq C_0 \int \varphi_{\beta}^2 |x|^{-n} |\chi (\tilde{W}(x) \cdot \nabla w + \tilde{V}(x) w)|^2 \, dx \\
+ C_0 \int \varphi_{\beta}^2 |x|^{-n} ||\tilde{P}, \chi||^2 w, 
\]  
(3.7)
where \( T \) denotes the domain \( \{ x : \frac{1}{4\lambda t} < |x| < \frac{1}{\lambda a} - \frac{3}{25\lambda t} \} \). To simplify the notations, we denote \( Y = \{ x : \frac{1}{8\lambda t} \leq |x| \leq \frac{1}{4\lambda t} \} \) and \( Z = \{ x : \frac{1}{a} - \frac{3}{25\lambda t \epsilon} \leq |x| \leq \frac{1}{a} - \frac{2}{25\lambda t \epsilon} \} \). By (3.2) and estimates (3.5), we deduce from (3.7) that
\[
\int_T (\log |x|)^{-2} \varphi_{\beta}^2 |x|^{-n} (\beta |x|^2 |\nabla w|^2 + \beta^3 |w|^2) \, dx \\
\leq C' \int_T \varphi_{\beta}^2 |x|^{-n} |x|^4 (a^2 t^{2-2\kappa_1+2\kappa_1 \delta} |\nabla w|^2 + a^4 t^{4-2\kappa_2+2\kappa_2 \delta} |w|^2) \, dx \\
+ C \int_Y \varphi_{\beta}^2 |x|^{-n} |\tilde{U}|^2 \, dx + C' \| \tilde{W} \|_{L^\infty} \int_Z \varphi_{\beta}^2 |x|^{-n} |x|^2 |\nabla w|^2 \, dx \\
+ C' \| \tilde{V} \|_{L^\infty} \int_Z \varphi_{\beta}^2 |x|^{-n} |w|^2 \, dx, 
\]  
(3.8)
where \( |\tilde{U}(x)|^2 = |x|^2 |\nabla w|^2 + |w|^2, C' = C'(\lambda_0, \lambda), C = C(\lambda_0, \lambda, a). \) From now on, \( \| \cdot \|_{L^\infty} \) is taken over \( \Omega_t \).

Taking a larger \( t_0 \) (recall \( a = t_0^\epsilon \)), if necessary, we can obtain that \( |x|^2 (\log |x|)^2 C' \leq \frac{1}{2} \) for all \( x \in T \). Additionally, we choose \( \beta \geq \beta_2 := a^2 t^{\kappa_0+\kappa_\delta}, \) where \( \kappa_\delta = \max \{ 2|\kappa_1|, 2|\kappa_2|/3, |\kappa_4|/2 \} \). then the first term on the right hand side of (3.8) can be absorbed by the left hand side of (3.8). With
the choices described above, we obtain from (3.8) that
\[
\beta^3(b_1)^{-n}(\log b_1)^{-2}\varphi_\beta(b_1) \int_{\frac{1}{2at} < |x| < b_1} |w|^2 \, dx \\
\leq \beta^3 \int_T (\log |x|)^{-2} \varphi_\beta|x|^{-n} |w|^2 \, dx \\
\leq C b_1^{-n} \varphi_\beta^2(b_2) \int_Y |\tilde{U}|^2 \, dx + C \|\tilde{W}\|_{L^\infty}^2 b_3^{-n} \varphi_\beta^2(b_3) \int_Z |x|^2 |\nabla w|^2 \, dx \\
+C \|\tilde{V}\|_{L^\infty}^2 b_3^{-n} \varphi_\beta^2(b_3) \int_Z |w|^2 \, dx, \tag{3.9}
\]
where \( b_1 = \frac{1}{a} - \frac{8}{20at^2}, b_2 = \frac{1}{8at} \) and \( b_3 = \frac{1}{a} - \frac{3}{20at^2} \).

Using (3.4), we can control \(|\tilde{U}|^2\) terms on the right hand side of (3.5). Indeed, let \( X = Y_1 := \{x : \frac{1}{16at} \leq |x| \leq \frac{1}{2at}\} \), then we can see that
\[
d(x) \geq C|x| \text{ for all } x \in Y,
\]
where \( C \) an absolute constant. Therefore, (3.4) implies
\[
\int_Y |x|^2 |\nabla w|^2 \, dx \\
\leq \ C \int_{Y_1} d(x)^2 |\nabla w|^2 \, dx \\
\leq \ C(C(\|\tilde{W}\|_{L^\infty}^4 + \|\tilde{V}\|_{L^\infty}^2 + \|\tilde{q}\|_{L^\infty}^2) \int_{Y_1} |w|^2 \, dx, \tag{3.10}
\]
where \( C = C(\lambda_0, \lambda, a) \). On the other hand, let \( X = Z_1 := \{x : \frac{1}{2a} \leq |x| \leq \frac{1}{a} - \frac{1}{20at^2}\} \), then
\[
d(x) \geq Ct^{-\delta}|x| \text{ for all } x \in Z,
\]
where \( C \) another absolute constant. Thus, it follows from (3.4) that
\[
\int_Z |x|^2 |\nabla w|^2 \, dx \\
\leq \ C t^{2\delta} \int_{Z_1} d(x)^2 |w|^2 \, dx \\
\leq \ C t^{2\delta}(\|\tilde{W}\|_{L^\infty}^4 + \|\tilde{V}\|_{L^\infty}^2 + \|\tilde{q}\|_{L^\infty}^2) \int_{Z_1} |w|^2 \, dx. \tag{3.11}
\]
Here, $C$ also depends on $\lambda_0$, $\lambda$. Combining (3.9), (3.10), and (3.11) leads to

$$b_1^{-2\beta-n} (\log b_1)^{-4\beta-2} \int_{\frac{1}{2b_1} < |x| < b_1} |w|^2 dx$$

$$\leq C'' t^{p_1} b_2^{-n} \varphi_\beta^2(b_2) \int_{Y_1} |w|^2 dx + C'' t^{p_2} b_3^{-n} \varphi_\beta^2(b_3) \int_{Z_1} |w|^2 dx, \quad (3.12)$$

where $C'' = C''(\lambda_0, \lambda, a)$, $p_1 = p_1(\kappa_1, \kappa_2, \kappa_3)$, $p_2 = p_2(\kappa_1, \kappa_2, \kappa_3)$. Notice that (3.12) holds for all $\beta \geq \beta_2$.

Changing $2\beta + n$ to $\beta$, (3.12) becomes

$$b_1^{-\beta} (\log b_1)^{-2\beta+2n-2} \int_{\frac{1}{2b_1} < |x| < b_1} |w|^2 dx$$

$$\leq C'' t^{p_1} b_2^{-\beta} (\log b_2)^{-2\beta+2n} \int_{Y_1} |w|^2 dx$$

$$+ C'' t^{p_2} b_3^{-\beta} (\log b_3)^{-2\beta+2n} \int_{Z_1} |w|^2 dx. \quad (3.13)$$

Recall that $\delta = \delta(t) = \frac{(\log \log t)^2}{\log t}$. By taking $t_0$ sufficiently large, if necessary, we can see that for $t \geq t_0$

$$\left\{ \begin{array}{l}
\frac{1}{a} - \frac{1}{at^\alpha} + \frac{1}{2a} \leq \frac{1}{a} - \frac{8}{20at^\alpha}, \\
\frac{1}{a} - \frac{1}{at^\alpha} - \frac{1}{at} \geq \frac{1}{2a}.
\end{array} \right. \quad (3.14)$$

In view of (3.14), dividing $b_1^{-\beta} (\log b_1)^{-2\beta+2n-2}$ on the both sides of (3.13) and noting that we can let $\beta_2 \geq n - 1$, i.e., $2\beta - 2n + 2 > 0$ for all $\beta \geq \beta_2$, we
obtain that
\[
\int_{|x + \frac{b_4 x_0}{1 - t}| < \frac{1}{1 - t}} |w(x)|^2 \, dx \\
\leq \int_{\frac{1}{2t} < |x| < b_1} |w(x)|^2 \, dx \\
\leq C'' t \beta (\log b_1)^2 (b_1/b_2)^{\beta} \int_{Y_1} \int_{|x| < \frac{1}{2t}} |w(x)|^2 \, dx \\
+ C'' t \beta (b_1/b_2)^{\beta} (\log b_3)^2 |\log b_1/\log b_3|^{2\beta - 2n + 2} \int_{Z_1} |w(x)|^2 \, dx
\]
\[
\leq C'' t \beta (\log b_1)^2 (8t)^{\beta} \int_{|x| < \frac{1}{2t}} |w(x)|^2 \, dx \\
+ C'' t \beta (b_1/b_2)^{\beta} \int_{Z_1} |w(x)|^2 \, dx,
\]
(3.15)

where \(b_4 = \frac{1}{a - \frac{1}{a \epsilon}}\) and \(b_5 = \frac{1}{a - \frac{6}{20at^{\epsilon}}}\). In deriving the third inequality above, we use the fact that if \(a = t_0^\epsilon\) is sufficiently large, then
\[
0 \leq \left(\frac{b_5}{b_3}\right)^2 \left(\log b_1/\log b_3\right)^2 \leq 1
\]
for all \(t \geq t_0\). From now on we fix \(a\) and hence \(t_0\). It is helpful to remind that \(t_0\) depends on \(\lambda_0\) and \(\lambda\). Having fixed constant \(a\), \(|\log b_1|\) and \(|\log b_3|\) can be bounded by a positive constant. Thus, (3.15) is reduced to
\[
\int_{|x + \frac{b_4 x_0}{1 - t}| < \frac{1}{1 - t}} |w(x)|^2 \, dx \leq C t \beta (8t)^{\beta} \int_{|x| < \frac{1}{2t}} |w(x)|^2 \, dx \\
+ C t \beta (b_1/b_2)^{\beta} \int_{Z_1} |w(x)|^2 \, dx,
\]
(3.16)

where \(C = C(\lambda_0, \lambda, \epsilon)\).

Using (3.16), (1.6), rescaling \(w\) back to \(v\), and replacing \(\beta\) by \(\beta - p_1\) (that is, taking \(\beta \geq \beta_2 + p_1\)), we have that
\[
I(t^{1-\delta} y_0) \leq C(8t)^{\beta} I(t y_0) + C t^{p_1} \left(\frac{t^{\delta}}{t^{\delta} + 0.1}\right)^{\beta},
\]
(3.17)

where \(x_0 = ty_0, \ C = C(\lambda_0, \lambda, \kappa_1, \kappa_2, \kappa_3, \epsilon), \) and \(p = p(\kappa_1, \kappa_2, \kappa_3, \alpha).\) For simplicity, by denoting
\[
A(t) = \log 8t, \quad B(t) = \log \left(\frac{t^{\delta} + 0.1}{t^{\delta}}\right),
\]

(3.17) becomes
\[ I(t^{1-\delta}y_0) \leq C\left\{ \exp(\beta A(t))I(ty_0) + t^p \exp(-\beta B(t)) \right\}. \] (3.18)

Now, we consider two cases. If
\[ \exp(\beta_2 A(t))I(ty_0) \geq t^p \exp(-\beta_2 B(t)), \]
then we have
\[ I(x_0) = I(ty_0) \geq t^p \exp(-\beta_2 (A(t) + B(t))) = t^p(8t)^{-\beta_2} \left( \frac{t^\delta + 0.1}{t^\delta} \right)^{-\beta_2}, \]
that is
\[ I(ty_0) \geq t^{-C\beta_2} = t^{-Ct^{\kappa_0+\kappa_3}} \geq \exp(-Ct^{\kappa_0+\kappa_3} \log t) \] (3.19)
for all \( t \geq t_0 \), where \( C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha) \). On the other hand, if
\[ \exp(\beta_2 A(t))I(ty_0) < t^p \exp(-\beta_2 B(t)), \]
then we can pick a \( \tilde{\beta} > \beta_2 \) such that
\[ \exp(\tilde{\beta} A(t))I(ty_0) = t^p \exp(-\tilde{\beta} B(t)). \] (3.20)
Solving \( \tilde{\beta} \) from (3.20) and using (3.18), we have that
\[ I(t^{1-\delta}y_0) \leq C \exp(\tilde{\beta} A(t))I(ty_0) \]
\[ = C (I(ty_0))^\tau (t^p)^{1-\tau} \]
\[ \leq C t^p (I(ty_0))^\tau, \] (3.21)
where \( \tau = \frac{B(t)}{A(t)+B(t)} \). This estimate will serve as a building block in the bootstrapping step in the next section.

4 Proof of Theorem 1.1 – Part II: bootstrapping

In the previous section, we see that (3.19) gives us the desired estimate. However, we need to work harder to derive the wanted estimate from (3.21).

We first observe that for \( t^{\log \log t} \leq \hat{t} \leq t \) we have.
\[
\begin{cases}
\frac{1}{a} - \frac{1}{at^\delta} + \frac{1}{at} \leq \frac{1}{a} - \frac{8}{20a t^\tau}, \\
\frac{1}{a} - \frac{1}{at^\delta} - \frac{1}{at} \geq \frac{1}{2at}.
\end{cases}
\]
We now let $|x_0| = t$ with
\[ t_0 \leq t^{\frac{2}{(\log \log t)^2}}, \]  
(4.1)
then we can write
\[ t = \mu^{(1-\delta)^s} \]  
(4.2)
for some positive integer $s$ and
\[ (t^{\frac{2}{(\log \log t)^2}})^{1-\delta} \leq \mu \leq t^{\frac{2}{(\log \log t)^2}}. \]  
(4.3)
For simplicity, we define $d_j = \mu^{(1-\delta)^s}$ and $\tau_j = \frac{B(d_j)}{A(d_j) + B(d_j)}$ for $j = 1, 2, \ldots, s$. Define
\[ \tilde{J} = \{1 \leq j \leq s : \exp(a^2d_j^{\kappa_0+\kappa_s}\lambda(d_j))I(d_jy_0) \geq (d_j)^p \exp(-a^2d_j^{\kappa_0+\kappa_s}\mu)B(d_j)\}, \]
where $y_0 = x_0/t$ as before. Note that $a$ is a fixed constant depending on $\lambda_0$, $\lambda$, $\epsilon$. Now, we divide it into two cases. If $\tilde{J} = \emptyset$, we only need to consider (3.21). Using (3.21) iteratively starting from $t = d_1$, we obtain that
\[ I(\mu y_0) \leq C_{d_1} (I(d_1y_0))^{\tau_1} \leq C_{d_1d_2\cdots d_s} (I(x_0))^{\tau_1\tau_2\cdots \tau_s}. \]  
(4.4)
It is easy to check that $s \leq C \log t (\log \log t/(\log \log t)^2)$ for some absolute constant $C$.
From (4.4) we have that
\[ I(\mu y_0) \leq C_{d_1d_2\cdots d_s} (I(x_0))^{\tau_1\tau_2\cdots \tau_s} \leq t^{C\log t/(\log \log t)^2} (I(x_0))^{\tau_1\tau_2\cdots \tau_s}. \]  
(4.5)
Hereafter, $C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha)$, unless indicated otherwise. We now estimate
\[ \frac{1}{\tau_j} = \frac{\log(8d_j) + \log(1 + 0.1d_j^{-\delta})}{\log(1 + 0.1d_j^{-\delta})} \leq \frac{2 \log(8d_j)}{\log(1 + 0.1d_j^{-\delta})} \leq 40d_j^p \log(d_j). \]
and thus
\[ \frac{1}{\tau_1\tau_2\cdots \tau_s} \leq 40^s (\log t)^s (d_1 \cdots d_s)^\delta \leq t \omega(t), \]  
(4.6)
where
\[ \omega(t) = t^{\frac{C \log \log \log t}{\log \log t}}. \]  
(4.7)
Raising both sides of (4.5) to the power $\frac{1}{\tau_1 \tau_2 \cdots \tau_s}$ and using (4.6), we obtain that
\[ (\min \{I(\mu y_0), 1\})^{\omega(t)} \leq t^{(C \log t/\log \log t^2)^{\omega(t)}} I(x_0), \]
i.e.,
\[ I(x_0) \geq \exp(-Ct^{\omega(t)}) (\min \{I(\mu y_0), 1\})^{\omega(t)}. \quad (4.8) \]

Next, if $\tilde{J} \neq \emptyset$, let $l$ be the largest integer in $J$. Estimate (3.19) implies
\[ I(d_1 y_0) \geq \exp(-Cd_1^{\kappa_0 + \kappa_1 \delta} \log d_1) \geq \exp(-Ct^{\kappa_0 + \kappa_1 \delta} \log t). \quad (4.9) \]

Note that here $\delta = \delta(t)$. As in (4.4), iterating (3.21) starting from $t = d_{l+1}$ yields
\[ I(d_1 y_0) \leq C^{s-l}(d_{l+1} \cdots d_s)^p (I(x_0))^{\tau_{l+1} \cdots \tau_s} \leq C^s(t/d_l)^{p/\delta} (I(x_0))^{\tau_{l+1} \cdots \tau_s} \leq t^{C \log t/\log \log t^2} (I(x_0))^{\tau_{l+1} \cdots \tau_s}. \quad (4.10) \]

It is enough to assume $I(d_1 y_0) < 1$. Repeating the computations in (4.6), we can see that
\[ \frac{1}{\tau_{l+1} \cdots \tau_s} \leq (t/d_l) \omega(t). \quad (4.11) \]
Hence, combining (4.9), (4.10) and using (4.11), we get that
\[ \exp(-Ct^{\omega_0} \omega(t)) \leq (I(x_0)). \quad (4.12) \]

Here $\omega(t)$ is given as in (4.7), but with $C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \alpha)$.

The last estimate (4.12) gives us the desired bound. We now focus on (4.8). In view of (4.3), if $\mu$ satisfies
\[ (t^{2 \log \log t^2})^{1-\delta} \leq \mu \leq t_0, \]
then we are done. Note that $t_0^{\frac{1}{1-\delta}} \leq (t^{2 \log \log t^2})^{1-\delta}$ due to (3.3) and (4.1). So we now consider $\mu > t_0$. We need another bootstrapping argument. Let us first rename $\tilde{t}_0 = t$, $\delta_1 = \delta(t)$, $s_1 = s$ (as in (4.2)). We then denote $\tilde{t}_1 = \mu$ and $\delta_2 = \delta_2(\tilde{t}_1) = \frac{\log \log \tilde{t}_1^2}{\log \log \tilde{t}_1}$. As before, we write
\[ \tilde{t}_1 = \tilde{t}_2^{\frac{(1-\delta_2)^{-\epsilon/2}}} \]
for some positive integer \( s_2 \) and \((\tilde{t}_1^{(\log \log t_1)^2})^{1-\delta_2} \leq \tilde{t}_2 \leq \tilde{t}_1^{(\log \log t_1)^2}\). Inductively, we denote \( \delta_k = \delta_k(\tilde{t}_{k-1} - 1) = \frac{(\log \log (\tilde{t}_{k-1} - 1))^2}{\log (\tilde{t}_{k-1})} \) and write
\[
\tilde{t}_{k-1} = \tilde{t}_k^{(1-\delta_k)^{s_k}}
\]
with positive constant \( s_k \) and \((\tilde{t}_{k-1}^{(\log \log \tilde{t}_{k-1})^2})^{1-\delta_k} \leq \tilde{t}_k \leq \tilde{t}_{k-1}^{(\log \log \tilde{t}_{k-1})^2}\) for \( k = 1, 2, \ldots \). It is easily seen that there exists an \( m \) such that
\[
\tilde{t}_1^{1+\epsilon_0} \leq \tilde{t}_m \leq t_0.
\]
Indeed, we have \( m \leq \log t \).

Now we are ready to perform bootstrapping using either (4.8) or (4.12). It is enough to treat the case where we have (4.8) all the way until \( \tilde{t}_m \), namely,
\[
I(x_0) \geq e^{-Ct_1 \omega(t_1) t \omega(t)} e^{-Ct_2 \omega(t_2) t_1 \omega(t_1) t \omega(t)} \cdots e^{-Ct_{m-1} \omega(t_{m-1}) \cdots t_1 \omega(t_1) t \omega(t)} \times \left( \min \{ I(\tilde{t}_m y_0), 1 \} \right) \tilde{t}_{m-1} \omega(\tilde{t}_{m-1}) \cdots \tilde{t}_1 \omega(\tilde{t}_1) t \omega(t).
\]
(4.13)

In view of
\[
\omega(\tilde{t}_k) \leq \tilde{t}_k, \quad \tilde{t}_{k+1}^2 \leq \tilde{t}_k \quad \text{for} \ 1 \leq k \leq (m - 1),
\]
we deduce that
\[
\tilde{t}_{m-1} \omega(\tilde{t}_{m-1}) \cdots \tilde{t}_1 \omega(\tilde{t}_1) t \omega(t) \leq \tilde{t}_1 \omega(t).
\]
Multiplying all terms in (4.13) implies
\[
I(x_0) \geq \exp(-Ct_1 \omega(t)),
\]
(4.14)

where \( C = C(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha) \) and
\[
\left| \log \left( \min \left\{ \inf_{t_0 < |x| < t_0} \int_{|y-x| < 1} |v(y)|^2 dy, 1 \right\} \right) \right|
\]
and \( \omega(t) \) is defined as in (4.7) with \( C \) depending on the same parameters.

On the other hand, we stop the bootstrapping process whenever (4.12) is satisfied. Similar computations give the following bound
\[
I(x_0) \geq \exp(-C' t_1 \omega(t)),
\]
(4.15)
where \( C' = C'(\lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \alpha) \) and the constant \( C \) in \( \omega(t) \) depends on \( \lambda_0, \lambda, \epsilon, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \alpha \). Notice that

\[
\omega(t) = t^{\frac{C \log \log \log t}{\log \log t}} = (\log t)^{\frac{C(\log \log \log t)}{(\log \log t)^2}}.
\]

Therefore, (4.12) gives the estimate for \( \kappa > 1 \) and (4.12), (4.14), (4.15) lead to the estimate for \( \kappa = 1 \).

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References


