Homework# 8 solutions

3. a. Pointwise, but not uniformly on $\mathbb{R}$. $\sum g_k(x)$ is not continuous. It has jumps at positive integers.

b. Uniformly (Weierstrass M-test with $1/k^2$). Continuous.

c. Converges uniformly on $[a,b] \subset (0,\pi)$ (use the Dirichlet test by computing $2\sin(x)S_n(x) = \sum_{k=1}^n (-1)^k 2\sin(x)\cos(kx) = \sum_{k=1}^n (-1)^k[\sin((k+1)x) - \sin((k-1)x)]$). Converges pointwise on $\mathbb{R} \setminus \{(2m+1)\pi : m \in \mathbb{Z}\}$. $\sum_{k=1}^\infty g_k(x)$ diverges at $\{(2m+1)\pi : m \in \mathbb{Z}\}$.

d. Pointwise, but not uniformly. Continuous.

4. a.& b. Observe that for $x \in [1,2]$

$$\sum_{n=1}^\infty \frac{x}{(1+x)^n} \le \sum_{n=1}^\infty \frac{x^n}{(1+x)^n} \le \sum_{n=1}^\infty (\frac{2}{3})^n.$$

Weierstrass test $\Rightarrow$ converges uniformly on $[1,2]$.

c. Unform convergence $\Rightarrow$ interchange of limit and summation.

8. No. The example given in the class provides a counterexample, i.e.,

$$f_n(x) = \begin{cases} n^2x, \quad 0 \le x < 1/n \\ 2n - n^2x, \quad 1/n \le x < 2/n \\ 0, \quad 2/n \le x \le 1. \end{cases}$$

$f_n \to 0$ pointwise for all $x \in [0,1]$, but not uniformly. Note that the example $f_n(x) = x^n$ for $x \in [0,1]$ doesn't work since f_n does not converge to a continuous function.

19. Splitting

$$(\frac{\sin nx}{n^2})x^3 = \frac{\sin nx}{n^{3/2}}\frac{x^3}{\sqrt{n}}.$$

Let $\phi_n(x) = x^3/\sqrt{n}$ and $f_n(x) = \sin nx/n^{3/2}$. For any fixed bounded set $\{\phi_n(x)\}$ are decreasing and uniformly bounded. By the Weierstrass test

$\sum f_n(x)$ uniformly converges on any set. So by Abel's test, $\sum f_n(x)\phi_n(x)$ converges uniformly on any bounded set. So the convergent function is continuous on any bounded set and therefore it is continuous in $\mathbb{R}$.

20. a. Straightforward. b. Obvious!

c. This is proved in Rudin's book (see Theorem 7.18 ??).

26. Easy application of the contraction mapping theorem. Define

$$\Phi(f) = A(x) + \int_0^1 k(x,y)f(y)dy$$

on $M = C([0,1], \mathbb{R})$ and check all the conditions.

37. Using the intermediate value theorem. If $f(x)$ is not a constant, then $f(x_1)$ and $f(x_2)$ different are rational numbers for some $x_1 < x_2$. Any irrational numbers between $f(x_1)$ and $f(x_2)$ must be in the range of (x_1, x_2). This is a contradiction.

45. a. K is compact $\Rightarrow$ K can be covered by a finite number of balls $D(x_j, \delta)$. It suffices to check Cauchy's criterion: for any ε, any $x \in K$, $x \in D(x_j, \delta)$ for some x_j and

$$\begin{aligned} & \rho(f_n(x), f_m(x)) \\ \leq\ & \rho(f_n(x), f_n(x_j)) + \rho(f_m(x), f_m(x_j)) + \rho(f_n(x_j), f_m(x_j)) \\ <\ & \varepsilon \end{aligned}$$

for all n, m large, where $\rho(f_n(x), f_n(x_j)) < \varepsilon/3$ and $\rho(f_m(x), f_m(x_j)) < \varepsilon/3$ by the equicontinuity and $\rho(f_n(x_j), f_m(x_j)) < \varepsilon/3$ by the pointwise convergence.

b. It is easy to see that $f_n(x)$ converges pointwise to 0. However

$$\max_{x\in[0,1]} |f_n(x)| = f_n(\frac{1}{n}) = 1.$$

So the convergence is not uniformly. From a., we can conclude that $f_n(x)$ is not equicontinuous. You can check this by yourselves.

47. Let K be a dense subset of A on which $f_n(x)$ converges (pointwise sense). Now for any $x \in A$ and any $\delta > 0$, we can find $y \in K$ such that $d(x, y) < \delta$ and equicontinuity implies $\rho(f_n(x), f_n(y)) < \varepsilon/3$. Then

$$\begin{aligned} & \rho(f_n(x), f_m(x)) \\ \leq \; & \rho(f_n(x), f_n(y)) + \rho(f_m(x), f_m(y)) + \rho(f_n(y), f_m(y)) \\ < \; & \varepsilon \end{aligned}$$

for all n, m large, where $\rho(f_n(y), f_m(y)) < \varepsilon/3$ is ensured by the convergence of f_n on K. In the proof of Theorem 5.6.2, one can replace the condition of pointwise compactness by that f_n converges on a countable dense subset of A.

57. Please refer to the solution on the back of the textbook. It is clear enough.