◇ **4E-22.** Give an alternative proof of the uniform continuity theorem using the Bolzano-Weierstrass Theorem as follows. First, show that if f is not uniformly continuous, there is an $\varepsilon > 0$ and there are sequences x_n, y_n such that $\rho(x_n, y_n) < 1/n$ and $\rho(f(x_n), f(y_n)) \geq \varepsilon$. Pass to convergent subsequences and obtain a contradiction to the continuity of f.

Solution. We are asked to prove that a continuous function on a compact set is uniformly continuous on that set by using the Bolzano-Weierstrass Theorem which says that a compact set is sequentially compact. So, suppose K is a compact subset of a metric space M with metric d and f is a continuous function from K into a metric space N with metric ρ. If f were not uniformly continuous, then there would be an $\varepsilon > 0$ for which no $\delta > 0$ would work in the definition of uniform continuity. In particular, $\delta = 1/n$ would not work. So there would be points x_n and y_n with $d(x_n, y_n) < 1/n$ and $\rho(f(x_n), f(y_n)) > \varepsilon$. Since K is a compact subset of the metric space M, it is sequentially compact by the Bolzano-Weierstrass Theorem (3.1.3). So there are indices $n(1) < n(2) < n(3) < \ldots$ and a point $z \in K$ such that $x_{n(k)} \to z$ as $k \to \infty$. Since $n(k) \to \infty$ and $d(x_{n(k)}, y_{n(k)})) < 1/n(k)$, we can compute

$$d(y_{n(k)}, z) \leq d(y_{n(k)}, x_{n(k)}) + d(x_{n(k)}, z) < \frac{1}{n(k)} + d(x_{n(k)}, z) \to 0.$$

So $y_{n(k)} \to z$ also. Since f is continuous on K, we should have $f(x_{n(k)}) \to f(z)$ and $f(y_{n(k)}) \to f(z)$ as $k \to \infty$. We can select k large enough so that $\rho(f(x_{n(k)}), f(z)) < \varepsilon/2$ and $\rho(f(y_{n(k)}), f(z)) < \varepsilon/2$. But this would give

$$\begin{aligned}\varepsilon &< \rho(f(x_{n(k)}), f(y_{n(k)})) \\ &\leq \rho(f(x_{n(k)}), f(z)) + \rho(f(y_{n(k)}), f(z)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This impossibility shows that f must, in fact, be uniformly continuous on K. ♦

◇ **4E-23.** Let X be a compact metric space and $f : X \to X$ an isometry; that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is a bijection.

Sketch. To show "onto" suppose $y_1 \in \mathcal{X} \backslash f(\mathcal{X})$ and consider the sequence $y_2 = f(y_1)$, $y_3 = f(y_2)$, ... ◇

Solution. If $f(x) = f(y)$, then $0 = d(f(x), f(y)) = d(x, y)$, so $x = y$. Thus f is one-to-one. If $\varepsilon > 0$, let $\delta = \varepsilon$. If $d(x, y) < \delta$, then $d(f(x), f(y)) = d(x, y) < \delta = \varepsilon$, so f is continuous, in fact uniformly continuous, on $\mathcal{X}$. It remains to show that f maps $\mathcal{X}$ onto $\mathcal{X}$. Since $\mathcal{X}$ is compact and f is continuous, the image, $f(\mathcal{X})$ is a compact subset of the metric space $\mathcal{X}$. So it must be closed. Its complement, $\mathcal{X} \setminus f(\mathcal{X})$ must be open. If there were a point x in $\mathcal{X} \setminus f(\mathcal{X})$, then there would be a radius $r > 0$ such that $D(x, r) \subseteq \mathcal{X} \setminus f(\mathcal{X})$. That is, $y \in f(\mathcal{X})$ implies $d(y, x) > r$. Consider the sequence defined by $x_0 = x$ and $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$. For positive integer k, let f^k denote the composition of f with itself k times. If n and p are positive integers, then

$$d(x_{n+p}, x_n) = d(f^n \circ f^p(x), f^n(x)) = d(f^p(x), x) > r$$

since $f^p(x) \in f(\mathcal{X})$. The points in the sequence are pairwise separated by distances of at least r. This would prevent any subsequence from converging. But $\mathcal{X}$ is sequentially compact by the Bolzano-Weierstrass Theorem. So there should be a convergent subsequence. This contradiction shows that there can be no such starting point x for our proposed sequence. The complement $\mathcal{X} \setminus f(\mathcal{X})$ must be empty. So $f(\mathcal{X}) = \mathcal{X}$ and f maps $\mathcal{X}$ onto $\mathcal{X}$ as claimed. ♦

◇ **4E-26.** Let $f :]a,b] \to \mathbb{R}$ be continuous and such that $f'(x)$ exists on $]a,b[$ and $\lim_{x\to a^+} f'(x)$ exists. Prove that f is uniformly continuous.

Suggestion. Use the limit of the derivative at a to get uniform continuity on a short interval $]a, a+2d]$. Use Theorem 4.6.2 to get uniform continuity on an overlapping interval $[a+d, b]$. Then combine the two results. ◇

Solution. Let $\varepsilon > 0$, and suppose $\lim_{x\to a^+} f(x) = \lambda$. There is a d such that $0 < 2d < b-a$ and $|f'(x)| \le |\lambda| + 1$ for $a < x < a+2d$. As is Example 4.6.4, f is uniformly continuous on $]a, a+2d]$, and there is a $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in $]a, a+2d]$ and $|x-y| < \delta_1$. Since f is continuous on $]a,b]$, it is continuous on the subinterval $[a+d, b]$. Since that interval is compact, f is uniformly continuous on it by the uniform continuity theorem, 4.6.2. There is a $\delta_2 > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in $[a+d, b]$ and $|x-y| < \delta_2$. Now we take advantage of the overlap we have carefully arranged between our two subdomains. If x and y are in $]a,b]$ and $|x-y| < \min(\delta_1, \delta_2, d/2)$, then either they are both in $]a, a+2d]$ or they are both in $[a+d, b]$, or both. If they are both in $]a, a+2d]$, then $|f(x) - f(y)| < \varepsilon$ since $|x-y| < \delta_1$. If they are both in $]a, a+2d]$, then $|f(x) - f(y)| < \varepsilon$ since $|x-y| < \delta_2$. In any case, $|f(x) - f(y)| < \varepsilon$ whenever x and y are in $]a,b]$ ans $|x-y| < \delta = \min(\delta_1, \delta_2, d/2)$. So f is uniformly continuous on $]a,b]$ as claimed. ♦

◇ **4E-28.** Let $f :]0,1[\to \mathbb{R}$ be uniformly continuous. Must f be bounded?

Answer. Yes. ◇

Solution. If f were not bounded on $]0,1[$, we could inductively select a sequence of points $\langle x_k \rangle_1^\infty$ in $]0,1[$ such that $|f(x_{k+1})| > |f(x_k)| + 1$ for each k. In particular, we would have $|f(x_k) - f(x_j)| > 1$ whenever $k \ne j$. But the points x_k are all in the compact interval $[0,1]$, so there should be a subsequence converging to some point in $[0,1]$. This subsequence would have to be a Cauchy sequence, so no matter how small a positive number δ were specified, we could get points x_k and x_j in the subsequence with $|x_k - x_j| < \delta$ and $|f(x_k) - f(x_j)| > 1$. This contradicts the uniform continuity of f on $]0,1[$. So the image $f(]0,1[)$ must, in fact, be bounded.

If we knew the result of Exercise 4E-24(c), then we would know that f has a unique continuous extension to the closure, $[0,1]$. There is a continuous $g : [0,1] \to \mathbb{R}$ such that $g(x) = f(x)$ for all x in $]0,1[$. Since g is continuous on the compact domain $[0,1]$, the image $g([0,1])$ is compact and hence bounded. So $f(]0,1[) = g(]0,1[) \subseteq g([0,1])$ is bounded. ♦

◇ **4E-29.** Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \le |x-y|^2$. Prove that f is a constant. [Hint: What is $f'(x)$?]

Suggestion. Divide by $x - y$ and let y tend to x to show that $f'(x) = 0$. ◇

Solution. Suppose $x_0 \in \mathbb{R}$. Then for $x \ne x_0$ we have $|f(x) - f(x_0)| \le |x - x_0|^2$, so

$$0 \le \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \le |x - x_0| .$$

Letting $x \to x_0$, we find that $\lim_{x\to x_0} (f(x) - f(x_0))/(x - x_0) = 0$. So $f'(x_0)$ exists and is equal to 0 for every $x_0 \in \mathbb{R}$. If $t \in \mathbb{R}$, there is, by the mean value theorem, a point x_0 between 0 and t such that $|f(t) - f(0)| = |f'(x_0)(t-0)| = |0(t-0)| = 0$. So $f(t) = f(0)$ for all $t \in \mathbb{R}$. Thus f is a constant function as claimed. ♦

◇ **4E-30.** (a) Let $f : [0,\infty[\to \mathbb{R}$, $f(x) = \sqrt{x}$. Prove that f is uniformly continuous.

(b) Let $k > 0$ and $f(x) = (x - x^k)/\log x$ for $0 < x < 1$ and $f(0) = 0$, $f(1) = 1 - k$. Show that $f : [0,1] \to \mathbb{R}$ is continuous. Is f uniformly continuous?

Suggestion. (a) Use Theorem 4.6.2 to show that f is uniformly continuous on $[0,3]$ and Example 4.6.4 to show that it is uniformly continuous on $[1,\infty[$. Then combine these results.

(b) Use L'Hôpital's Rule. ◊

Solution. (a) Let $\varepsilon > 0$. We know that $f(x) = \sqrt{x}$ is continuous on $[0,\infty[$, so it is certainly continuous on the compact domain $[0,3]$. By the uniform continuity theorem, 4.6.2, it is uniformly continuous on that set. There is a $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in $[0,3]$ and $|x - y| < \delta_1$.

We also know that f is differentiable for $x > 0$ with $f'(x) = 1/(2\sqrt{x})$. So $|f'(x)| \leq 1/2$ for $x \geq 1$. As in Example 4.6.4, we can use the mean value theorem to conclude that if $\delta_2 = 2\varepsilon$, and x and y are in $[1,\infty[$ with $|x-y| < \delta_2$, then there is a point c between x and y such that $|f(x) - f(y)| = |f'(c)(x-y)| < (1/2)(2\varepsilon) = \varepsilon$.

Now take advantage of the overlap of our two domains. If x and y are in $[0,\infty[$ and $|x-y| < \delta = \min(1, \delta_1, \delta_2)$, then either x and y are both in $[0,3]$ or both are in $[1,\infty[$ or both. If they are both in $[0,3]$, then $|f(x) - f(y)| < \varepsilon$ since $|x - y| < \delta_1$. If they are both in $[1,\infty[$, then $|f(x) - f(y)| < \varepsilon$ since $|x-y| < \delta_2$. In either case, $|f(x) - f(y)| < \varepsilon$. So f is uniformly continuous on $[0,\infty[$ as claimed.

(b) Suppose k is a positive integer and $f(x) = (x - x^k)/\log x$ for $0 < x < 1$, $f(0) = 0$, and $f(1) = 1 - k$. The numerator, $x - x^k$, is continuous for all x. The denominator, $\log x$, is continuous for $x > 0$. So f is continuous on $x > 0$ except possibly at $x = 1$ where the denominator is 0, However, the numerator is also 0 at $x = 1$. To apply L'Hôpital's Rule, we consider the ratio of the derivatives

$$\frac{1 - kx^{k-1}}{1/x} = x - kx^k \to 1 - k = f(1) \quad \text{as} \quad x \to 1.$$

By L'Hôpital's Rule, $\lim_{x\to 1}(x - x^k)/\log x = \lim_{x\to 1} f(x)$ also exists and is equal to $f(1)$. So f is continuous at 1. As $x \to 0^+$, the numerator of $f(x)$ tends to 0 and the denominator to $-\infty$. So $\lim_{x\to 0^+} f(x) = 0 = f(0)$. So f is continuous from the right at 0. So f is continuous on $[0,\infty[$ and on the smaller domain $[0,1]$. Since the latter is compact, f is uniformly continuous on it by the uniform continuity theorem, 4.6.2. ♦

◊ **4E-37.** Prove the following intermediate value theorem for derivatives: If f is differentiable at all points of $[a,b]$, and if $f'(a)$ and $f'(b)$ have opposite signs, then there is a point $x_0 \in]a,b[$ such that $f'(x_0) = 0$.

Sketch. Suppose $f'(a) < 0 < f'(b)$. Since f is continuous on $[a,b]$ (why?), it has a minimum at some x_0 in $[a,b]$. (Why?) $x_0 \neq a$ since $f(x) < f(a)$ for x slightly larger than a. (Why?) $x_0 \neq b$ since $f(x) < f(b)$ for x slightly smaller that b. (Why?) So $a < x_0 < b$. So $f'(x_0) = 0$. (Why?) The case of $f'(a) > 0 > f'(b)$ is similar. ◊

Solution. We know from Proposition 4.7.2 that f is continuous on the compact domain $[a,b]$. By the maximum-minimum theorem, 4.4.1, it attains a finite minimum, m, and a finite maximum, M, at points x_1 and x_2 in $[a,b]$.

CASE ONE: $f'(a) < 0 < f'(b)$: Since $f'(b) > 0$, and it is the limit of the difference quotients at b, we must have $(f(x) - f(b))/(x-b) > 0$ for x slightly smaller than b. Since $x - b < 0$, we must have $f(x) < f(b)$ for such x. So the minimum doe not occur at b. Since $f'(a) < 0$, and it is the limit of the difference quotients at a, we must have $(f(x) - f(a))/(x-a) < 0$ for x slightly larger than a. Since $x - a > 0$, we must have $f(x) < f(a)$ for such x. So the minimum does not occur at a. Thus the minimum must occur at a point $x_0 \in]a,b[$. By Proposition 4.7.9, we must have $f'(x_0) = 0$.

CASE TWO: $f'(a) > 0 > f'(b)$: Since $f'(b) < 0$, and it is the limit of the difference quotients at b, we must have $(f(x) - f(b))/(x-b) < 0$ for x slightly smaller than b. Since $x - b < 0$, we must have $f(x) > f(b)$ for such x. So the maximum does not occur at b. Since $f'(a) > 0$, and it is the limit of the difference quotients at a, we must have $(f(x) - f(a))/(x-a) > 0$ for x slightly larger than a. Since $x - a > 0$, we must have $f(x) > f(a)$ for such x. So the maximum does not occur at a. Thus the maximum must occur at a point $x_0 \in]a,b[$. By Proposition 4.7.9, we must have $f'(x_0) = 0$.

a, b with $0 \le a < b \le 1$ there is a c, $a < c < b$, with $f(c) = 0$. Prove $\int_0^1 f = 0$. Must f be zero? What if f is continuous?

Suggestion. Show that the upper and lower sums are both 0 for every partition of $[0, 1]$. Consider a function which is 0 except at finitely many points. ♢

Solution. Since f is integrable on $[0, 1]$, the upper and lower integrals are the same and are equal to the integral. Let $P = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}$ be any partition of $[0, 1]$. For each subinterval $[x_{j-1}, x_j]$ there is a point c_j in it with $f(c_j) = 0$. So

$$m_j = \inf\{f(x) \mid x \in [x_{j-1}, x_j]\} \le 0 \le \sup\{f(x) \mid x \in [x_{j-1}, x_j]\} = M_j.$$

So

$$L(f, P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \le 0 \le \sum_{j=1}^{n} M_j(x_j - x_{j-1}) = U(f, P).$$

This is true for every partition of $[0, 1]$. So

$$\begin{aligned} \int_0^1 f(x)\,dx = \underline{\int_0^1} f(x)\,dx &= \sup_{P \text{ a partition of } [0,1]} L(f, P) \le 0 \\ &\le \inf_{P \text{ a partition of } [0,1]} U(f, P) = \overline{\int_0^1} f(x)\,dx = \int_0^1 f(x)\,dx. \end{aligned}$$

So we must have $\int_0^1 f(x)\,dx = 0$.

The function f need not be identically 0. We could, for example, have $f(x) = 0$ for all but finitely many points at which $f(x) = 1$.

If f is continuous and satisfies the stated condition, then f must be identically 0. Let $x \in [0, 1]$. By hypothesis there is, for each integer $n > 0$, at least one point c_n in $[0, 1]$ with $x - (1/n) \le c_n \le x + (1/n)$ and $f(c_n) = 0$. Since $c_n \to 0$ and f is continuous, we must have $0 = f(c_n) \to f(x)$. So $f(x) = 0$. ♦

♢ **4E-45.** Prove the following *second mean value theorem. Let f and g be defined on $[a, b]$ with g continuous, $f \ge 0$, and f integrable. Then there is a point $x_0 \in\,]a, b[$ such that*

$$\int_a^b f(x)g(x)\,dx = g(x_0) \int_a^b f(x)\,dx.$$

Sketch. Let $m = \inf(g([a, b]))$ and $M = \sup(g([a, b]))$. Then

$$m \int_a^b f(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M \int_a^b f(x)\,dx.$$

(Why?) Since $t \int_a^b f(x)\,dx$ depends continuously on t, the intermediate value theorem gives t_0 in $[m, M]$ with

$$\int_a^b f(x)g(x)\,dx = t_0 \int_a^b f(x)\,dx.$$

Now apply that theorem to g to get x_0 with $g(x_0) = t_0$. (Supply details.) ♢

Solution. Since g is continuous on the compact interval $[a, b]$, we know that $m = \inf(g([a, b]))$ and $M = \sup(g([a, b]))$ exist as finite real numbers and that there are points x_1 and x_2 in $[a,b]$ where $g(x_1) = m$ and $g(x_2) = M$. Since $m \le g(x) \le M$ and $f(x) \ge 0$, we have $mf(x) \le f(x)g(x) \le Mf(x)$ for all x in $[a,b]$. Assuming that f and fg are integrable on $[a,b]$, Proposition 4.8.5(iii) gives

$$\begin{aligned} m\int_a^b f(x)\,dx &= \int_a^b mf(x)\,dx \\ &\le \int_a^b f(x)g(x)\,dx \\ &\le \int_a^b Mf(x)\,dx \\ &= M\int_a^b f(x)\,dx. \end{aligned}$$

The function $h(t) = t\int_a^b f(x)\,dx$ is a continuous function of t in the interval $m \le t \le M$, and $\int_a^b f(x)g(x)\,dx$ is a number between $h(m)$ and $h(M)$. By the intermediate value theorem there is a number t_0 in $[m, M]$ with $h(t_0) = \int_a^b f(x)g(x)\,dx$. Since g is continuous between x_1 and x_2 and t_0 is between $m = g(x_1)$ and $M = g(x_2)$, another application of the intermediate value theorem gives a point x_0 between x_1 and x_2 with $g(x_0) = t_0$. So

$$\int_a^b f(x)g(x)\,dx = h(t_0) = t_0\int_a^b f(x)\,dx = g(x_0)\int_a^b f(x)\,dx$$

as desired.