4E-22. Give an alternative proof of the uniform continuity theorem using the Bolzano-Weierstrass Theorem as follows. First, show that if $f$ is not uniformly continuous, there is an $\varepsilon > 0$ and there are sequences $x_n, y_n$ such that $\rho(x_n, y_n) < 1/n$ and $\rho(f(x_n), f(y_n)) \geq \varepsilon$. Pass to convergent subsequences and obtain a contradiction to the continuity of $f$.

Solution. We are asked to prove that a continuous function on a compact set is uniformly continuous on that set by using the Bolzano-Weierstrass Theorem which says that a compact set is sequentially compact. So, suppose $K$ is a compact subset of a metric space $M$ with metric $d$ and $f$ is a continuous function from $K$ into a metric space $N$ with metric $\rho$. If $f$ were not uniformly continuous, then there would be an $\varepsilon > 0$ for which no $\delta > 0$ would work in the definition of uniform continuity. In particular, $\delta = 1/n$ would not work. So there would be points $x_n$ and $y_n$ with $d(x_n, y_n) < 1/n$ and $\rho(f(x_n), f(y_n)) > \varepsilon$. Since $K$ is a compact subset of the metric space $M$, it is sequentially compact by the Bolzano-Weierstrass Theorem (3.1.3). So there are indices $n(1) < n(2) < n(3) < \ldots$ and a point $z \in K$ such that $x_{n(k)} \to z$ as $k \to \infty$. Since $n(k) \to \infty$ and $d(x_{n(k)}, y_{n(k)}) < 1/n(k)$, we can compute

$$d(y_{n(k)}, z) \leq d(y_{n(k)}, x_{n(k)}) + d(x_{n(k)}, z) < \frac{1}{n(k)} + d(x_{n(k)}, z) - 0.$$ 

So $y_{n(k)} \to z$ also. Since $f$ is continuous on $K$, we should have $f(x_{n(k)}) \to f(z)$ and $f(y_{n(k)}) \to f(z)$ as $k \to \infty$. We can select $k$ large enough so that $\rho(f(x_{n(k)}), f(z)) < \varepsilon/2$ and $\rho(f(y_{n(k)}), f(z)) < \varepsilon/2$. But this would give

$$\varepsilon < \rho(f(x_{n(k)}), f(y_{n(k)})) \leq \rho(f(x_{n(k)}), f(z)) + \rho(f(y_{n(k)}), f(z)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

This impossibility shows that $f$ must, in fact, be uniformly continuous on $K$.

4E-23. Let $X$ be a compact metric space and $f : X \to X$ an isometry; that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that $f$ is a bijection.

Sketch. To show "auto" suppose $y_1 \in X \setminus f(X)$ and consider the sequence $y_2 = f(y_1), y_3 = f(y_2), \ldots$. 

Solution. If $f(x) = f(y)$, then $0 = d(f(x), f(y)) = d(x, y)$, so $x = y$. Thus $f$ is one-to-one. If $\varepsilon > 0$, let $\delta = \varepsilon$. If $d(x, y) < \delta$, then $d(f(x), f(y)) = d(x, y) < \delta = \varepsilon$, so $f$ is continuous, in fact uniformly continuous, on $X$. It remains to show that $f$ maps $X$ onto $X$. Since $X$ is compact and $f$ is continuous, the image, $f(X)$, is a compact subset of the metric space $X$. So it must be closed. Its complement, $X \setminus f(X)$, must be open. If there were a point $x$ in $X \setminus f(X)$, then there would be a radius $r > 0$ such that $D(x, r) \subseteq X \setminus f(X)$. That is, $y \in f(X)$ implies $d(y, x) > r$. Consider the sequence defined by $x_0 = x$ and $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$. For positive integer $k$, let $f^k$ denote the composition of $f$ with itself $k$ times. If $n$ and $p$ are positive integers, then

$$d(x_{n+p}, x_n) = d(f^n \circ f^p(x), f^n(x)) = d(f^p(x), x) > r$$

since $f^p(x) \in f(X)$. The points in the sequence are pairwise separated by distances of at least $r$. This would prevent any subsequence from converging. But $X$ is sequentially compact by the Bolzano-Weierstrass Theorem. So there should be a convergent subsequence. This contradiction shows that there can be no such starting point $x$ for our proposed sequence. The complement $X \setminus f(X)$ must be empty. So $f(X) = X$ and $f$ maps $X$ onto $X$ as claimed.
4E-26. Let \( f : [a, b] \to \mathbb{R} \) be continuous and such that \( f'(x) \) exists on \([a, b]\) and \( \lim_{y \to x} f'(x) \) exists. Prove that \( f \) is uniformly continuous.

**Suggestion.** Use the limit of the derivative at \( a \) to get uniform continuity on a short interval \([a, a+2d]\). Use Theorem 4.6.2 to get uniform continuity on an overlapping interval \([a+d, b]\). Then combine the two results.

**Solution.** Let \( \varepsilon > 0 \), and suppose \( \lim_{y \to x} f'(x) = \lambda \). There is a \( \delta \) such that \( \varepsilon < 2d < a - b \) and \( |f'(x)| \leq |\lambda| + 1 \) for \( a < x < a + 2d \). As is Example 4.6.4, \( f \) is uniformly continuous on \([a, a+2d]\), and there is a \( \delta_1 > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( x \) and \( y \) are in \([a, a+2d]\) and \( |x - y| < \delta_1 \). Since \( f \) is continuous on \([a, b]\), it is continuous on the subinterval \([a + d, b]\). Since that interval is compact, \( f \) is uniformly continuous on it by the uniform continuity theorem, 4.6.2. There is a \( \delta_2 > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( x \) and \( y \) are in \([a+2d, b]\) and \( |x - y| < \delta_2 \). Now we take advantage of the overlap we have carefully arranged between our two subdomains. If \( x \) and \( y \) are in \([a, b]\) and \( |x - y| < \min(\delta_1, \delta_2, d/2) \), then either they are both in \([a, a+2d]\) or they are both in \([a+d, b]\) or both. If they are both in \([a, a+2d]\), then \( |f(x) - f(y)| < \varepsilon \) since \( |x - y| < \delta_1 \). If they are both in \([a+d, b]\), then \( |f(x) - f(y)| < \varepsilon \) since \( |x - y| < \delta_2 \). In any case, \( |f(x) - f(y)| < \varepsilon \) whenever \( x \) and \( y \) are in \([a, b]\) and \( |x - y| < \delta = \min(\delta_1, \delta_2, d/2) \). So \( f \) is uniformly continuous on \([a, b]\) as claimed.

4E-28. Let \( f : [0, 1] \to \mathbb{R} \) be uniformly continuous. Must \( f \) be bounded?

**Answer.** Yes.

**Solution.** If \( f \) were not bounded on \([0, 1]\), we could inductively select a sequence \( \{x_k\} \subseteq [0, 1] \) such that \( |f(x_{k+1})| > |f(x_k)| + 1 \) for each \( k \). In particular, we would have \( |f(x_k) - f(x_j)| > 1 \) whenever \( k \neq j \). But the points \( x_k \) are all in the compact interval \([0, 1]\), so there should be a subsequence converging to some point in \([0, 1]\). This subsequence would have to be a Cauchy sequence, so no matter how small a positive number \( \delta \) were specified, we could get points \( x_k \) and \( x_j \) in the subsequence with \( |x_k - x_j| < \delta \) and \( |f(x_k) - f(x_j)| > 1 \). This contradicts the uniform continuity of \( f \) on \([0, 1]\). So the image \( f([0, 1]) \) must, in fact, be bounded.

If we knew the result of Exercise 4E-24(c), then we would know that \( f \) has a unique continuous extension to the closure, \([0, 1]\). There is a continuous \( g : [0, 1] \to \mathbb{R} \) such that \( g(x) = f(x) \) for all \( x \) in \([0, 1]\). Since \( g \) is continuous on the compact domain \([0, 1]\), the image \( g([0, 1]) \) is compact and hence bounded. So \( f([0, 1]) = g([0, 1]) \subseteq g([0, 1]) \) is bounded.

4E-29. Let \( f : \mathbb{R} \to \mathbb{R} \) satisfy \( |f(x) - f(y)| \leq |x - y|^2 \). Prove that \( f \) is a constant. [Hint: What is \( f'(x) \)?]

**Suggestion.** Divide by \( x - y \) and let \( y \) tend to \( x \) to show that \( f'(x) = 0 \).

**Solution.** Suppose \( x_0 \in \mathbb{R} \). Then for \( x \neq x_0 \) we have \( |f(x) - f(x_0)| \leq |x - x_0|^2 \), so

\[
|f(x) - f(x_0)| \leq |x - x_0|^2.
\]

Letting \( x \to x_0 \), we find that \( \lim_{x \to x_0} (f(x) - f(x_0))/(x - x_0) = 0 \). So \( f'(x_0) \) exists and is equal to 0 for every \( x_0 \in \mathbb{R} \). If \( t \in \mathbb{R} \), there is, by the mean value theorem, a point \( x_0 \) between 0 and \( t \) such that \( |f(t) - f(0)| = |f'(x_0)(t - 0)| = |0(t - 0)| = 0 \). So \( f(t) = f(0) \) for all \( t \in \mathbb{R} \). Thus \( f \) is a constant function as claimed.

4E-30. (a) Let \( f : [0, \infty] \to \mathbb{R} \), \( f(x) = \sqrt{x} \). Prove that \( f \) is uniformly continuous.

(b) Let \( k > 0 \) and \( f(x) = (x - x^k)/\log x \) for \( 0 < x < 1 \) and \( f(0) = 0, f(1) = 1 - k \). Show that \( f : [0, 1] \to \mathbb{R} \) is continuous. Is \( f \) uniformly continuous?
Suggestion. (a) Use Theorem 4.6.2 to show that $f$ is uniformly continuous on $[0, 3]$ and Example 4.6.4 to show that $g$ is uniformly continuous on $[1, \infty)$. Then combine these results.

(b) Use L'Hôpital's Rule.

Solution. (a) Let $\epsilon > 0$. We know that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$, so it is certainly continuous on the compact domain $[0, 3]$. By the uniform continuity theorem, 4.6.2, it is uniformly continuous on that set. There is a $\delta_1 > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x$ and $y$ are in $[0, 3]$ and $|x - y| < \delta_1$.

We also know that $f$ is differentiable for $x > 0$ with $f'(x) = 1/(2\sqrt{x})$. So $|f'(x)| \leq 1/2$ for $x \geq 1$. As in Example 4.6.4, we can use the mean value theorem to conclude that if $b_2 = 2\epsilon$, and $x$ and $y$ are in $[1, \infty]$ with $|x - y| < \delta_2$, then there is a point $c$ between $x$ and $y$ such that $|f(x) - f(y)| = |f'(c)(x-y)| < (1/2)(2\epsilon) = \epsilon$.

Now take advantage of the overlap of our two domains. If $x$ and $y$ are in $[0, \infty]$ and $|x - y| < \delta = \min(\delta_1, \delta_2)$, then either $x$ and $y$ are both in $[0, 3]$ or both in $[1, \infty]$ or both. If they are both in $[0, 3]$, then $|f(x) - f(y)| < \epsilon$ since $|x - y| < \delta_1$. If they are both in $[1, \infty]$, then $|f(x) - f(y)| < \epsilon$ since $|x - y| < \delta_2$. In either case, $|f(x) - f(y)| < \epsilon$.

So $f$ is uniformly continuous on $[0, \infty]$ as claimed.

(b) Suppose $k$ is a positive integer and $f(x) = (x-x^k)/\log x$ for $0 < x < 1$, $f(0) = 0$, and $f(1) = 1-k$. The numerator, $x - x^k$, is continuous for all $x$. The denominator, $\log x$, is continuous for $x > 0$. So $f$ is continuous on $x > 0$ except possibly at $x = 1$ where the denominator is 0. However, the numerator is also 0 at $x = 1$. To apply L'Hôpital's Rule, we consider the ratio of the derivatives

$$
\frac{1-kx^{k-1}}{1/x} = x - kx^k \to 1 - k = f(1) \quad \text{as} \quad x \to 1.
$$

By L'Hôpital's Rule, $\lim_{x \to 1} (x-x^k)/\log x = \lim_{x \to 1} f(x)$ also exists and is equal to $f(1)$. So $f$ is continuous at 1. As $x \to 0^+$, the numerator of $f(x)$ tends to 0 and the denominator to $-\infty$. So $\lim_{x \to 0^+} f(x) = 0 = f(0)$. So $f$ is continuous from the right at 0. So $f$ is continuous on $[0, \infty]$ and on the smaller domain $[0, 1]$. Since the latter is compact, $f$ is uniformly continuous on it by the uniform continuity theorem, 4.6.2.

4E-37. Prove the following intermediate value theorem for derivatives: If $f$ is differentiable at all points of $[a, b]$, and if $f'(a)$ and $f'(b)$ have opposite signs, then there is a point $x_0 \in [a, b]$ such that $f'(x_0) = 0$.

Sketch. Suppose $f'(a) < 0 < f'(b)$. Since $f$ is continuous on $[a, b]$ (why?), it has a minimum at some $x_0$ in $[a, b]$. (Why?) $x_0 \neq a$ since $f(x) < f(a)$ for $x$ slightly larger than $a$. (Why?) $x_0 \neq b$ since $f(x) < f(b)$ for $x$ slightly smaller than $b$. (Why?) So $a < x_0 < b$. So $f'(x_0) = 0$. (Why?) The case of $f'(b) > 0 > f'(b)$ is similar.

Solution. We know from Proposition 4.7.2 that $f$ is continuous on the compact domain $[a, b]$. By the maximum-minimum theorem, 4.4.1, it attains a finite minimum, $m$, and a finite maximum, $M$, at points $x_1$ and $x_2$ in $[a, b]$.

CASE ONE: $f'(a) < 0 < f'(b)$: Since $f'(b) > 0$, and it is the limit of the difference quotients at $b$, we must have $(f(x) - f(b))/(x-b) > 0$ for $x$ slightly smaller than $b$. Since $x - b < 0$, we must have $f(x) < f(b)$ for such $x$. So the minimum does not occur at $b$. Since $f'(a) < 0$, and it is the limit of the difference quotients at $a$, we must have $(f(x) - f(a))/(x-a) < 0$ for $x$ slightly larger than $a$. Since $x - a > 0$, we must have $f(x) > f(a)$ for such $x$. So the minimum does not occur at $a$. Thus the minimum must occur at a point $x_0 \in [a, b]$. By Proposition 4.7.9, we must have $f'(x_0) = 0$.

CASE TWO: $f'(a) > 0 > f'(b)$: Since $f'(b) < 0$, and it is the limit of the difference quotients at $b$, we must have $(f(x) - f(b))/(x-b) < 0$ for $x$ slightly smaller than $b$. Since $x - b < 0$, we must have $f(x) > f(b)$ for such $x$. So the maximum does not occur at $b$. Since $f'(a) > 0$, and it is the limit of the difference quotients at $a$, we must have $(f(x) - f(a))/(x-a) > 0$ for $x$ slightly larger than $a$. Since $x - a > 0$, we must have $f(x) > f(a)$ for such $x$. So the maximum does not occur at $a$. Thus the maximum must occur at a point $x_0 \in [a, b]$. By Proposition 4.7.9, we must have $f'(x_0) = 0$. 
Suggestion. Show that the upper and lower sums are both 0 for every partition of \([0, 1]\). Consider a function which is 0 except at finitely many points.

Solution. Since \(f\) is integrable on \([0, 1]\), the upper and lower integrals are the same and are equal to the integral. Let \(P = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}\) be any partition of \([0, 1]\). For each subinterval \([x_{j-1}, x_j]\) there is a point \(c_j\) in it with \(f(c_j) = 0\). So

\[
\eta_j = \inf \{f(x) \mid x \in [x_{j-1}, x_j]\} \leq \sup \{f(x) \mid x \in [x_{j-1}, x_j]\} = M_j.
\]

So

\[
L(f, P) = \sum_{j=1}^{n} \eta_j (x_j - x_{j-1}) \leq 0 \leq \sum_{j=1}^{n} M_j (x_j - x_{j-1}) = U(f, P).
\]

This is true for every partition of \([0, 1]\). So

\[
\int_0^1 f(x) \, dx = \int_0^1 f(x) \, dx = \sup_{P \text{ a partition of } [0, 1]} L(f, P) \leq 0 \leq \inf_{P \text{ a partition of } [0, 1]} U(f, P) = \int_0^1 f(x) \, dx = \int_0^1 f(x) \, dx.
\]

So we must have \(\int_0^1 f(x) \, dx = 0\).

The function \(f\) need not be identically 0. We could, for example, have \(f(x) = 0\) for all but finitely many points at which \(f(x) = 1\).

If \(f\) is continuous and satisfies the stated condition, then \(f\) must be identically 0. Let \(x \in [0, 1]\). By hypothesis there is, for each integer \(n > 0\), at least one point \(c_n\) in \([0, 1]\) with \(x - (1/n) \leq c_n \leq x + (1/n)\) and \(f(c_n) = 0\). Since \(c_n \to 0\) and \(f\) is continuous, we must have \(0 = f(c_n) \to f(x)\). So \(f(x) = 0\).

4E-45. Prove the following second mean value theorem. Let \(f\) and \(g\) be defined on \([a, b]\) with \(g\) continuous, \(f \geq 0\), and \(f\) integrable. Then there is a point \(x_0 \in [a, b]\) such that

\[
\int_a^b f(x)g(x) \, dx = g(x_0) \int_a^b f(x) \, dx.
\]

Sketch. Let \(m = \inf(g([a, b]))\) and \(M = \sup(g([a, b]))\). Then

\[
m \int_a^b f(x) \, dx \leq \int_a^b f(x)g(x) \, dx \leq M \int_a^b f(x) \, dx.
\]

(Why?) Since \(\int_a^b f(x) \, dx\) depends continuously on \(f\), the intermediate value theorem gives \(t_0\) in \([m, M]\) with

\[
\int_a^b f(x)g(x) \, dx = t_0 \int_a^b f(x) \, dx.
\]

Now apply that theorem to \(g\) to get \(x_0\) with \(g(x_0) = t_0\). (Supply details.)

Solution. Since \(g\) is continuous on the compact interval \([a, b]\), we know that \(m = \inf(g([a, b]))\) and \(M = \sup(g([a, b]))\) exist as finite real numbers.
and that there are points $x_1$ and $x_2$ in $[a, b]$ where $g(x_1) = m$ and $g(x_2) = M$. Since $m \leq g(x) \leq M$ and $f(x) \geq 0$, we have $m \int_a^b f(x) \, dx \leq \int_a^b f(x)g(x) \, dx \leq M \int_a^b f(x) \, dx$ for all $x$ in $[a, b]$. Assuming that $f$ and $f \circ g$ are integrable on $[a, b]$, Proposition 4.8.3(iii) gives

$$m \int_a^b f(x) \, dx = \int_a^b mf(x) \, dx \leq \int_a^b f(x)g(x) \, dx \leq \int_a^b Mf(x) \, dx = M \int_a^b f(x) \, dx.$$

The function $h(t) = t \int_a^b f(x) \, dx$ is a continuous function of $t$ in the interval $m \leq t \leq M$, and $\int_a^b f(x)g(x) \, dx$ is a number between $h(m)$ and $h(M)$. By the intermediate value theorem there is a number $t_0$ in $[m, M]$ with $h(t_0) = \int_a^b f(x)g(x) \, dx$. Since $g$ is continuous between $x_1$ and $x_2$ and $t_0$ is between $m = g(x_1)$ and $M = g(x_2)$, another application of the intermediate value theorem gives a point $x_0$ between $x_1$ and $x_2$ with $g(x_0) = t_0$. So

$$\int_a^b f(x)g(x) \, dx = h(t_0) = t_0 \int_a^b f(x) \, dx = g(x_0) \int_a^b f(x) \, dx$$

as desired.