◇ **4.2-4.** Let $A, B \subset \mathbb{R}$, and suppose $A \times B \subset \mathbb{R}^2$ is connected.

(a) Prove that A is connected.
(b) Generalize to metric spaces.

Suggestion. Use Exercise 4.1-1(b). ◇

Solution. (a) Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. That is, $\pi_1((x, y)) = x$. Then π_1 is continuous. (See Exercise 4.1-1(b).) If B is not empty, then $\pi_1(A \times B) = A$. So A is connected by Theorem 4.2.1. If $B = \emptyset$, then $A \times B = \emptyset$, and $\pi_1(A \times B) = \emptyset$. We are still all right provided we realize that the empty set is connected. (Could there be sets U and V which disconnect it?)

(b) To generalize to metric spaces, let M_1 and M_2 be metric spaces with metrics d_1 and d_2. One way to put a metric on the cross product $M = M_1 \times M_2$ is to imitate the "taxicab" metric on $\mathbb{R}^2$

$$d((x, y), (a, b)) = d_1(x, a) + d_2(y, b)$$

so that a set $S \subseteq M$ would be open if and only if for each $(a, b) \in S$ there is an $r > 0$ such that $(x, y) \in S$ whenever $d_1(x, a) + d_2(y, b) < r$. We could also use a formula analogous to the Euclidean metric on $\mathbb{R}^2$:

$$\rho((x, y), (a, b)) = \sqrt{d_1(x, a)^2 + d_2(y, b)^2}.$$

Then

$$\begin{aligned} d((x, y), (a, b))^2 &= (d_1(x, a) + d_2(y, b))^2 \\ &= d_1(x, a)^2 + 2d_1(x, a)d_2(x, a) + d_2(x, a)^2 \\ &\geq d_1(x, a)^2 + d_2(x, a)^2 \\ &= (\rho((x, y), (a, b)))^2 . \end{aligned}$$

On the other hand,

$$\begin{aligned} d_1(x, a) + d_2(y, b) &\leq \sqrt{d_1(x, a)^2 + d_2(y, b)^2} + \sqrt{d_1(x, a)^2 + d_2(y, b)^2} \\ &\leq 2\rho((x, y), (a, b)). \end{aligned}$$

So

$$\rho((x, y), (a, b)) \leq d((x, y), (a, b)) \leq 2\rho((x, y), (a, b)).$$

These inequalities show that d and ρ produce the same open sets, the same closed sets, and the same convergent sequences with the same limits in M. Also, since

$$\begin{aligned} &d_1(x, a) \leq d((x, y), (a, b)) \text{ and } d_2(y, b) \leq d((x, y), (a, b)), \\ &\text{and } d((x, y), (a, b)) \leq \max(d_1(x, a), d_2(y, b)), \end{aligned}$$

we see that

$$\begin{aligned} (x_k, y_k) \to (a, b) \text{ in } M &\iff \rho((x_k, y_k), (a, b)) \to 0 \\ &\iff d((x_k, y_k), (a, b)) \to 0 \\ &\iff d_1(x_k, a) \to 0 \text{ and } d_2(y_k, b) \to 0 \\ &\iff x_k \to a \text{ in } M_1 \text{ and } y_k \to b \text{ in } M_2. \end{aligned}$$

This last observation shows that if we define the projections $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ by

$$\pi_1((x, y)) = x \in M_1 \text{ and } \pi_2((x, y)) = y \in M_2,$$

then π_1 and π_2 are continuous by Theorem 4.1.4(ii).
If $A \subseteq M_1$ and $B \subseteq M_2$, then $A = \pi_1(A \times B)$ and $B = \pi_2(A \times B)$. So Theorem 4.2.1 says that if $A \times B$ is connected, then so are A and B. If $A \times B$ is compact, then so are A and B.

For implications in the opposite direction, see Exercise 3E-15. ◆

◇ **4.2-5.** Let A and B be subsets of $\mathbb{R}$ with B not empty. If $A \times B \subseteq \mathbb{R}^2$ is open, must A be open?

Answer. Yes. ◇

Solution. Since B is not empty, there is a point $b \in B$. If $a \in A$, then $(a, b) \in A \times B$. Since $A \times B$ is open, there is an $r > 0$ such that $\sqrt{(x-a)^2 + (y-b)^2} < r$ implies $(x, y) \in A \times B$. If $|x - a| < r$, then $\sqrt{(x-a)^2 + (b-b)^2} = |x - a| < r$, so $(x, b) \in A \times B$. Thus $x \in A$. For each $a \in A$ there is an $r > 0$ such that $x \in A$ whenever $|x - a| < r$. So A is open. ♦

◇ **4.4-3.** Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be continuous on a compact set K and let $M = \{x \in K \mid f(x) \text{ is the maximum of } f \text{ on } K\}$. Show that M is a compact set.

Suggestion. $M = f^{-1}(\sup(f(x)))$. Why does this exist; why compact? ◇

Solution. The function f is a continuous function from the compact set K into $\mathbb{R}$. We know from the Maximum-Minimum Theorem that $b = \sup\{f(x) \in \mathbb{R} \mid x \in K\}$ exists as a finite real number and that there is at least one point x_0 in K such that $f(x_0) = b$. So $M = \{x \in K \mid f(x) \text{ is the maximum of } f \text{ on } K\}$ is not empty. In particular, $x_0 \in M$. The single point set $\{b\}$ is a closed set in $\mathbb{R}$. So $M = f^{-1}(\{b\})$ is a closed set in K. Since it is a closed subset of the compact set K, it is also a compact set by Lemma 2 to Theorem 3.1.3 (p. 165 of the text). ♦

◇ **4.5-3.** Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point.

Suggestion. Apply the intermediate value theorem to $g(x) = f(x) - x$. ◇

Solution. For $x \in [0, 1]$, let $g(x) = f(x) - x$. Since f is continuous and $x \mapsto -x$ is continuous, g is continuous. We have

$$g(0) = f(0) - 0 = f(0) \geq 0 \quad \text{and} \quad g(1) = f(1) - 1 \leq 1 - 1 = 0$$

By the intermediate value theorem, there must be at least one point c in $[0, 1]$ with $g(c) = 0$. For such a point we have $f(c) - c = 0$. So $f(c) = c$. Thus c is a fixed point for the mapping f as required. ♦

◇ **4.5-4.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that the range of f is a bounded closed interval.

Sketch. The interval $[a, b]$ is compact and connected and f is continuous. ◇

Solution. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. The interval $[a, b]$ is a connected set and f is continuous. So the image $f([0, 1])$ must be a connected subset of $\mathbb{R}$. So it must be an interval, a half-line, or the whole line. But $[a, b]$ is a closed, bounded subset of $\mathbb{R}$, so it is compact. Since f is continuous, the image $f([0, 1])$ must be compact in $\mathbb{R}$. It must thus be closed and bounded. Of the possibilities listed above, this leaves a closed, bounded interval. ♦

◇ **4.5-5.** Prove that there is no continuous map taking $[0, 1]$ onto $]0, 1[$.

Sketch. $f([0, 1])$ would be compact, and $]0, 1[$ is not compact. ◇

Solution. The interval $[0, 1]$ is a closed bounded subset of $\mathbb{R}$ and so it is compact. If f were a continuous function on it, then the image would have to be compact. The open interval $]0, 1[$ is not compact. So it cannot be the image of such a map. ♦

◇ **4.6-6.** (a) Show that $f : \mathbb{R} \to \mathbb{R}$ is *not* uniformly continuous iff there exist an $\varepsilon > 0$ and sequences x_n and y_n such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Generalize this statement to metric spaces.

(b) Use (a) on $\mathbb{R}$ to prove that $f(x) = x^2$ is not uniformly continuous.

Solution. (a) First suppose that $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous. Then there is an $\varepsilon > 0$ for which no $\delta > 0$ will work in the definition of uniform continuity. In particular, $\delta = 1/n$ will not work. So there must be a pair of numbers x_n and y_n such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| > \varepsilon$. These x_n and y_n form the required sequences. Conversely, if there are such sequences, then for that ε no $\delta > 0$ can work in the definition of uniform continuity. No matter what $\delta > 0$ is proposed, we can, by the Archimedean Principle, select an integer n with $0 < 1/n < \delta$. For the corresponding x_n and y_n we have $|f(x_n) - f(y_n)| > \varepsilon$ even though $|x_n - y_n| < 1/n < \delta$. Since this can be done for every $\delta > 0$, the function f cannot be uniformly continuous on $\mathbb{R}$.

◇ **4E-2.** (a) Prove that if $f : A \to \mathbb{R}^m$ is continuous and $B \subset A$, then the restriction $f|B$ is continuous.

(b) Find a function $g : A \to \mathbb{R}$ and a set $B \subset A$ such that $g|B$ is continuous but g is continuous at no point of A.

Solution. (a) Suppose $\varepsilon > 0$ and $x_0 \in B$. Then $x_0 \in A$, so there is a $\delta > 0$ such that $\| f(x) - f(x_0) \| < \varepsilon$ whenever $x \in A$ and $\| x - x_0 \| < \delta$. If $x \in B$, then it is in A, so

$$(x \in B \text{ and } \| x - x_0 \| < \delta) \implies \| f(x) - f(x_0) \| .$$

So f is continuous at x_0. Since x_0 was arbitrary in B, f is continuous on B.

(b) Let $A = \mathbb{R}$, $B = \mathbb{Q}$, and define $g : A \to \mathbb{R}$ by $g(x) = 1$ if $x \in \mathbb{Q}$ and $g(x) = 0$ if $x \notin \mathbb{Q}$. The restriction of g to $B = \mathbb{Q}$ is constantly equal to 1 on B. So it is continuous on B. (See Exercise 4E-1(b).) But, if $x_0 \in \mathbb{R}$, then there are rational and irrational points in every short interval around x_0. So g takes the values 1 and 0 in every such interval. The values of $g(x)$ cannot be forced close to any single value by restricting to a short interval around x_0. So, as a function on $\mathbb{R}$, g is not continuous at x_0. This is true for every $x_0 \in \mathbb{R}$. ♦

◇ **4E-3.** (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is connected, is $f^{-1}(K)$ necessarily connected?

(b) Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on all of $\mathbb{R}^n$ and $B \subset \mathbb{R}^n$ is bounded, then $f(B)$ is bounded.

Sketch. (a) No. Let $f(x) = \sin x$ and $K = \{1\}$.

(b) f is continuous on all of $\mathbb{R}^n$, so f is continuous on $\text{cl}(B)$ which is compact. $f(\text{cl}(B))$ is compact and thus bounded. So, since $f(B) \subseteq f(\text{cl}(B))$, $f(B)$ is also bounded. ◇

Solution. (a) The continuous image of a connected set must be connected, but not necessarily a *preimage*. For example. Let $f(x) = \sin x$ for all $x \in \mathbb{R}$, and let $K = \{1\}$. The one point set K is certainly connected. but $f^{-1}(K) = \{(4n+1)\pi/2 \mid n \in \mathbb{Z}\}$. This discrete set of points is not connected.

For an easier example, let $f(x) = x^2$ and $K = \{1\}$. Then f is continuous and $f^{-1}(K) = \{-1, 1\}$. This two point set is not connected.

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and B is a bounded subset of $\mathbb{R}^n$. Since B is bounded, there is a radius R such that $\| v \| < R$ for every $v \in B$. If $w \in \text{cl}(B)$, then there is a $v \in B$ with $v \in D(w, 1)$. So $\| w \| \leq \| w - v \| + \| v \| \leq 1 + R$. So $\text{cl}(B)$ is a bounded set in $\mathbb{R}^n$. It is also closed, so it is compact. The function f is continuous on all of $\mathbb{R}^n$, so it is continuous on $\text{cl}(B)$. (See Exercise 4E-2(a).) So the image $f(\text{cl}(B))$ is compact by Theorem 4.2.2. Since it is compact, it must be bounded. Since $B \subseteq \text{cl}(B)$, we have $f(B) \subseteq f(\text{cl}(B))$. Since it is a subset of a bounded set, the image $f(B)$ is bounded. ♦

◇ **4E-7.** Consider a compact set $B \subset \mathbb{R}^n$ and let $f : B \to \mathbb{R}^m$ be continuous and one-to-one. Then prove that $f^{-1} : f(B) \to B$ is continuous. Show by example that this may fail if B is connected but not compact. (To find a counterexample, it is necessary to take $m > 1$.)

Sketch. Suppose C is a closed subset of B. Then C is compact. (Why?) So $f(C)$ is closed. (Why?) Thus f^{-1} is continuous. (Why?) For a counterexample with $n = 2$ consider $f : [0, 2\pi[\to \mathbb{R}^2$ given by $f(t) = (\sin t, \cos t)$.

Solution. FIRST PROOF: We use the characterization of continuity in terms of closed sets. To show that $f^{-1} : f(B) \to B$ is continuous on $f(B)$, we need to show that if C is a closed subset of the metric space B, then $(f^{-1})^{-1}(C)$ is closed relative to $f(B)$. Since B is a compact subset of $\mathbb{R}^n$, it is closed, and a subset C of it is closed relative to B if and only if it is closed in $\mathbb{R}^n$. Since it is a closed subset of the compact set B it is closed. (In $\mathbb{R}^n$ this follows since it is closed and bounded. However, it is true more generally. See Lemma 2 to the proof of the Bolzano-Weierstrass Theorem, 3.1.3, at the end of Chapter 3: *A closed subset of a compact space is compact.*) Since C is a compact subset of B and f is continuous on B and hence on C, the image $f(C)$ is compact. Since it is a compact subset of a metric space, it is closed. (See Lemma 1 to the proof of 3.1.3.) But since f is one-to-one, $f(C) = (f^{-1})^{-1}(C)$. Thus $(f^{-1})^{-1}(C)$ is closed for every closed subset C of $f(B)$. The inverse f^{-1} is thus a continuous function from $f(B)$ to B.

SECOND PROOF: Here is a proof using sequences. Suppose $y \in f(B)$ and $\langle y_k \rangle_1^\infty$ is a sequence in $f(B)$ with $y_k \to y$. We want to show that $x_k = f^{-1}(y_k) \to x = f^{-1}(y)$ in B. Since B is compact, there is a subsequence $x_{k(1)}, x_{k(2)}, x_{k(3)}, \ldots$ converging to some point $\hat{x} \in B$. Since f is continuous on B, we must have $y_{k(j)} = f(x_{k(j)}) \to f(\hat{x})$. But $y_{k(j)} \to y$. Since limits are unique in the metric space $f(B)$, we must have $y = f(\hat{x})$. But $y = f(x)$ and f is one-to-one, so $x = \hat{x}$. Not only does this argument show that there must be some subsequence of the x_k converging to x, it shows that x is the only possible limit of a subsequence. Since B is compact, every subsequence would have to have a sub-subsequence converging to something, and the only possible "something" is x. Thus $x_k \to x$ as needed.

If the domain is not compact, for example the half-open interval $B = [0, 2\pi[$, then we can get a counterexample. The map $f : [0, 2\pi[\to \mathbb{R}^2$ given by $f(t) = (\sin t, \cos t)$ takes $[0, 2\pi[$ onto the unit circle. The point $(0, 1)$ has preimages near 0 and near 2π. So the inverse function is not continuous at $(0, 1)$.

It turns out that a continuous map from a half-open interval one-to-one into $\mathbb{R}$ must have a continuous inverse. Challenge: Prove it. ♦

◇ **4E-8.** Define maps $s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as addition and scalar multiplication defined by $s(x, y) = x + y$ and $m(\lambda, x) = \lambda x$. Show that these mappings are continuous.

Solution. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then $s(x, y) = x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. and $m(\lambda, x) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$.

We compute

$$\begin{aligned}\| s(x,y) - s(u,v) \|_{\mathbb{R}^n} &= \| (x+y) - (u+v) \|_{\mathbb{R}^n} = \| (x-u) + (y-v) \|_{\mathbb{R}^n} \\ &\le \| x-u \|_{\mathbb{R}^n} + \| y-v \|_{\mathbb{R}^n} \\ &\le 2\sqrt{\| x-u \|^2_{\mathbb{R}^n} + \| y-v \|^2_{\mathbb{R}^n}} \\ &\le 2 \| (x-u, y-v) \|_{\mathbb{R}^n \times \mathbb{R}^n} = 2 \| (x,y) - (u,v) \|_{\mathbb{R}^n \times \mathbb{R}^n} .\end{aligned}$$

So, if $\| (x,y) - (u,v) \|_{\mathbb{R}^n \times \mathbb{R}^n} < \varepsilon/2$, then $\| s(x,y) - s(u,v) \|_{\mathbb{R}^n} < \varepsilon$. So s is continuous.

Fix $(\mu, u) \in \mathbb{R} \times \mathbb{R}^n$. We have

$$\| (\lambda, x) - (\mu, u) \|^2_{\mathbb{R}^n \times \mathbb{R}^n} = \| (\lambda - \mu, x-u) \|^2_{\mathbb{R}^n \times \mathbb{R}^n} = (\lambda - \mu)^2 + \| x-u \|^2_{\mathbb{R}^n} .$$

Thus if $\| (\lambda, x) - (\mu, u) \|_{\mathbb{R}^n \times \mathbb{R}^n} < \delta$ then $|\lambda - \mu| < \delta$ and $\| x-u \|_{\mathbb{R}^n} < \delta$.

$$\begin{aligned}\| m(\lambda, x) - m(\mu, u) \|_{\mathbb{R}^n} &= \| \lambda x - \mu u \|_{\mathbb{R}^n} = \| \lambda x - \lambda u + \lambda u - \mu u \|_{\mathbb{R}^n} \\ &\le \| \lambda x - \lambda u \|_{\mathbb{R}^n} + \| \lambda u - \mu u \|_{\mathbb{R}^n} = |\lambda| \, \| x-u \|_{\mathbb{R}^n} + |\lambda - \mu| \, \| u \|_{\mathbb{R}^n} \\ &\le |\lambda| \, \delta + \delta \, \| u \|_{\mathbb{R}^n} \le (|\mu| + \delta)\delta + +\delta \, \| u \|_{\mathbb{R}^n} .\end{aligned}$$

If we require that $\delta < 1$ and $\delta < \varepsilon/2(|\mu|+1)$ and $\delta < \varepsilon/2(\| u \| + 1)$, we have

$$\begin{aligned}\| m(\lambda, x) - m(\mu, u) \|_{\mathbb{R}^n} &\le (|\mu| + \delta)\delta + \delta \, \| u \|_{\mathbb{R}^n} \\ &\le (|\mu|+1)\frac{\varepsilon}{2(|\mu|+1)} + \frac{\varepsilon}{2(\| u \| + 1)} \| u \| < \varepsilon.\end{aligned}$$

Thus m is continuous. ♦

◇ **4E-10.** Show that $f : A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, is continuous iff for every set $B \subset A$, $f(\mathrm{cl}(B) \cap A) \subset \mathrm{cl}(f(B))$.

Solution. Suppose f is continuous on A and $B \subseteq A$. Then $B \subseteq f^{-1}(f(B)) \subseteq f^{-1}(\mathrm{cl}(f(B)))$. So $f^{-1}(\mathrm{cl}(f(B))) = A \cap F$ for some closed set F since f is continuous on A and $\mathrm{cl}(f(B))$ is closed. So $B \subseteq F$ and $\mathrm{cl}(B) \subseteq F$. Thus

$$\mathrm{cl}(B) \cap A \subseteq F \cap A = f^{-1}(\mathrm{cl}(f(B))).$$

This implies that $f(\mathrm{cl}(B) \cap A) \subseteq f(f^{-1}\mathrm{cl}(f(B))) = \mathrm{cl}(f(B))$ as required.

For the converse, suppose $f(\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B))$ for every subset B of A. Let C be any closed set in $\mathbb{R}^m$ and put $B = f^{-1}(C)$. The hypothesis gives

$$f(\mathrm{cl}(f^{-1}(C)) \cap A) \subseteq \mathrm{cl}(f(f^{-1}(C))) \subseteq \mathrm{cl}(C) = C.$$

So

$$\mathrm{cl}(f-1(C)) \cap A \subseteq f^{-1}(f(\mathrm{cl}(f-1(C)) \cap A)) \subseteq f^{-1}(C) \subseteq \mathrm{cl}(f-1(C)) \cap A.$$

We must have $f^{-1}(C) = \mathrm{cl}(f-1(C)) \cap A$. So $f^{-1}(C)$ is closed relative to A. Since this is true for every closed set C in $\mathbb{R}^m$, f is continuous on A. ♦

◇ **4E-11.** (a) For $f :]a, b[\to \mathbb{R}$, show that if f is continuous, then its graph Γ is path-connected. Argue intuitively that if the graph of f is path-connected, then f is continuous. (The latter is true, but it is a little more difficult to prove.)

(b) For $f : A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, show that for $n \ge 2$, connectedness of the graph does not imply continuity. [Hint: For $f : \mathbb{R}^2 \to \mathbb{R}$, cut a slit in the graph.]

(c) Discuss (b) for $m = n = 1$. [Hint: On $\mathbb{R}$, consider $f(x) = 0$ if $x = 0$ and $f(x) = \sin(1/x)$ if $x > 0$.]

Sketch. (a) If $(c, f(c))$ and $(d, f(d))$ are on the graph, put $\gamma(t) = (t, f(t))$. ◊

Solution. (a) If $(c, f(c))$ and $(d, f(d))$ are two points on the graph, then either $a < c < d < b$ or $a < d < c < b$. We may as well assume the first. The map $\gamma : [c, d] \to \mathbb{R}^2$ given by $\gamma(t) = (t, f(t))$ is a continuous path in the graph joining $(c, f(c))$ to $d, f(d))$ by Worked Example 4WE-1 since both coordinate functions are continuous. Thus the graph is path-connected.

An intuitive argument for the converse is that the only paths possible in the graph must be of the form $\gamma(t) = (u(t), f(u(t)))$. For this to be continuous, u and f should be continuous.

(b) Represent a point $v \in \mathbb{R}^2$ in polar coordinates $v = (r, \vartheta)$ where $r = \| v \|$ and $0 \leq \vartheta < 2\pi$ is the polar angle counterclockwise from the positive horizontal axis. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(v) = r\vartheta$ and $f(0, 0) = 0$. Then the graph of f is connected since it is path-connected. But f is not continuous since it has a jump discontinuity across the positive x-axis.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Although the graph of f is not path-connected, it is connected since any open set containing $(0, 0)$ would also contain points in the path-connected portion of the graph. The function is not continuous since it attain all values in the interval $[-1, 1]$ in every neighborhood of 0. ◆

◊ **4E-13.** Let f be a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. Prove that $f(U)$ is open for all open sets $U \subset \mathbb{R}^n$ iff for all nonempty open sets $V \subset \mathbb{R}^n$,

$$\inf_{x \in V} f(x) < f(y) < \sup_{x \in V} f(x)$$

for all $y \in V$.

Sketch. If $\inf(f(V))$ or $\sup(f(V))$ were in $f(V)$, then $f(V)$ could not be open since it could not contain an interval around either of these points. ◊

Solution. First suppose that $f(u)$ is an open subset of $\mathbb{R}$ for every open $U \subseteq \mathbb{R}^n$, and let V be a nonempty subset of $\mathbb{R}^n$. If $\inf(f(V)) = a = f(y)$ for some $y \in V$, then $a = f(y)$ cannot be an interior point of $f(V)$ since $f(V)$ can contain no points smaller than a. So $f(V)$ would not be open contrary to hypothesis. Similarly, if $\sup(f(V)) = b = f(y)$ for some $y \in V$, then b would be in $f(V)$ but could not be an interior point of $f(V)$ since $f(V)$ could contain no points larger than b. Again $f(V)$ would not be open contrary to hypothesis. Since we always have $\inf_{x \in V} f(x) \leq f(y) \leq \sup_{x \in V} f(x)$ for all y in V and we have just shown that neither equality can occur, we must have $\inf_{x \in V} f(x) < f(y) < \sup_{x \in V} f(x)$ as claimed.

For the converse, suppose that for every open set V in $\mathbb{R}^n$ we have $\inf_{x \in V} f(x) < f(y) < \sup_{x \in V} f(x)$ for every y in V, and let U be an open subset of $\mathbb{R}^n$. We want to show that $f(U)$ is open. So let $y = f(x) \in f(U)$. Since $x \in U$ and U is open, there is an $r > 0$ with $D(x, 2r) \subseteq U$. So $\text{cl}(D(x, r)) \subseteq U$. Let $K = \text{cl}(D(x, r))$. The closed disk K is a closed bounded set in $\mathbb{R}^n$ and so is compact. Since f is continuous, the image, $f(K)$, is a connected, compact set in $\mathbb{R}$. So it must be a closed, bounded interval containing y. $y \in f(K) = [a, b]$. Let V be the open disk $D(x, r)$. We have $y \in f(V) \subseteq f(K) = [a, b]$. By hypothesis,

$$a = \inf(f(K)) \leq \inf(f(V)) < y < \sup(f(V)) \leq \sup(F(K)) = b.$$

Let ρ be the smaller of $(b - y)/2$ and $(y - a)/2$. Then $]a + \rho, b - \rho[$ is an open interval in $f(U)$ containing y. Since this can be done for any y in $f(U)$, the set $F(U)$ is open as claimed. ◆

◇ **4E-18.** Let $A \subset M$ be connected and let $f : A \to \mathbb{R}$ be continuous with $f(x) \neq 0$ for all $x \in A$. Show that $f(x) > 0$ for all $x \in A$ or else $f(x) < 0$ for all $x \in A$.

Solution. If there were a points x_1 and x_2 with $f(x_1) < 0$ and $f(x_2) > 0$, then by the intermediate value theorem, 4.5.1, there would be a point z in A with $f(z) = 0$. By hypothesis, this does not happen. So $f(x)$ must have the same sign for all x in A. ♦