◇ **3E-2.** Prove that a set $A \subset \mathbb{R}^n$ is not connected iff we can write $A \subset F_1 \cup F_2$, where F_1, F_2 are closed, $A \cap F_1 \cap F_2 = \varnothing$, $F_1 \cap A \neq \varnothing$, $F_2 \cap A \neq \varnothing$.

Suggestion. Suppose U_1 and U_2 disconnect A and consider the sets $F_1 = M \setminus U_1$ and $F_2 = M \setminus U_2$. ◇

Solution. By definition, A is not connected if and only if there are open sets U_1 and U_2 such that

1. $U_1 \cap U_2 \cap A = \emptyset$
2. $U_1 \cap A$ is not empty
3. $U_2 \cap A$ is not empty
4. $A \subseteq U_1 \cup U_2$.

But U_1 and U_2 are open if and only if their complements $F_1 = M \setminus U_1$ and $F_2 = \setminus U_2$ are closed. Using DeMorgan's Laws, our four conditions translate to

1. $F_1 \cup F_2 \cup (M \setminus A) = M$
2. $F_1 \cup A$ is not all of M
3. $F_2 \cup A$ is not all of M
4. $M \setminus A \supseteq F_1 \cap F_2$.

Since none of the points in A are in $M \setminus A$, condition 1 is equivalent to $A \subseteq F_1 \cup F_2$. Condition 4 is equivalent to $A \cap F_1 \cap F_2 = \emptyset$. So, with a bit more manipulations, our conditions become

1. $A \subseteq F_1 \cup F_2$
2. $F_1 \cap A = (M \setminus U_1) \cap A = (M \setminus U_1) \cap (M \setminus (M \setminus A)) = M \setminus (U_1 \cup (M \setminus A))$. This is not empty since there are points in $U_2 \cap A$ and these cannot be in U_1.
3. $F_2 \cap A = (M \setminus U_2) \cap A = (M \setminus U_2) \cap (M \setminus (M \setminus A)) = M \setminus (U_2 \cup (M \setminus A))$. This is not empty since there are points in $U_1 \cap A$ and these cannot be in U_2.
4. $A \cap F_1 \cap F_2 = \emptyset$.

These are exactly the conditions required of the closed sets F_1 and F_2 in the problem.

In the converse direction, if we have closed sets F_1 and F_2 satisfying

1. $A \subset F_1 \cup F_2$
2. $F_1 \cap A$ not empty
3. $F_2 \cap A$ not empty
4. $A \cap F_1 \cap F_2 = \emptyset$,

we can consider the open sets $U_1 = M \setminus F_1$ and $U_2 = M \setminus F_2$ and work backwards through these manipulations to obtain

1. $U_1 \cap U_2 \cap A = \emptyset$
2. $U_1 \cap A$ is not empty
3. $U_2 \cap A$ is not empty
4. $A \subseteq U_1 \cup U_2$,

so that A is not connected. ◆

◇ **3E-16.** If $x_k \to x$ in a normed space, prove that $||x_k|| \to ||x||$. Is the converse true? Use this to prove that $\{x \in \mathbb{R}^n \mid ||x|| \le 1\}$ is closed, using sequences.

Sketch. Use the alternative form of the triangle inequality:

$$\Big| \| v \| - \| w \| \Big| \le \| v - w \| . \qquad \Diamond$$

Solution. Recall that we have an alternative form of the triangle inequality.

Lemma. *If v and w are vectors in a normed space $\mathcal{V}$, then* $\big| \| v \| - \| w \| \big| \le \| v - w \|$.

Proof: First we compute $\| v \| = \| v - w + w \| \le \| v - w \| + \| w \|$. So

$$\| v \| - \| w \| \le \| v - w \| . \tag{1}$$

Reversing the roles of v and w, we find that $\| w \| = \| w - v + v \| \le \| w - v \| + \| v \| = \| v - w \| + \| v \|$. So that

$$\| w \| - \| v \| \le \| v - w \| . \tag{2}$$

Combining (1) and (2), we find that $\big| \| v \| - \| w \| \big| \le \| v - w \|$ as claimed.

With this version of the triangle inequality, the first assertion of the exercise is almost immediate. Let $\varepsilon > 0$. Since $x_k \to x$, there is an N such that $\| x_k - x \| < \varepsilon$ whenever $k \ge N$. For such k we have $\big| \| x_k \| - \| x \| \big| \le \| x_k - x \| < \varepsilon$. So $\| x_k \| \to \| x \|$ as claimed.

The converse is not true, even in one dimension. Let $x_k = (-1)^k$ and $x = 1$ then $\| x_k \| = 1 = \| x \|$ for each index k. We certainly have $\| x_k \| \to \| x \|$, but the points x_k do not tend to x. Now suppose $\mathcal{V}$ is a normed space and let $F = \{x \in \mathcal{V} \mid \| x \| \le 1\}$. If $x \in \mathrm{cl}(F)$, then there is a sequence $\langle x_k \rangle_1^\infty$ in F with $x_k \to x$. We now know that this implies that $\| x_k \| \to \| x \|$. But $\| x_k \| \le 1$ for each k, so we must have $\| x \| \le 1$ also. Thus $x \in F$. This shows that $\mathrm{cl}(F) \subseteq F$ so that F is closed as claimed. ♦

◇ **3E-19.** Let $V_n \subset M$ be open sets such that $\mathrm{cl}(V_n)$ is compact, $V_n \ne \varnothing$, and $\mathrm{cl}(V_n) \subset V_{n-1}$. Prove $\bigcap_{n=1}^\infty V_n \ne \varnothing$.

Sketch. $\mathrm{cl}(V_{n+1}) \subseteq V_n \subseteq \mathrm{cl}(V_n)$. Use the nested set property. $\Diamond$

Solution. Let $K_n = \mathrm{cl}(V_n)$, we have assumed that the sets K_n are compact, not empty, and that $\mathrm{cl}(V_k) \subseteq V_{k-1}$ for each k. Applying this with $k = n + 1$ gives

$$K_{n+1} = \mathrm{cl}(V_{n+1}) \subseteq V_n \subseteq \mathrm{cl}(V_n) = K_n .$$

So we have a nested sequence of nonempty compact sets. By the nested set property, there must be at least one point x_0 in the intersection $\bigcap_1^\infty K_n$. For each n we have $x_0 \in K_n - 1 \subseteq V_n$. Thus $x_0 \in \bigcap_1^\infty V_n$, and this intersection is not empty. ♦

◇ **3E-23.** Let $\mathbb{Q}$ denote the rationals in $\mathbb{R}$. Show that both $\mathbb{Q}$ and the irrationals $\mathbb{R}\backslash\mathbb{Q}$ are not connected.

Sketch. $\mathbb{Q} \subset]-\infty, \sqrt{2}[\cup]\sqrt{2}, \infty[$; both intervals are open, they are disjoint. They disconnect $\mathbb{Q}$. Similarly $\mathbb{R}\backslash\mathbb{Q} \subset]-\infty, 0[\cup]0, \infty[$ disconnects $\mathbb{R} \setminus \mathbb{Q}$. ◇

Solution. To show that $\mathbb{Q} \subseteq \mathbb{R}$ is not connected, recall that $\sqrt{2}$ is not rational. The two open half lines $U = \{x \in \mathbb{R} \mid x < \sqrt{2}\}$ and $V = \{x \in \mathbb{R} \mid x > \sqrt{2}\}$ are disjoint. Each intersects $\mathbb{Q}$ since $0 \in U$ and $3 \in V$. Their union is $\mathbb{R} \setminus \{\sqrt{2}\}$ which contains $\mathbb{Q}$. Thus U and V disconnect $\mathbb{Q}$.

To show that $\mathbb{R} \setminus \mathbb{Q}$ is not connected, we do essentially the same thing but use a rational point such as 0 as the separation point. Let $U = \{x \in \mathbb{R} \mid x < 0\}$ and $V = \{x \in \mathbb{R} \mid x > 0\}$. Then U and V are disjoint open half lines. Each intersects $\mathbb{R} \setminus \mathbb{Q}$ since $-\sqrt{2} \in U$ and $\sqrt{2} \in V$. Their union is $\mathbb{R} \setminus \{0\}$ which contains $\mathbb{R} \setminus \mathbb{Q}$. Thus U and V disconnect $\mathbb{R} \setminus \mathbb{Q}$. ♦

◇ **3E-26.** Show that the completeness property of $\mathbb{R}$ may be replaced by the *Nested Interval Property*: If $\{F_n\}_1^\infty$ is a sequence of closed bounded intervals in $\mathbb{R}$ such that $F_{n+1} \subset F_n$ for all $n = 1, 2, 3, \ldots$, then there is at least one point in $\bigcap_{n=1}^\infty F_n$.

Solution. We want to show that the following two assertions are equivalent:

Completeness Property: *If $\langle a_n \rangle_1^\infty$ is a monotonically increasing sequence bounded above in $\mathbb{R}$, then it converges to some point in $\mathbb{R}$.*

Nested Interval Property: *If $\langle F_n \rangle_1^\infty$ is a sequence of closed bounded intervals in $\mathbb{R}$ such that $F_{n+1} \subseteq F_n$ for all $n = 1, 2, 3, \ldots$, then there is at least one point in $\bigcap_{n=1}^\infty F_n$.*

The development of the text starts with the assumption of the completeness property and proves that closed bounded intervals are compact. The nested interval property as stated then follows from the Nested Set Property (Theorem 3.3.1) which says that the intersection of a nested sequence of nonempty compact sets is not empty.

What is needed now is to start with the assumption of the nested interval property and to show that the completeness property can be proved from it. So, suppose the nested interval property is true in $\mathbb{R}$ and $\langle a_n \rangle_1^\infty$ is an increasing sequence in $\mathbb{R}$ with $a_n \leq B \in \mathbb{R}$ for all n. We want to show that the a_n must converge to a limit in $\mathbb{R}$. The idea is to use the points a_n as the left ends of a nested family of intervals. The right ends, b_n, are to be selected so that the lengths of the intervals $[a_n, b_n]$ tend to 0. If we can manage this we will apply the following variation on the nested interval property.

Lemma. *If $F_n = [a_n, b_n]$, $n = 1, 2, 3, \ldots$ are closed bounded intervals in $\mathbb{R}$ such that $F_{n+1} \subseteq F_n$ for each n and $b_n = a_n \to 0$ as $n \to \infty$, then there is exactly one point λ in $\bigcap_{n=1}^\infty F_n$ and $a_n \to \lambda$ and $b_n \to \lambda$.*

Proof: That there is at least one point λ in the intersection follows from the nested interval property. Any other point μ would be excluded from any of the intervals with $b_n - a_n < |\lambda - \mu|$, so there is exactly one point in the intersection. If $\varepsilon > 0$, then there is an index N with $b_N - a_N < \varepsilon$. If $n \geq N$, we have $a_N \leq a_n \leq \lambda \leq b_n \leq b_N$. So $|\lambda - a_n| < \varepsilon$, and $|b_n - \lambda| < \varepsilon$. Thus $a_n \to \lambda$ and $b_n \to \lambda$ as claimed.

Returning to our original problem, we let $b_0 = B$ so that $b_0 \geq a_m$ for all m. The right hand endpoints b_n are produced inductively. Having selected $b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_n$ with $b_n \geq a_m$ for all m, we need to specify b_{n+1}. If there is a positive integer k with $b_n - (1/k) \geq x_m$ for all m, then we let k_n be the smallest such integer and put $b_{n+1} = b_n - (1/k_n)$. If there is no such integer, put $b_{n+1} = b_n$. This produces a sequence $\langle b_n \rangle_1^\infty$ such that $b_{n+1} \leq b_n$ for each n and $b_n \geq a_m$ for all indices n and m. We will be able to use the lemma to conclude that there is a real number λ to which the points a_n converge as soon as we know that $b_n - a_n \to 0$.

If $b_n - a_n$ did not tend to 0, there would be an $\varepsilon > 0$ such that $b_m - a_m > \varepsilon$ for all indices m. Focus attention on one of the b_n. If $m \geq n$ we would have $a_n \leq a_m < b_m \leq b_n$ so $b_n - a_m \geq b_m - a_m > \varepsilon$. If $m < n$ we would have $a_m \leq a_n < b_n \leq b_m$, so again $b_n - a_m \geq b_n - a_n > \varepsilon$. So $b_n - \varepsilon > a_m$ for all indices m and n. Fix a positive integer N with $0 < 1/N < \varepsilon$. Then $b_n - (1/N) > x_m$ for all m, so $1 \leq k_n \leq N$. This would imply that

$$b_{n+1} = b_n - \frac{1}{k_n} \leq b_n - \frac{1}{N} \qquad \text{for every index } n.$$

This would mean that the points b_n would tend to $-\infty$ and in a finite number of steps would be smaller than a_1. Since this is not the case, we conclude that $b_n - a_n \to 0$. The length of our closed intervals tends to 0, we can apply the lemma to conclude that there is exactly one point in their intersection and that the right ends, a_n converge to that point.

◇ **3E-28.** Let $A \subset M$ be connected and contain more than one point. Show that every point of A is an accumulation point of A.

Solution. Suppose a and b are different points in A and that a is not an accumulation point of A. Then there is a radius $r > 0$ such that $D(a, r) \cap A = \{a\}$. Let $U = \{x \in M \mid d(x, a) < r/3\}$ and $V = \{x \in M \mid d(x, a) > 2r/3\}$. Then U and V are open (why?) and disjoint. Neither $A \cap U$ nor $A \cap V$ is empty since $a \in A \cap U$ and $b \in A \cap V$. But $A \subseteq U \cup V$ since there are no points x in A with $r/3 \leq d(x, a) \leq 2r/3$. So the sets U and V would disconnect A. Since A is connected, this is not possible. Thus a must be an accumulation point of A. Since a was arbitrary in A assuming that there was at least one other point in A, this establishes our assertion. ♦

◇ **3E-31.** Suppose $A \subset \mathbb{R}^n$ is not compact. Show that there exists a sequence $F_1 \supset F_2 \supset F_3 \cdots$ of closed sets such that $F_k \cap A \neq \varnothing$ for all k and

$$\left(\bigcap_{k=1}^{\infty} F_k\right) \bigcap A = \varnothing.$$

Suggestion. The set A must be either not closed or not bounded or both. Treat these cases separately. If A is not closed, there must be an accumulation point of A which is not in A. ◇

Solution. If A is not bounded, let $F_k = \{v \in \mathbb{R}^n \mid \| v \| \geq k\}$. Since A is not bounded, $F_k \cap A$ is not empty. However, $\bigcap_{k=1}^{\infty} F_k = \emptyset$, so $(\bigcap_{k=1}^{\infty} F_k) \cap A$ is certainly empty.

If A is not closed, then there is a point $v_0 \in \mathrm{cl}(A) \setminus A$. Such a point must be an accumulation point of A. For $k = 1, 2, 3, 4, \ldots$, let $F_k = \{v \in \mathbb{R}^n \mid \| v - v_0 \| \leq 1/k\}$. Then each F_k is closed and each F_k intersects A since v_0 is an accumulation point of A. We certainly have $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$. But $\bigcap_{k=1}^{\infty} F_k = \{v_0\}$ and v_0 is not in A. So $(\bigcap_{k=1}^{\infty} F_k) \cap A = \emptyset$.

If A is not compact then it is either not bounded or not closed or both, so at least one of the previous paragraphs applies. ♦

◇ **3E-33.** ***Baire category theorem.*** A set S in a metric space is called ***nowhere dense*** if for each nonempty open set U, we have $\mathrm{cl}(S) \cap U \neq U$, or equivalently, $\mathrm{int}(\mathrm{cl}(S)) = \varnothing$. Show that $\mathbb{R}^n$ cannot be written as the countable union of nowhere dense sets.

Sketch. If $A_1, A_2, A_3, \ldots$ are closed nowhere dense sets, put $B_k = \mathbb{R}^n \setminus A_k$ to see that the assertion is implied by the following theorem.

Theorem. *If $B_1, B_2, \ldots$ are open dense subsets of $\mathbb{R}^n$, then $B = \bigcap_k B_k$ is dense in $\mathbb{R}^n$.*

Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Inductively define a nested sequence of closed disks $B(x_k, r_k) = \{y \mid \| x_k - y \| \leq r_k\}$ by

$$D_1 = B(x_1, r_1) \text{ where } x_1 \in B_1, \quad \| x - x_1 \| < \varepsilon/3, \quad r_1 < \varepsilon/3, \text{ and } B(x_1, r_1) \subset B_1$$

$$D_2 = B(x_2, r_2) \text{ where } x_2 \in B_2, \quad \| x_1 - x_2 \| < r_1/3, \quad r_2 < r_1/3, \text{ and } B(x_2, r_2) \subset B_2$$

etc.

Check that $D_k \subseteq B_k$ for each k and $D(x, \varepsilon) \supseteq \mathrm{cl}(D_1) \supseteq D_1 \supseteq \mathrm{cl}(D_2) \supseteq D_2 \supseteq \cdots$. Existence of $y \in \bigcap_k \mathrm{cl}(D_k)$ shows that B is dense in $\mathbb{R}^n$. (Why?) ♢

Solution. To say that a set is "nowhere dense" is to say that its closure has empty interior.

$$A \text{ is nowhere dense} \iff (U \text{ open} \implies U \cap \mathrm{cl}(A) \neq U) \iff \mathrm{int}(\mathrm{cl}(A)) = \emptyset.$$

Suppose A is such a set and B is the complement of A. If $v_0 \in A$, then every neighborhood of v_0 must intersect B since otherwise v_0 would be an interior point of A, and there are none. So $v_0 \in \mathrm{cl}(B)$. Thus $A \subseteq \mathrm{cl}(B)$. Since $B \subseteq \mathrm{cl}(B)$ also, we see that the whole space is contained in the closure of B. That is, B is "dense". Conversely, if B is dense, then its complement can have no interior. If A is closed so that $A = \mathrm{cl}(A)$, this says that A is nowhere dense and that B is open.

◇ **3E-34.** Prove that each closed set $A \subset M$ is an intersection of a countable family of open sets.

Suggestion. Consider $U_k = \bigcup_{x \in A} D(x, 1/k)$. ♢

Solution. For each positive integer k, let $U_k = \bigcup_{x \in A} D(x, 1/k)$. Since each of the disks $D(x, 1/k)$ is open, their union U_k is also open. Since $x \in D(x, 1/k)$, we certainly have $A \subseteq \bigcup_{x \in A} D(x, 1/k) = U_k$. Since this is true for each k, we have $A \subseteq \bigcap_{k+1}^{\infty} U_k$. On the other hand, if y is in this intersection and $\varepsilon > 0$, then we can find an integer k with $0 < 1/k < \varepsilon$. Since $y \in U_k$, there is an $x \in A$ with $y \in D(x, 1/k) \subseteq D(x, \varepsilon)$. Since this can be done for each $\varepsilon > 0$, we have $y \in \mathrm{cl}(A)$. But since A is closed, we have $\mathrm{cl}(A) = A$, and $y \in A$. Thus $\bigcap_{k=1}^{\infty} U_k \subseteq A$. We have inclusion in both directions, so $A = \bigcap_{k=1}^{\infty} U_k$ which is the intersection of a countable collection of open sets. ♦

◇ **3E-36.** Let $A \subset \mathbb{R}^n$ be uncountable. Prove that A has an accumulation point.

Suggestion. Consider the sets $A_k = A \cap \{v \in \mathbb{R}^n \mid \| v \| \leq k\}$. ♢

Solution. For $k = 1, 2, 3, \ldots$, let $A_k = A \cap \{v \in \mathbb{R}^n \mid \| v \| \leq k\}$. Then $A = \bigcup_{k=1}^{\infty} A_k$. If each of the countably many sets A_k were finite, then their union A would be countable. Since it is not, at least one of the sets A_k must have infinitely many points. We can choose a sequence of points $a_1, a_2, a_3, \ldots$ in A_k all different. This is a bounded sequence in $\mathbb{R}^n$, so by the Bolzano-Weierstrass property of $\mathbb{R}^n$, it must have a subsequence converging to some point $w \in \mathbb{R}^n$. (The closed bounded set $\mathrm{cl}(A_k) \subseteq \mathbb{R}^n$ is sequentially compact.) Points in A_k are in A, so there is a sequence of distinct points of A converging to w and w must be an accumulation point of A. ♦

◇ **3E-40.** Let F_k be a nest of compact sets (that is, $F_{k+1} \subset F_k$). Furthermore, suppose each F_k is connected. Prove that $\cap_{k=1}^{\infty}\{F_k\}$ is connected. Give an example to show that compactness is an essential condition and we cannot just assume that "F_k is a nest of closed connected sets."

Suggestion. Use the Nested Set Property. ◇

Solution. From the nested set property, there is at least one point x_0 in the intersection $F = \bigcap_{k=1}^{\infty} F_k$. Suppose U and V are open sets with $F \cap U \cap V = \emptyset$ and $F \subseteq U \cup V$.

Claim. *There is an integer k such that $F_k \subset U \cap V$.*

Proof of claim: For each k let $C_k = F_k \cap (M \setminus (U \cup V))$. Then C_k is a closed subset of the compact set F_k, so C_k is compact. (Why?) Furthermore $C_{k+1} \subseteq C_k$ for each k. (Why?) If each of the sets C_k were nonempty, then by the nested set property there would be at least one point y in their intersection. But such a point would be in all of the F_k and so in F. We would have $y \in F \cap U \cap V$ which was supposed to be empty. Thus at least one of the sets C_k must be empty and for that k we have $F_k \subseteq U \cap V$.

Let k be an index with $F_k \subseteq U \cap V$. The point x_0 must be in one of the sets U or V. Say $x_0 \in U$. Then $x_0 \in U \cap F_k$, so $V \cap F_k$ must be empty or the sets U and V would disconnect the connected set F_k. Thus $F \subseteq F_k \subseteq U$. So $F \subseteq U$ and $F \cap V = \emptyset$. The sets U and V cannot disconnect F. Thus there can be no pair of open sets which disconnect F and F must be connected.

For a counterexample in which the sets are closed and connected but not compact, let $F_k = \{(x, y) \in \mathbb{R}^2 \mid |x| \geq 1 \text{ or } |y| \geq k\}$ for $k = 1, 2, 3, \ldots$. Each of these is closed and connected. But their intersection is $\{(x, y) \in \mathbb{R}^2 \mid |x| \geq 1\}$ which is not connected. ♦