◇ **2E-2.** Determine the interiors, closures, and boundaries of the sets in Exercise 2E-1.

Answer. (a) If $A =]1,2[$ in $\mathbb{R}^1 = \mathbb{R}$, then $\text{int}(A) = A =]1,2[$, $\text{cl}(A) = [1,2]$, and $\text{bd}(A) = \{1,2\}$.

(b) If $B = [2,3]$ in $\mathbb{R}$, then $\text{int}(B) =]2,3[$, $\text{cl}(B) = B = [2,3]$, and $\text{bd}(B) = \{2,3\}$.

(c) If $C = \bigcap_{n=1}^{\infty} [-1, 1/n[$ in $\mathbb{R}$, then $\text{int}(C) =]-1,0[$, $\text{cl}(C) = [-1,0]$, and $\text{bd}(C) = \{-1,0\}$.

(d) If $D = \mathbb{R}^n$ in $\mathbb{R}^n$, then $\text{int}(D) = \text{cl}(D) = D$ and $\text{bd}(D) = \emptyset$.

(e) If E is a hyperplane in $\mathbb{R}^n$, then $\text{int}(E) = \emptyset$ and $\text{cl}(E) = \text{bd}(E) = E$.

(f) If $F = \{r \in]0,1[\mid r \in \mathbb{Q}\}$ in $\mathbb{R}$, then $\text{int}(F) = \emptyset$ and $\text{cl}(F) = \text{bd}(F) = [0,1]$.

(g) If $G = \{(x,y) \in \mathbb{R}^2 \mid 0 < x \le 1\}$ in $\mathbb{R}^2$, then $\text{int}(G) = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1\}$, $\text{cl}(G) = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\}$, and $\text{bd}(G) = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ or } x = 1\}$.

(h) If $H = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ in $\mathbb{R}^n$, then $\text{int}(H) = \emptyset$ and $\text{cl}(H) = \text{bd}(H) = H$. ◇

Solution. (a) If $A =]1,2[$ in $\mathbb{R}^1 = \mathbb{R}$, then we know from Exercise 2E-1(a) that A is open, so $\text{int}(A) = A$. The endpoints 1 and 2 are certainly accumulation points, so they are in the closure. From Exercise 2E-1(b), we know that closed intervals are closed sets, so $[1,2]$ is closed. Thus $\text{cl}(A) = [1,2]$. Similarly $\text{cl}(\mathbb{R} \setminus A) = \{x \mid x \le 1\} \cup \{x \mid x \ge 2\}$, so $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(\mathbb{R} \setminus A) = \{1,2\}$.

(b) If $B = [2,3]$ in $\mathbb{R}$, then we know from Exercise 2E-1(b) that B is closed, so $\text{cl}(B) = B$. The situation is just like that of part (a). We have a finite interval in $\mathbb{R}$. The interior is the open interval, the closure is the closed interval, and the boundary is the set consisting of the two endpoints. So $\text{int}(B) =]2,3[$, $\text{cl}(B) = B = [2,3]$, and $\text{bd}(B) = \{2,3\}$.

(c) If $C = \bigcap_{n=1}^{\infty} [-1, 1/n[$ in $\mathbb{R}$, then as we saw in Exercise 2E-1(c), $C = [-1,0]$. Again we have an interval in $\mathbb{R}$ and, as discussed above, we must have $\text{int}(C) =]-1,0[$, $\text{cl}(C) = [-1,0]$, and $\text{bd}(C) = \{-1,0\}$.

(d) If $D = \mathbb{R}^n$ in $\mathbb{R}^n$, then we know that D is both open and closed, so $\text{int}(D) = \text{cl}(D) = D$. Since the complement of D is empty, the closure of the complement is empty, and we must have $\text{bd}(D) = \emptyset$.

(e) Suppose E is a hyperplane in $\mathbb{R}^n$. As in Exercise 2E-1(e), E is closed, so $\text{cl}(E) = E$. Let e_n be a unit vector orthogonal to E, then if $w \in E$, the points $v_k = w + (1/k)e_n$ are in the complement of E and converge to w. So w is in the closure of the complement and is not in the interior of E. This is true for every $w \in E$, so $\text{int}(E) = \emptyset$ and $\text{bd}(E) = \text{cl}(E) \cap \text{cl}(\mathbb{R}^n \setminus E) = E \cap \mathbb{R}^n = E$.

(f) Let $F = \{r \in]0,1[\mid r \in \mathbb{Q}\}$ in $\mathbb{R}$. If $r \in F$ and $\varepsilon > 0$, then we have seen that the interval $]r-\varepsilon, r+\varepsilon[$ contains irrational numbers. So r cannot be an interior point. So $\text{int}(F) = \emptyset$. On the other hand, if $0 \le x_0 \le 1$, then there are rational numbers between $x_0 - \varepsilon$ and x_0 and between x_0 and $x_0 + \varepsilon$. So x_0 is an accumulation point of F. We must have $\text{cl}(F) = [0,1]$. But for similar reasons, $\text{cl}(\mathbb{R} \setminus F) = \mathbb{R}$, so $\text{bd}(F) = \text{cl}(F) \cap \text{cl}(\mathbb{R} \setminus F) = [0,1] \cap \mathbb{R} = [0,1]$.

(g) Consider the sets

$$
\begin{aligned}
G &= \{(x,y) \in \mathbb{R}^2 \mid 0 < x \le 1\} \\
S &= \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1\} \\
T &= \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\} \\
U &= \{(x,y) \in \mathbb{R}^2 \mid x < 0\} \\
V &= \{(x,y) \in \mathbb{R}^2 \mid x > 1\} \\
W &= \{(x,y) \in \mathbb{R}^2 \mid x \ge 1\} \\
P &= \{(x,y) \in \mathbb{R}^2 \mid x \le 0\}.
\end{aligned}
$$

We certainly have $S \subseteq G \subseteq T$. As in Example 2.1.4 of the text, the set S is open. So $S \subseteq \mathrm{int}(G)$. As in Example 2.1.5, none of the points (x,y) with $\mathrm{int}(H) = S = \{(x,y) \in \mathbb{R}^2 \mid 0, x < 1\}$. The half planes U and V are open by an argument similar to that of Example 2.1.5 which we have seen before. For example, suppose $(a,b) \in U$. Then $a < 0$. If $\| (x,y) - (a,b) \| < |a|$, then

$$|x - a| \le \sqrt{(x-a)^2 + (y-b)^2} < |a|\,.$$

So

$$a = -|a| < x - a < |a| = -a.$$

Adding a to both sides of the second inequality gives $x < 0$. Thus $(x,y) \in U$. Thus $D((a,b),|a|) \subseteq U$. The set U contains an open disk around each of its points and so is open. The argument for the half plane V is similar. Since U and V are open, so is their union. But that union is the complement of T. So T is closed. Thus $\mathrm{cl}(G) \subseteq T$. On the other hand, the points $(1/n, y)$ are in G for each $n = 1, 2, \ldots$ and converge to $(0,y)$. So $(0,y) \in \mathrm{cl}(G)$. We must have $\mathrm{cl}(G) = T = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\}$. Finally, W and P are closed since they are the complements of V and U. So their union is closed and must be the closure of the complement of G. So $\mathrm{bd}(G) = T \cap (W \cup P) = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ or } x = 1\}$.

(h) Let $H = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ in $\mathbb{R}^n$. Thus H is closed (Exercise 2E-1(h)). So $\mathrm{cl}(H) = H$. If $x \in H$, then the points $w_n = (1 + (1/n)x$ are not in H but do converge to x. So x cannot be an interior point and $\mathrm{int}(H) = \emptyset$. This also shows that $x \in \mathrm{cl}(\mathbb{R}^n \setminus H)$. So $H \subseteq \mathrm{cl}(\mathbb{R}^n \setminus H)$, and $\mathrm{bd}(H) = \mathrm{cl}(H) \cap \mathrm{cl}(\mathbb{R}^n \setminus H) = H \cap \mathrm{cl}(\mathbb{R}^n \setminus H) = H$. ♦

◇ **2E-10.** Determine which of the following statements are true.

(a) $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(A)$.
(b) $\mathrm{cl}(A) \cap A = A$.
(c) $\mathrm{cl}(\mathrm{int}(A)) = A$.
(d) $\mathrm{bd}(\mathrm{cl}(A)) = \mathrm{bd}(A)$.
(e) If A is open, then $\mathrm{bd}(A) \subset M \setminus A$.

Solution. (a) The equality $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(A)$ is not always true. Consider the example $A = [-1,1] \backslash \{0\} \subseteq \mathbb{R}$. Then $\mathrm{cl}(A)$ is the closed interval $[-1,1]$, and $\mathrm{int}(\mathrm{cl}(A))$ is the open interval $]-1,1[$. But $\mathrm{int}(A)$ is the open interval with zero deleted. $\mathrm{int}(A) =]-1,1[\backslash\{0\}$.

(b) True: Since $A \subseteq \mathrm{cl}(A)$, we always have $\mathrm{cl}(A) \cap A = A$.

(c) The proposed equality, $\mathrm{cl}(\mathrm{int}(A)) = A$, is not always true. Consider the example of a one point set with the usual metric on $\mathbb{R}$. Take $A = \{0\} \subseteq \mathbb{R}$. Then $\int A = \emptyset$. So $\mathrm{cl}(\mathrm{int}(A)) = \emptyset$. But A is not empty.

(d) The proposed equality, $\mathrm{bd}(\mathrm{cl}(A)) = \mathrm{bd}(A)$, is not always true. Consider the same example as in part (a). $A = [-1,1] \setminus \{0\} \subseteq \mathbb{R}$. Then $\mathrm{cl}(A)$ is the closed interval $[-1,1]$ and $\mathrm{bd}(\mathrm{cl}(A))$ is the two point set $\{-1,1\}$. But $\mathrm{bd}(A)$ is the three point set $\{-1,0,1\}$.

(e) The proposed inclusion, $\mathrm{bd}(A) \subseteq M \setminus A$, is true if A is an open subset of the metric space M. The set A is open, so its complement, $M \setminus A$, is closed. Thus

$$\mathrm{bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A) = \mathrm{cl}(A) \cap (M \setminus A) \subseteq M \setminus A$$

◇ **2E-15.** Prove the following for subsets of a metric space M:

(a) $\mathrm{bd}(A) = \mathrm{bd}(M \setminus A)$.
(b) $\mathrm{bd}(\mathrm{bd}(A)) \subset \mathrm{bd}(A)$.
(c) $\mathrm{bd}(A \cup B) \subset \mathrm{bd}(A) \cup \mathrm{bd}(B) \subset \mathrm{bd}(A \cup B) \cup A \cup B$.
(d) $\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) = \mathrm{bd}(\mathrm{bd}(A))$.

Sketch. (a) Use $M \setminus (M \setminus A) = A$ and the definition of boundary.
(b) Use the fact that $\mathrm{bd}(A)$ is closed. (Why?)
(c) The facts $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ from Exercise 2E-14 are useful.
(d) One approach is to show that $\mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A))) = M$ or, equivalently, that $\mathrm{int}(\mathrm{bd}(\mathrm{bd}(A))) = \emptyset$, and use that to compute $\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A)))$. ◇

Solution. (a) If A is a subset of a metric space M, we can compute

$$\begin{aligned}\mathrm{bd}(M \setminus A) &= \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus (M \setminus A)) = \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(A)\\ &= \mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A) = \mathrm{bd}(A)\end{aligned}$$

as claimed.

(b) Since $\mathrm{bd}(A)$ is the intersection of $\mathrm{cl}(A)$ and $\mathrm{cl}(M \setminus A)$, both of which are closed, it is closed. In particular, $\mathrm{cl}(\mathrm{bd}(A)) = \mathrm{bd}(A)$, and we have

$$\mathrm{bd}(\mathrm{bd}(A)) = \mathrm{cl}(\mathrm{bd}(A)) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A)) = \mathrm{bd}(A) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A)).$$

Since $\mathrm{bd}(\mathrm{bd}(A))$ is the intersection of $\mathrm{bd}(A)$ with something else, we have $\mathrm{bd}(\mathrm{bd}(A)) \subseteq \mathrm{bd}(A)$ as claimed.

(c) A key to part (c) are the observations that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ for any subsets A and B of a metric space M. (See Exercise 2E-14.) Using them we can compute

$$\begin{aligned}\mathrm{bd}(A \cup B) &= \mathrm{cl}(A \cup B) \cap \mathrm{cl}(M \setminus (A \cup B))\\ &= \mathrm{cl}(A \cup B) \cap \mathrm{cl}((M \setminus A) \cap (M \setminus B))\\ &\subseteq \mathrm{cl}(A \cup B) \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B)\\ &= \big[\mathrm{cl}(A) \cup \mathrm{cl}(B)\big] \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B)\\ &= \big[\mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B)\big]\\ &\qquad \cup \big[\mathrm{cl}(B) \cap \mathrm{cl}(M \setminus A) \cap \mathrm{cl}(M \setminus B)\big]\\ &\subseteq \big[\mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A)\big] \cup \big[\mathrm{cl}(B) \cap \mathrm{cl}(M \setminus B)\big] = \mathrm{bd}(A) \cup \mathrm{bd}(B).\end{aligned}$$

This is the first inclusion claimed. Now suppose $x \in \mathrm{bd}(A) \cup \mathrm{bd}(B)$. Then $x \in \mathrm{cl}(A)$ or $x \in \mathrm{cl}(B)$. So $x \in \mathrm{cl}(A) \cup \mathrm{cl}(B) = \mathrm{cl}(A \cup B)$. If x is in neither A nor B, then $x \in (M \setminus A) \cap (M \setminus B) = M \setminus (A \cup B) \subseteq \mathrm{cl}((M \setminus (A \cup B))$. Since we also have $x \in \mathrm{cl}(A \cup B)$, this puts x in $\mathrm{bd}(A \cup B)$. So x must be in at least one of the three sets A, B, or $\mathrm{bd}(A \cup B)$. That is, $\mathrm{bd}(A) \cup \mathrm{bd}(B) \subseteq \mathrm{bd}(A \cup B) \cup A \cup B$ as claimed.

(d) From part (a) we know that $\mathrm{bd}(\mathrm{bd}(C)) \subseteq \mathrm{bd}(C)$ for every set C. So, in particular, $\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) \subseteq \mathrm{bd}(\mathrm{bd}(A))$. But now we want equality:

$$\begin{aligned}\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) &= \mathrm{cl}(\mathrm{bd}(\mathrm{bd}(A))) \cap \mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A)))\\ &= \mathrm{bd}(\mathrm{bd}(A)) \cap \mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A))).\end{aligned}$$

But

$$\begin{aligned}\mathrm{cl}(M \setminus \mathrm{bd}(\mathrm{bd}(A))) &= \mathrm{cl}(M \setminus (\mathrm{cl}(\mathrm{bd}(A)) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A))))\\ &= \mathrm{cl}(M \setminus (\mathrm{bd}(A) \cap \mathrm{cl}(M \setminus \mathrm{bd}(A))))\\ &= \mathrm{cl}((M \setminus \mathrm{bd}(A)) \cup (M \setminus \mathrm{cl}(M \setminus \mathrm{bd}(A))))\\ &= \mathrm{cl}(M \setminus \mathrm{bd}(A)) \cup \mathrm{cl}(M \setminus \mathrm{cl}(M \setminus \mathrm{bd}(A)))\\ &\subseteq \mathrm{cl}(M \setminus \mathrm{bd}(A)) \cup \mathrm{cl}(M \setminus (M \setminus \mathrm{bd}(A)))\\ &\subseteq (M \setminus \mathrm{bd}(A)) \cup (M \setminus (M \setminus \mathrm{bd}(A))) = M.\end{aligned}$$

Combining the last two displays gives

$$\mathrm{bd}(\mathrm{bd}(\mathrm{bd}(A))) = \mathrm{bd}(\mathrm{bd}(A)) \cap M = \mathrm{bd}(\mathrm{bd}(A))$$

as claimed.

Remark: The identity in the next to last display is equivalent to the assertion that

$$\mathrm{int}(\mathrm{bd}(\mathrm{bd}(A))) = \emptyset.$$ ♦

◇ **2E-16.** Let $a_1 = \sqrt{2}$, $a_2 = (\sqrt{2}\,)^{a_1}, \dots, a_{n+1} = (\sqrt{2}\,)^{a_n}$. Show that $a_n \to 2$ as $n \to \infty$. (You may use any relevant facts from calculus.)

Suggestion. Show the sequence is increasing and bounded above by 2. Conclude that the limit λ exists and is a solution to the equation $\lambda = \left(\sqrt{2}\right)^{\lambda}$. Show $\lambda = 2$ and $\lambda = 4$ are the only solutions of this equation. Conclude that the limit is 2. ◇

Solution. First: $a_1 > 1$, and if $a_k > 1$, then $a_{k+1} = \left(\sqrt{2}\right)^{a_k} > \sqrt{2} > 1$ also. By induction we have

$$a_n > 1 \qquad \text{for every } n = 1, 2, 3, \dots.$$

Second: The sequence is increasing: $a_1 = \sqrt{2}$ and $a_2 = \left(\sqrt{2}\right)^{\sqrt{2}} > \sqrt{2} = a_1$. If $a_{k-1} < a_k$, we can compute

$$\frac{a_{k+1}}{a_k} = \frac{\left(\sqrt{2}\right)^{a_k}}{\left(\sqrt{2}\right)^{a_{k-1}}} = \left(\sqrt{2}\right)^{a_k - a_{k-1}} > 1.$$

So $a_{k+1} > a_k$. By induction we find that $a_{n+1} > a_n$ for each n so the sequence is increasing.

Third: The sequence is bounded above by 2: $a_1 = \sqrt{2} < 2$. If $a_k < 2$, then $a_{k+1} = \left(\sqrt{2}\right)^{a_k} < \left(\sqrt{2}\right)^2 = 2$ also. By induction we conclude that $a_n < 2$ for every n.

Fourth: Since the sequence $\langle a_n \rangle_1^\infty$ is increasing and bounded above by 2, the completeness of $\mathbb{R}$ implies that it must converge to some $\lambda \in \mathbb{R}$.

Fifth: $a_{n+1} \to \lambda$. But also $a_{n+1} = \left(\sqrt{2}\right)^{a_n} \to \left(\sqrt{2}\right)^{\lambda}$. Since limits in $\mathbb{R}$ are unique we must have

$$\lambda = \left(\sqrt{2}\right)^{\lambda} \qquad \text{or equivalently} \qquad \log \lambda = \frac{\log 2}{2}\lambda.$$

These equations have the two solutions $\lambda = 2$ and $\lambda = 4$ as may be checked directly. There can be no other solutions since the solutions occur at the intersection of a straight line with an exponential curve (or a logarithm curve in the second equation). The exponential curve is always concave up and the logarithm curve is always concave down. Either can cross a straight line at most twice. Thus the limit must be either 2 or 4. But all terms of the sequence are smaller than 2, so the limit cannot be 4. It must be 2. $\lim_{n\to\infty} a_n = 2$. ♦

◇ **2E-20.** For a set A in a metric space M and $x \in M$, let

$$d(x, A) = \inf\{d(x, y) \mid y \in A\},$$

and for $\varepsilon > 0$, let $D(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$.

(a) Show that $D(A, \varepsilon)$ is open.
(b) Let $A \subset M$ and $N_\varepsilon = \{x \in M \mid d(x, A) \leq \varepsilon\}$, where $\varepsilon > 0$. Show that N_ε is closed and that A is closed iff $A = \cap \{N_\varepsilon \mid \varepsilon > 0\}$.

Suggestion. For (a), show that $D(A, \epsilon)$ is a union of open disks. For the first part of (b), consider convergent sequences in $N(A, \varepsilon)$ and their limits in M. ◇

Solution. (a) First suppose $x \in D(A, \varepsilon)$. Then $d(x, A) = \inf\{d(x, y) \mid y \in A\} = r < \varepsilon$. So there is a point $y \in A$ with $r \leq d(x, y) < \varepsilon$. Thus $x \in D(y, \varepsilon)$. This can be done for each $x \in D(A, \varepsilon)$. We conclude that

$$D(A, \varepsilon) \subseteq \bigcup_{y \in A} D(y, \varepsilon).$$

Conversely, if there is a y in A with $x \in D(y, \varepsilon)$, then $d(x, y) < \varepsilon$. So $d(x, A) = \inf\{d(x, y) \mid y \in A\} < \varepsilon$, and $x \in D(A, \varepsilon)$. This proves inclusion in the other direction. We conclude that

$$D(A, \varepsilon) = \bigcup_{y \in A} D(y, \varepsilon).$$

Each of the disks $D(y, \varepsilon)$ is open by Proposition 2.1.2, so their union, $D(A, \varepsilon)$ is also open by Proposition 2.1.3(ii).

(b) To show that $N(A,\varepsilon)$ is closed we will show that it contains the limits of all convergent sequences in it. Suppose $\langle x_k\rangle_1^\infty$ is a sequence in $N(A,\varepsilon)$ and that $x_k \to x \in M$. Since each $x_k \in N(A,\varepsilon)$, we have $d(x_k, A) \le \varepsilon < \varepsilon + (1/k)$. So there are points y_k in A with $d(x_k, y_k) < \varepsilon + (1/k)$. Since $x_k \to x$, we know that $d(x, x_k) \to 0$, and can compute

$$d(x, y_k) \le d(x, x_k) + d(x_k, y_k) < d(x, x_k) + \varepsilon + \frac{1}{k} \to \varepsilon \quad \text{as } k \to \infty.$$

Thus

$$d(x, A) = \inf\{d(x,y) \mid y \in A\} \le \varepsilon.$$

So $x \in N(A,\varepsilon)$. We have shown that if $\langle x_k\rangle_1^\infty$ is a sequence in $N(A,\varepsilon)$ and $x_k \to x \in M$, then $x \in N(A,\varepsilon)$. So $N(A,\varepsilon)$ is a closed subset of M by Proposition 2.7.6(i).

We have just shown that each of the sets $N(A,\varepsilon)$ is closed, and we know that the intersection of any family of closed subsets of M is closed. So if $A = \bigcap_{\varepsilon>0} N(A,\varepsilon)$, then A is closed.

Conversely, if A is closed and $y \in M \setminus A$, then there is an $r > 0$ such that $D(y,r) \subset M \setminus A$ since the latter set is open. Thus y is not in $N(A, r/2)$. So y is not in $\bigcap_{\varepsilon>0} N(A,\varepsilon)$. This establishes the opposite inclusion $\bigcap_{\varepsilon>0} N(A,\varepsilon) \subseteq A$.

If A is closed, we have inclusion in both directions, so $A = \bigcap_{\varepsilon>0} N(A,\varepsilon)$ as claimed. ♦

◇ **2E-24.** Identify $\mathbb{R}^{n+m}$ with $\mathbb{R}^n \times \mathbb{R}^m$. Show that $A \subset \mathbb{R}^{n+m}$ is open iff for each $(x,y) \in A$, with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, there exist open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ with $x \in U$, $y \in V$ such that $U \times V \subset A$. Deduce that the product of open sets is open.

Suggestion. Try drawing the picture in $\mathbb{R}^2$ and writing the norms out in coordinates in $\mathbb{R}^{n+m}$, $\mathbb{R}^n$, and $\mathbb{R}^m$. ◇

Solution. The spaces $\mathbb{R}^{n+m}$ and $\mathbb{R}^n \times \mathbb{R}^m$ are identified as sets by identifying the element $(x,y) = ((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_m))$ of $\mathbb{R}^n \times \mathbb{R}^m$ with the point
$(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$ of $\mathbb{R}^{n+m}$. With this identification, we have

$$\|(x,y)\|^2 = x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_m^2 = \|x\|^2 + \|y\|^2. \qquad (1)$$

Using this we see that if x and s are in $\mathbb{R}^n$ and y and t are in $\mathbb{R}^m$, then

$$\|(s,t) - (x,y)\|^2 = \|s - x\|^2 + \|t - y\|^2 \qquad (2)$$

and this certainly implies that

$$\|s - x\|^2 \le \|(s,t) - (x,y)\|^2 \quad \text{and} \quad \|t - y\|^2 \le \|(s,t) - (x,y)\|^2$$

so that

$$\|s - x\| \le \|(s,t) - (x,y)\| \quad \text{and} \quad \|t - y\| \le \|(s,t) - (x,y)\|. \qquad (3)$$

Now Let $A \subseteq \mathbb{R}^{n+m}$, and let $(x,y) \in A$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

First suppose that A is open. Then there is an $r > 0$ such that $D((x,y),r) \subseteq A$. Let $\rho = \sqrt{r/2}$, and set $U = D(x,\rho) \subseteq \mathbb{R}^n$ and $V = D(y,\rho) \subsetneq \mathbb{R}^m$. We want to show that $U \times V \subseteq D((x,y),r) \subseteq A$. Since $x \in D(x,\rho) = U$, and $y \in D(y,\rho) = V$, we certainly have $(x,y) \in U \times V$. If $(s,t) \in U \times V$, then using (2) we have $\|(s,t) - (x,y)\|^2 = \|s - x\|^2 + \|t - y\|^2 < 2\rho^2 = r$. So $(s,t) \in D((x,y),r) \subseteq A$. Thus $U \times V \subseteq D((x,y),r) \subseteq A$ as we wanted.

For the converse, suppose that there are open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ with $(x,y) \in U \times V \subseteq A$. Since U and V are open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, there are radii $\rho_1 > 0$ and $\rho_2 > 0$ such that $D(x,\rho_1) \subseteq U$ and $D(y,\rho_2) \subseteq V$. So

$$(x,y) \in D(x,\rho_1) \times D(y,\rho_2) \subseteq U \times V \subseteq A.$$

Let $r = \min(\rho_1, \rho_2)$, and suppose that $(s,t) \in D((x,y),r)$, then by (3) we have $\|s - x\| \le \|(s,t) - (x,y)\| < r \le \rho_1$. So $s \in D(x,\rho_1) \subseteq U$. Also $\|t - y\| \le \|(s,t) - (x,y)\| < r \le \rho_2$. So $t \in D(y,\rho_2) \subseteq V$. Thus $(s,t) \in D(x,\rho_1) \times D(y,\rho_2) \subseteq U \times V$. This is true for every such (s,t). So

$$(x,y) \in D((x,y),r) \subseteq D(x,\rho_1) \times D(y,\rho_2) \subseteq U \times V \subseteq A.$$

The set A contains an open disk around each of its points, so A is open.

◇ **2E-25.** Prove that a set $A \subset M$ is open iff we can write A as the union of some family of ε-disks.

Sketch. Since ε-disks are open, so is any union of them. Conversely, if A is open and $x \in A$, there is an $\varepsilon_x > 0$ with $x \in D(x, \varepsilon_x) \subseteq A$. So $A = \bigcup_{x \in A} D(x, \varepsilon_x)$. ◊

Solution. To say that A is a union of ε-disks is to say that there is a set of points $\{x_\beta \mid \beta \in B\} \subseteq A$ and a set of positive radii $\{r_\beta \mid \beta \in B\}$ such that $A = \bigcup_{\beta \in B} D(x_\beta, r_\beta)$. ($B$ is just any convenient index set for listing these things.) We know from Proposition 2.1.2 that each of the disks $D(x_\beta, r_\beta)$ is open. By 2.1.3(ii), the union of any family of open subsets of M is open. So A must be open.

For the converse, suppose A is an open subset of A. Then for each x in A, there is a radius $r_x > 0$ such that $D(x, r_x) \subseteq A$. Since $x \in D(x, r_x) \subseteq A$, we have

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} D(x, r_x) \subseteq A.$$

So we must have $A = \bigcup_{x \in A} D(x, r_x)$, a union of open disks, as required. ♦

◇ **2E-27.** Suppose $a_n \geq 0$ and $a_n \to 0$ as $n \to \infty$. Given any $\varepsilon > 0$, show that there is a subsequence b_n of a_n such that $\sum_{n=1}^{\infty} b_n < \varepsilon$.

Suggestion. Pick b_n with $b_n = |b_n| < \varepsilon/2^n$. (Why can you do this?) ◊

Solution. Select the subsequence inductively.

STEP ONE: Since $a_n \geq 0$ and $a_n \to 0$, there is an index $n(1)$ such that $0 \leq a_{n(1)} < \varepsilon/2$.

STEP TWO: Since $a_n \geq 0$ and $a_n \to 0$, there is an index $n(2)$ such that $n(2) > n(1)$ and $0 \leq a_{n(2)} < \varepsilon/4$.

STEP THREE: Since $a_n \geq 0$ and $a_n \to 0$, there is an index $n(3)$ such that $n(3) > n(2)$ and $0 \leq a_{n(3)} < \varepsilon/8$.

⋮

STEP k+1: Having selected indices $n(1) < n(2) < \cdots < n(k)$ such that $0 \leq a_{n(j)} < \varepsilon/2^j$ for $j = 1, 2, \ldots, k$, we observe that since $a_n \geq 0$ and $a_n \to 0$, there is an index $n(k+1)$ such that $n(k+1) > n(k)$ and $0 \leq a_{n(k+1)} < \varepsilon/2^{k+1}$.

⋮

This process inductively generates a subsequence with indices $n(1) < n(2) < n(3) < \cdots$ such that $0 \leq a_{n(k)} < \varepsilon/2^k$ for each k. So $0 \leq \sum_{k=1}^{\infty} a_{n(k)} < \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon \sum_{k=1}^{\infty} (1/2^k) = \varepsilon$, as we wanted. ♦

◇ **2E-31.** Let A' denote the set of accumulation points of a set A. Prove that A' is closed. Is $(A')' = A'$ for all A?

Sketch. What needs to be done is to show that an accumulation point of A' must be an accumulation point of A. $(A')'$ need not be equal to A'. Consider $A = \{1/2, 1/3, \ldots\}$. ◊

Solution. To show that A' is closed, we show that it contains all of its accumulation points. That is, $(A')' \subseteq A'$. Suppose x is an accumulation point of A', and let U be an open set containing x. Then U contains a point y in A' with y not equal to x. Let $V = U \setminus \{x\}$. Then V is an open set containing y. Since $y \in A'$, there is a point z in $V \cap A$ with z not equal to y. Since x is not in V, we also know that z is not equal to x. Since $V \subseteq U$, we know that $z \in U$. Every neighborhood U of x contains a point z of A which is not equal to x. So x is an accumulation point of A. This works for every x in A'. So $(A')' \subseteq A'$. Since the set A' contains all of its accumulation points, it is closed as claimed.

Although we now know that $(A')' \subseteq A'$ for all subsets of a metric space M, the inclusion might be proper. Consider $A = \{1, 1/2, 1/3 \ldots\} \subseteq \mathbb{R}$. Then $A' = \{0\}$, and $(A')' = \emptyset$. ♦

◇ **2E-41.** Let A_n be subsets of a metric space M, $A_{n+1} \subset A_n$, and $A_n \neq \varnothing$, but assume that $\cap_{n=1}^{\infty} A_n = \varnothing$. Suppose $x \in \cap_{n=1}^{\infty} \text{cl}(A_n)$. Show that x is an accumulation point of A_1.

Sketch. Let U be a neighborhood of x. There is an n with $x \in \text{cl}(A_n) \setminus A_n$. (Why?) So $U \cap (A_n \setminus \{x\})$ is not empty. ◇

Solution. Let U be an open set containing x. We must show that U contains some point of A_1 not equal to x. Since $\bigcap_{n=1}^{\infty} A_n = \emptyset$, there is an n_0 such that x is not in A_{n_0}. But $x \in \bigcap_{n=1}^{\infty} \text{cl}(A_n)$. So $x \in \text{cl}(A_{n_0})$ Since x is in the closure of A_{n_0} but not in A_{n_0}, it must be an accumulation point of A_{n_0}. So there is a y in $U \cap (A_{n_0} \setminus \{x\})$. But $A_{n_0} \subseteq A_{n_0-1} \subseteq \cdots \subseteq A_1$. So $y \in U \cap (A_1 \setminus \{x\})$. This can be done for every open set containing x. So x is an accumulation point of A_1 as claimed. ♦

◇ **2E-48.** Prove the following generalizations of the ratio and root tests:

(a) If $a_n > 0$ and $\limsup_{n\to\infty} a_{n+1}/a_n < 1$, then $\sum a_n$ converges, and if $\liminf_{n\to\infty} a_{n+1}/a_n > 1$, then $\sum a_n$ diverges.
(b) If $a_n \geq 0$ and if $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$ (respectively, > 1), then $\sum a_n$ converges (respectively, diverges).

(c) In the ratio comparison test, can the limits be replaced by lim sup's?

Suggestion. (a) For large n, the series is comparable to an appropriately selected geometric series.
(b) For large n, the series is comparable to an appropriately selected geometric series.
(c) Not quite, but we can replace the limit in the convergence statement by lim sup and the one in the divergence statement by lim inf. ◇

Solution. (a) Suppose each $a_n > 0$ and that $\limsup_{n\to\infty} a_{n+1}/a_n < 1$. Select a number r such that $\limsup_{n\to\infty} a_{n+1}/a_n < r < 1$. Then there is an integer N such that $a_{n+1}/a_n < r$ whenever $n \geq N$. Since the numbers are nonnegative, this gives $0 \leq a_{n+1} \leq ra_n$ for $n = N, N+1, N+2, \ldots$. Apply this repeatedly.

$$\begin{aligned}
&0 \leq a_{N+1} \leq ra_N \\
&0 \leq a_{N+2} \leq ra_{N+1} \leq r^2 a_N \\
&0 \leq a_{N+3} \leq ra_{N+2} \leq r^3 a_N \\
&\quad\vdots \\
&0 \leq a_{N+p} \leq ra_{N+p-1} \leq r^p a_N \\
&\quad\vdots
\end{aligned}$$

Since $0 \leq r < 1$, we know that the geometric series $\sum_{p=0}^{\infty} r^p a_N$ converges. By comparison, we conclude that $\sum_{k=N}^{\infty} a_k = \sum_{p=0}^{\infty} a_{N+p}$ also converges. Reintroducing the finitely many terms $a_1 + \cdots + a_{N-1}$ does not change the fact of convergence. We conclude that $\sum_{n=1}^{\infty} a_n$ converges.

Now suppose each $a_n > 0$ and that $\liminf_{n\to\infty} a_{n+1}/a_n > 1$. Select a number r such that $1 < r < \liminf_{n\to\infty} a_{n+1}/a_n$ Then there is an integer N such that $a_{n+1}/a_n > r > 1$ whenever $n \geq N$. Since the numbers are nonnegative, this gives $a_{n+1} \geq ra_n \geq a_n > 0$ for $n = N, N+1, N+2, \ldots$. Apply this repeatedly.

$$\begin{aligned}
&a_{N+1} \geq ra_N \geq a_N > 0 \\
&a_{N+2} \geq ra_{N+1} \geq a_N > 0 \\
&a_{N+3} \geq ra_{N+2} \geq a_N > 0 \\
&\quad\vdots \\
&a_{N+p} \geq ra_{N+p-1} \geq a_N > 0 \\
&\quad\vdots
\end{aligned}$$

Since $a_N > 0$, we know that the constant series $\sum_{p=0}^{\infty} a_N$ diverges to $+\infty$. By comparison, we conclude that $\sum_{k=N}^{\infty} a_k = \sum_{p=0}^{\infty} a_{N+p}$ also diverges. Reintroducing the finitely many terms $a_1 + \cdots + a_{N-1}$ does not change the fact of divergence. We conclude that $\sum_{n=1}^{\infty} a_n$ diverges.

(b) Suppose each $a_n \geq 0$ and that $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$. Select a number r such that $\limsup_{n\to\infty} \sqrt[n]{a_n} < r < 1$ Then there is an integer N such that $0 \leq \sqrt[n]{a_n} < r < 1$ whenever $n \geq N$. Taking n^{th} powers gives $0 \leq a_n \leq r^n$ for $n = N. N+1, N+2, \ldots$. That is, $0 \leq a_{N+p} \leq r^{N+p}$ for $p = 1, 2, 3, \ldots$. Since $0 \leq r < 1$, we know that the geometric series $\sum_{p=0}^{\infty} r^{N+p} = r^N \sum_{p=0}^{\infty} r^p$ converges. By comparison, the series $\sum_{n=N}^{\infty} a_n = \sum_{p=0}^{\infty} a_{N+p}$ also converges. Reintroducing the finitely many terms $a_1 + \cdots + a_{N-1}$ does not change the fact of convergence. We conclude that $\sum_{n=1}^{\infty} a_n$ converges.

Now suppose each $a_n \geq 0$ and that $\limsup_{n\to\infty} \sqrt[n]{a_n} > 1$. Select a number r such that $1 < r < \limsup_{n\to\infty} \sqrt[n]{a_n}$. Then there is an integer N such that $1 < r < \sqrt[n]{a_n}$ whenever $n \geq N$. Taking n^{th} powers gives $1 < r^n <\leq a_n$ for $n = N, N+1, N+2, \ldots$. That is, $1 < a_{N+p}$ for $p = 1, 2, 3, \ldots$. The constant series $\sum_{p=0}^{\infty} 1$ certainly diverges. By comparison, the series $\sum_{n=N}^{\infty} a_n = \sum_{p=0}^{\infty} a_{N+p}$ also diverges. Reintroducing the finitely many terms $a_1 + \cdots + a_{N-1}$ does not change the fact of divergence. We conclude that $\sum_{n=1}^{\infty} a_n$ diverges.

(c) Following the pattern in the ratio test of part (a), we can replace the limits by lim sup in the convergence part of the result and by lim inf in the divergence part.

Proposition. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $b_n > 0$ for each n.*

(1) If $|a_n| \leq b_n$ for each n or $\limsup_{n\to\infty} |a_n|/b_n < \infty$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges (in fact absolutely).

(2) If $a_n \geq b_n$ for each n or $\liminf_{n\to\infty} a_n/b_n > 0$, and if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Proof: Suppose each $0 \leq |a_n| \leq b_n$ and that $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} |a_n|$ also converges by comparison. Since $\mathbb{R}$ is complete, absolute convergence implies convergence, and $\sum_{n=1}^{\infty} a_n$ also converges.

Now suppose $\limsup_{n\to\infty} |a_n|/b_n < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges. Select a number $r > 0$ such that $\limsup_{n\to\infty} |a_n|/b_n < r < \infty$. Then there is an integer N such that $|a_n|/b_n < r$ whenever $n \geq N$. Since the numbers b_n are positive, this gives $0 \leq |a_n| \leq rb_n$ for $n = N, N+1, N+2, \ldots$. Since the series $\sum_{n=N}^{\infty} b_n$ converges, it still does after each term is multiplied by the constant r. By comparison, we conclude that $\sum_{n=N}^{\infty} |a_n|$ also converges. Reintroducing the finitely many terms $|a_1| + \cdots + |a_{N-1}|$ does not change the fact of convergence. We conclude that $\sum_{n=1}^{\infty} |a_n|$ converges. We use again the fact that absolute convergence implies convergence to conclude that $\sum_{n=1}^{\infty} a_n$ also converges.

◇ **2E-53.** Given a set A in a metric space, what is the maximum number of distinct subsets that can be produced by successively applying the operations closure, interior, and complement to A (in any order)? Give an example of a set achieving your maximum.

Answer. 14. ◇

Solution. It is convenient to set up some shorthand. Let I, C, N, and E stand for the operations of taking the interior, closure, or complement of a set S in a metric space M, or of leaving it alone.

$$E(S) = S, \quad I(S) = \text{int}(S), \quad C(S) = \text{cl}(S), \quad \text{and} \quad N(S) = M \setminus S$$

We know rather a lot about these operations. For example, if S is any subset of M, then

$$\text{int}(\text{int}(S)) = \text{int}(S), \quad \text{cl}(\text{cl}(S)) = \text{cl}(S), \quad \text{and } M \setminus (M \setminus S) = S.$$

In our new shorthand these become

$$II(S) = I(S), \quad CC(S) = C(S), \quad \text{and} \quad NN(S) = E(S).$$

So, as operators on the subsets of M, we have

$$II = I, \quad CC = C, \quad \text{and} \quad NN = E.$$

What we want to know is how many different operators we can get by compositions of these four, in effect, by forming words from the four letters E, N, C, and I. There are infinitely many different words, but different words may give the same operator. We have just seen, for example, that the words CC and C represent the same operator on sets. Four more such facts are in the next lemma.

Lemma. *If S is a subset of a metric space M, then*

(1) $CN(S) = NI(S)$.
(2) $NC(S) = IN(S)$.
(3) $CICI(S) = CI(S)$.
(4) $ICIC(S) = IC(A)$.

Proof: For (1)

$$\begin{aligned} x \notin CN(S) &\iff x \notin \text{cl}(M \setminus S) \\ &\iff D(X, r) \cap (M \setminus S) = \emptyset \text{ for some } r > 0 \\ &\iff D(x, r) \subseteq S \text{ for some } r > 0 \\ &\iff x \in \text{int}(S) \\ &\iff x \in I(S). \end{aligned}$$

So $x \in CN(S) \iff x \notin I(S) \iff x \in NI(S)$.
For (2), one can give a proof like that for (1), or apply (1) to the set $N(S)$. Since $NN(S) = S$, this gives

$$C(S) = CNN(S) = CN(N(S)) = NI(N(S)).$$

So

$$NC(S) = NNI(N(S)) = I(N(S)) = IN(S).$$

For (3) we start with the fact that any set is contained in its closure applied to the interior of S. This becomes $I(S) \subseteq CI(S)$. If $A \subseteq B$, then $\text{int}(A) \subseteq \text{int}(B)$, and $\text{cl}(A) \subseteq \text{cl}(B)$. So $I(S) = I(I(S)) \subseteq ICI(S)$, and $CI(S) \subseteq CICI(S)$. But also the interior of any set is contained in the set, so, $ICI(S) \subseteq CI(S)$, and $CICI(S) \subseteq CCI(S) = CI(S)$. We have inclusion in both directions, so $CICI(S) = CI(S)$ as claimed.
The proof of (4) is similar to that of (3). Start with the fact that the interior of a set is contained in the set. $I(C(S)) \subseteq C(S)$. Now take closures: $C(I(C(S))) \subseteq C(C(S)) = C(S)$, and then interiors: $ICIC(S) = I(C(I(C(S)))) \subseteq I(C(S)) = IC(S)$. In the other direction, start with the fact that any set is contained in its closure: $IC(S) \subseteq CIC(S)$. Then take interiors: $IC(S) = I(IC(S)) \subseteq I(CIC(S)) = ICIC(S)$. We have inclusion in both directions, so the sets are equal as claimed.
Using these facts we can look at compositions of the operators as words in the letters E, N, C, and I, noting possibly new ones and grouping ones which we know produce the same operator (that is, which are the same for all sets). If we look at the sixteen possible two letter words, we see that ten of them collapse to one letter words, while the remaining six group

themselves into at most four new operators.

$$\begin{array}{cccc} EE=E & NE=E & CE=C & IE=I \\ EN=E & NN=E & CN=NI & IN=NC \\ EC=C & NC=IN & CC=C & IC \\ EI=I & NI=CN & CI & II=I \end{array}$$

In addition to the four operators with which we started,

$$o_1 = E, \quad o_2 = N, \quad o_3 = C, \quad \text{and} \quad o_4 = I,$$

we have at most four new ones,

$$o_5 = NC = IN, \quad o_6 = CN = NI, \quad o_7 = CI, \quad o_8 = IC.$$

To find three letter words, we append a letter to each of the noncollapsing two letter words. Of the twenty-four resulting words, three collapse to one letter and thirteen collapse to two letters leaving eight noncollapsing three letter words.

$$\begin{array}{cccc} ENC=NC & NNC=C & CNC=NIN & INC=IIN=IN \\ EIN=IN & NIN=NNC=NC & CIN=CNC & IIN=IN \\ ECN=CN & NCN=NNI=I & CCN=CN & ICN=INI \\ ENI=NI & NNI=I & CNI=NII=NI & INI=NCI \\ ECI=CI & NCI & CCI=CI & ICI \\ EIC=IC & NIC & CIC & IIC=IC \end{array}$$

Again we have at most four new operators:

$$o_9 = NCI = ICN = INI \quad , \quad o_{10} = NIC = CIN = CNC,$$
$$o_{11} = CIC \quad , \quad o_{12} = ICI.$$

To seek four letter words which might give new operators, we append a letter to one representative from each of the sets of equal three letter words. This will mean that we do not list all four letter words, but those we do not list will be equal as operators to something among those we do list. Also, we do not bother with the column appending E since it always disappears as before.

$$\begin{array}{ccc} NNCI=CI & CNCI=NICI & INCI=NCCI=NCI \\ NNIC=IC & CNIC=NIIC=NIC & INIC=NCIC \\ NCIC & CCIC=CIC & ICIC=IC \\ NICI & CICI=CI & IICI=ICI \end{array}$$

Notice where we have used parts (4) and (5) of the lemma. We find at most two new operators.

$$o_{13} = NCIC = INIC \quad \text{and} \quad o_{14} = NICI = CNCI$$

To see if we get any new operators with five letters, it is enough to try adding N, C, or I to one of the four letter words giving each of o_{13} and o_{14}.

$$\begin{array}{cc} NNCIC = CIC & NNICI = ICI \\ CNCIC = NICIC = NIC & CNICI = NIICI = NICI \\ INCIC = NCCIC = NCIC & INICI = NCICI = NCI \end{array}$$

Since these collapse, all five letter words can be collapsed to four or fewer letters. We get no new operators.

The argument above shows that composition of the operations of complementation, closure, and interior, together with the identity operator, generate at most fourteen different operators on the powerset of a metric space. Some of these might be equal in some metric spaces. If the metric space has only three points, then there cannot be very many different operators generated. In any set with the discrete metric all sets are both open and closed, so the closure operator and the interior operators are the same as the identity operator, and we get only two operators. But in most reasonable spaces they are different. In fact, even in $\mathbb{R}$ we can get one set S such that the fourteen sets $o_k(S)$ for $1 \le k \le 14$ are all different. The set must be fairly complicated to keep generating new sets up through the four letter words above. But it need not be completely outrageous. One example which works is

$$S = \{-\frac{1}{n} \mid n \in \mathbb{N}\} \cup [0,1[\cup \left( ]1,2[\setminus \{1 + \frac{1}{n} \mid n \in \mathbb{N}\} \right) \cup (]2,3[\cap \mathbb{Q}) .$$

Possibly the easiest way to see that this works is to sketch the fourteen sets on the line. See Figure 2-17. ♦