◇ **1.4-4.** Let x_n be a Cauchy sequence. Suppose that for every $\varepsilon > 0$ there is some $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$. Prove that $x_n \to 0$.

Discussion. The assumption that $\langle x_n \rangle_1^\infty$ is a Cauchy sequence says that far out in the sequence, all of the terms are close to each other. The second assumption, that for every $\varepsilon > 0$ there is an $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$, says, more or less, that no matter how far out we go in the sequence there will be at least one term out beyond that point which is small. Combining these two produces the proof. If some of the points far out in the sequence are small and all of the points far out in the sequence are close together, then all of the terms far out in the sequence must be small. The technical tool used to merge the two assumptions is the triangle inequality. ◇

Solution. Let $\varepsilon > 0$. Since the sequence is a Cauchy sequence, there is an N_1 such that $|x_n - x_k| < \varepsilon/2$ whenever $n \geq N_1$ and $k \geq N_1$.

Pick $N_2 > N_1$ large enough so that $1/N_2 < \varepsilon/2$. By hypothesis there is at least one index $n > 1/(1/N_2) = N_2$ with $|x_n| < 1/N_2$.

If $k \geq N_1$, then both k and n are at least as large as N_1 and we can compute

$$|x_k| = |x_k - x_n + x_n| \leq |x_k - x_n| + |x_n| < \frac{\varepsilon}{2} + \frac{1}{N_2} < \varepsilon$$

Thus $x_k \to 0$ as claimed. ♦

◇ **1.4-5.** True or false: If x_n is a Cauchy sequence, then for n and m large enough, $d(x_{n+1}, x_{m+1}) \leq d(x_n, x_m)$.

Answer. False. ◇

Solution. If this were true, it would hold in particular with m selected as $m = n + 1$. That is, we should have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$ for large enough n. But this need not be true. Consider the sequence

$$1, 0, 0, \frac{1}{2}, 0, 0, \frac{1}{3}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{5}, \ldots.$$

Then

$$\begin{aligned}
d(x_1, x_2) &= 1 \\
d(x_2, x_3) &= 0 \\
d(x_3, x_4) &= \frac{1}{2} \\
d(x_4, x_5) &= \frac{1}{2} \\
d(x_5, x_6) &= 0 \\
d(x_6, x_7) &= \frac{1}{3} \\
d(x_7, x_8) &= \frac{1}{3} \\
d(x_8, x_9) &= 0 \\
&\vdots
\end{aligned}$$

This sequence converges to 0 and is certainly a Cauchy sequence. But the differences of succeeding terms keep dropping to 0 and then coming back up a bit. ♦

◇ **1.7-5.** Show that $\| \cdot \|_\infty$ is not the norm defined by the inner product in Example 1.7.7.

Sketch. For $f(x) = x$, $\| f \|_\infty = 1$, but $\langle f, f \rangle^{1/2} = 1/\sqrt{3}$. So these norms are different. ◇

Solution. All we have to do to show that the norms are different is to display a function to which they assign different numbers. Let $f(x) = x$. The norm defined by the inner product of Example 1.7.7 assigns to f the number

$$\| f \|_2 = \left(\int_0^1 |f|^2 \, dx \right)^{1/2} = \left(\int_0^1 x^2 \, dx \right)^{1/2} = \frac{1}{\sqrt{3}}$$

But

$$\| f \|_\infty = \sup\{|f(x)| \mid x \in [0, 1]\} = \sup\{x \mid x \in [0, 1]\} = 1.$$

Since $1/\sqrt{3}$ is not equal to 1, these norms are different. ♦

◇ **1E-28.** Let x_n be a convergent sequence in $\mathbb{R}$ and define $A_n = \sup\{x_n, x_{n+1}, \dots\}$ and $B_n = \inf\{x_n, x_{n+1}, \dots\}$. Prove that A_n converges to the same limit as B_n, which in turn is the same as the limit of x_n.

Solution. Suppose $x_n \to a \in \mathbb{R}$. Let $\varepsilon > 0$. There is an index N such that $a - \varepsilon < x_k < a + \varepsilon$ whenever $k \geq N$. Thus if $n \geq N$ we have $a - \varepsilon < x < a + \varepsilon$ for all x in the set $S_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$. That is, $a - \varepsilon$ is a lower bound for S_n, and $a + \varepsilon$ is an upper bound. Hence

$$a - \varepsilon \leq B_n = \inf S_n \leq \sup S_n = A_n \leq a + \varepsilon.$$

So

$$|A_n - a| \leq \varepsilon \qquad \text{and} \qquad |B_n - a| \leq \varepsilon$$

whenever $n \geq N$. We conclude that

$$\lim_{n \to \infty} A_n = a = \lim_{n \to \infty} B_n$$

as claimed. ♦

◇ **1E-30.** Let $\mathcal{V}$ be the vector space $\mathcal{C}([0, 1])$ with the norm $\| f \|_\infty = \sup\{|f(x)| \mid x \in [0, 1]\}$. Show that the parallelogram law fails and conclude that this norm does not come from any inner product. (Refer to Exercise 1E-12.)

Solution. Let $f(x) = x$ and $g(x) = 1 - x$. Both of these functions are in $\mathcal{C}([0, 1])$, and $\| f \|_\infty = \sup\{x \mid x \in [0, 1]\} = 1$ and $\| g \|_\infty = \sup\{1 - x \mid x \in [0, 1]\} = 1$. For the sum and difference we have, $(f + g)(x) = 1$ and $(f - g)(x) = 2x - 1$. So

$$\| f + g \|_\infty + \| f - g \|_\infty = 1 + 1 = 2$$

while

$$2 \| f \|_\infty^2 + 2 \| g \|_\infty^2 = 2 \cdot 1^2 + 2 \cdot 1^2 = 4.$$

Since 2 and 4 are not the same, we see that this norm does not satisfy the parallelogram law. If there were any way to define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{C}([0, 1])$ in such a way that $\| h \|_\infty^2 = \langle h, h \rangle$ for each h in the space, then the parallelogram law would have to hold by the work of Exercise 1E-12(a). Since it does not, there can be no such inner product.

◇ **1E-31.** Let $A, B \subset \mathbb{R}$ and $f : A \times B \to \mathbb{R}$ be bounded. Is it true that

$$\sup\{ f(x,y) \mid (x,y) \in A \times B\} = \sup\{\sup\{ f(x,y) \mid x \in A\} \mid y \in B\}$$

or, the same thing in different notation,

$$\sup_{(x,y)\in A\times B} f(x,y) = \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right)?$$

Answer. True. ◇

Solution. First suppose (s,t) is a point in $A \times B$. Then $s \in A$ and $t \in B$, so

$$\begin{aligned} f(s,t) &\le \sup_{x\in A} f(x,t) \\ &\le \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right) \end{aligned}$$

Since this is true for every point (s,t) in $A \times B$, we can conclude that

$$\sup_{(x,y)\in A\times B} f(x,y) \le \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right).$$

For the opposite inequality, let $\varepsilon > 0$. There is a $t \in B$ such that

$$\sup_{x\in A} f(x,t) > \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right) - \varepsilon/2.$$

So there must be an $s \in A$ with

$$f(s,t) > \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right) - \varepsilon/2 - \varepsilon/2.$$

So

$$\sup_{(x,y)\in A\times B} f(x,y) \ge \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right) - \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we must have

$$\sup_{(x,y)\in A\times B} f(x,y) \ge \sup_{y\in B}\left(\sup_{x\in A} f(x,y)\right).$$

We have inequality in both directions, and thus equality as claimed. ◆

◇ **1E-32.** (a) Give a reasonable definition for what $\lim_{n\to\infty} x_n = \infty$ should mean.

(b) Let $x_1 = 1$ and define inductively $x_{n+1} = (x_1 + \cdots + x_n)/2$. Prove that $x_n \to \infty$.

Solution. (a) **Definition.** *We say the sequence $\langle x_n\rangle_1^\infty$ tends to infinity and write $\lim_{n\to\infty} x_n = \infty$ if for each $B > 0$ there is an N such that $x_n > B$ whenever $n \ge N$.*

(b) Computation of the first few terms of the sequence shows that

$$x_1 = 1 \qquad x_2 = 1/2 \qquad x_3 = 3/4 \qquad x_4 = 9/8 \qquad \ldots$$

If $x_1, x_2, x_3, \ldots, x_n$ are all positive, then $x_{n+1} = (x_1 + \cdots + x_n)/2$ must be positive also. Since the first few terms are positive, it follows by induction that all terms of the sequence are positive. With this in hand we can get a lower estimate for the terms. If $n \ge 4$, then

$$x_{n+1} = \frac{x_1 + x_2 + \cdots + x_n}{2} > \frac{x_1 + x_4}{2} > 1.$$

If we feed this information back into the formula we find

$$\begin{aligned} x_{n+1} &= \frac{x_1 + x_2 + \cdots + x_n}{2} > \frac{1 + 1/2 + 3/4 + \overbrace{1 + 1 + 1 + \cdots + 1}^{n-3 \text{ terms}}}{2} \\ &> \frac{n-1}{2}. \end{aligned}$$

Thus $x_n > (n/2) - 1$ for $n = 5, 6, 7, \ldots$. If $B > 0$ and $n > 2B + 2$, then $x_n > B$. Thus $\lim_{n\to\infty} = \infty$ as claimed.

Method Two: Observation of the first few terms might lead to the conjecture that $x_n = 3^{n-2}/2^{n-1} = (1/2)(3/2)^{n-2}$ for $n = 2, 3, 4, \ldots$. We can confirm this by induction. It is true by direct computation for $n =$ 2, 3, and 4. If it is true for $1 \le k \le n$, then

$$\begin{aligned} x_{n+1} &= \frac{x_1 + x_2 + \cdots + x_n}{2} \\ &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2}\left(\frac{3}{2}\right) + \frac{1}{2}\left(\frac{3}{2}\right)^2 + \cdots + \frac{1}{2}\left(\frac{3}{2}\right)^{n-2}\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{2} \cdot \frac{1 - (3/2)^{n-1}}{1 - (3/2)}\right) \\ &= \frac{1}{2}\left(1 + \frac{1 - (3/2)^{n-1}}{2 - 3}\right) = \frac{1}{2}\left(\frac{3}{2}\right)^{n-1} \end{aligned}$$

This is exactly the desired form for the $(n+1)^{st}$ term. Since $x_n = (1/2)(3/2)^{n-2}$, a straightforward computation shows that $x_n > B$ whenever $n > \frac{\log(2B)}{\log(3/2)} + 2$. Again we can conclude that $\lim_{n\to\infty} x_n = \infty$.

Method Three: A bit of clever manipulation with the formula first along with a much easier induction produces the same formula for the n^th term.

$$\begin{aligned} x_{n+1} &= \frac{x_1 + \cdots + x_n}{2} = \frac{x_1 + \cdots + x_{n-1}}{2} + \frac{x_n}{2} \\ &= x_n + \frac{x_n}{2} = \frac{3}{2}x_n \qquad \text{for } n = 2, 3, \ldots \end{aligned}$$

With this and the first two terms, a much simpler induction produces the same general formula for x_n as in Method Two.

♦

◇ **1E-46.** Prove that each nonempty set S of $\mathbb{R}$ that is bounded above has a least upper bound as follows: Choose $x_0 \in S$ and M_0 an upper bound. Let $a_0 = (x_0 + M_0)/2$. If a_0 is an upper bound, let $M_1 = a_0$ and $x_1 = x_0$; otherwise let $M_1 = M_0$ and $x_1 > a_0$, $x_1 \in S$. Repeat, generating sequences x_n and M_n. Prove that they both converge to $\sup(S)$.

Solution. Since S is bounded, there is an M_0 with $x \le M_0$ for all x in S. Since S is not empty, we can select x_0 in S and let $a_0 = (x_0 + M_0)/2$. If this is an upper bound, let $M_1 = a_0$ and $x_1 = M_0$. If not, then there is an x in S with $x > a_0$. Let $x_1 = x$ and $M_1 = M_0$. Repeat with $a_1 = (x_1 + M_1)/2$ and $a_2 = (x_2 + M_2)/2$, and so forth. Each M_{k+1} is either M_k or $(x_k + M_k)/2$. Since M_k is an upper bound and $x_k \in S$, we have $x_k \le M_k$, and $M_{k+1} \le M_k$. The M_k form a monotone decreasing sequence and since every term is greater than some x in S, it is bounded below and must converge to some number M in $\mathbb{R}$ by completeness.

Similarly, the x_k form a monotone increasing sequence bounded above by any of the M's. So they must converge to something. We claim that they converge to M and that M is a least upper bound for S.

Let $d_0 = M_0 - x_0 = d(M_0, x_0)$. For each k we have that $M_k - x_k = d(M_k, x_k)$ is equal either to $x_{k-1} + M_{k-1})/2 - x_{k-1}$ or $M_{k-1} - (x_{k-1} + M_{k-1})/2$. Both of these are equal to $(M_{k-1} - x_{k-1})/2 = d(M_{k-1}, x_{k-1})/2$. Inductively we obtain $d(M_n, x_n) = d_0/2^n$ for each n. This tends to 0 as $n \to \infty$, so the limits of the two convergent sequences must be the same.

To show that M is an upper bound for S, suppose that there were a point x in S with $x > M$. Select k with $M_k - M < x - M$. This would force $M_k < x$ contradicting the fact that M_k is an upper bound for S. To show that M is a least upper bound for S, suppose that b were an upper bound with $b < M$. Since the points x_k converge to M, there is a k with $|M - x_k| < M - b$. This forces $b < x_k$ contradicting the supposition that it was an upper bound. ♦