◇ **1.2-4.** Let x_n be a monotone increasing sequence such that $x_{n+1} - x_n \leq 1/n$. Must x_n converge?

Answer. No, not necessarily. ◇

Solution. If we put $x_1 = 1$ and suppose that $x_{n+1} = x_n + (1/n)$, then

$$\begin{aligned}
x_2 &= 1 + \frac{1}{1} \\
x_3 &= x_2 + \frac{1}{2} = 1 + \frac{1}{1} + \frac{1}{2} \\
x_4 &= x_3 + \frac{1}{3} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \\
&\vdots \\
x_{n+1} &= x_n + \frac{1}{n} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}
\end{aligned}$$

Since we know that the harmonic series diverges to infinity, we see that the sequence $\langle x_n \rangle_1^\infty$ cannot converge. ♦

◇ **1.2-5.** Let $\mathbb{F}$ be an ordered field in which every *strictly* monotone increasing sequence bounded above converges. Prove that $\mathbb{F}$ is complete.

Sketch. From a (nontrivial) monotone sequence $\langle x_n \rangle_1^\infty$, extract a subsequence which is strictly monotone. ◇

Solution. We need to show that every increasing sequence which is bounded above must converge to an element of F whether the increase is strict or not. If the sequence is constant beyond some point, then it certainly converges to that constant. If not, then we can inductively extract a strictly increasing subsequence from it as follows. Let $\langle x_n \rangle_1^\infty$ be the sequence, and net $n(1) = 1$. Let $n(2)$ be the first integer larger than 1 with $x_{n(1)} < x_{n(2)}$. There must be such an index since the sequence never again gets smaller than $x_{n(1)}$ and it is not constantly equal to $x_{n(1)}$ beyond that point. Repeat this process. Having selected $n(1) < n(2) < \cdots < n(k) < \ldots$ with $x_{n(1)} < x_{n(2)} < \cdots < x_{n(k)} \ldots$, let $n(k+1)$ be the first index larger than $n(k)$ with $x_{n(k)} < x_{n(k+1)}$. Such an index exists since the sequence never again is smaller than $x_{(n(k)}$, and it is not constantly equal to $x_{n(k)}$ beyond this point. This inductively produces indices $n(1) < n(2) < n(3) < \ldots$ with $x_{n(1)} < x_{n(2)} < x_{n(3)} < \ldots$. By hypothesis, this strictly increasing sequence must converge to some element λ of F since it is bounded above by the same bound as the original sequence. We claim that the whole sequence must converge to λ. If $\varepsilon > 0$, then there is a J such that $|x_{n(k)} - \lambda| < \varepsilon$ whenever $k \geq J$. Put $N = n(J)$. If $n \geq N$, there is a $k > J$ with $n(J) \leq n \leq n(k)$. So $x_{n(J)} \leq x_n \leq x_{n(k)} \leq \lambda$. Thus $|x_n - \lambda| = \lambda - x_n \leq \lambda - x_{n(J)} < \varepsilon$. Thus $\lim_{n\to\infty} x_n = \lambda$ as claimed. ♦

◇ **1.3-4.** Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded below and define $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$. Is it true that $\inf(A + B) = \inf A + \inf B$?

Answer. Yes. ◇

Solution. First suppose $z \in A + B$, then there are points $x \in A$ and $y \in B$ with $z = x + y$. Certainly $\inf A \leq x$ and $\inf B \leq y$. So

$$\inf A + \inf B \leq x + y = z.$$

Thus $\inf A + \inf B$ is a lower bound for the set $A + B$. So $\inf A + \inf B \leq \inf(A + B)$.

To get the opposite inequality, let $\varepsilon \geq 0$. There must be points $x \in A$ and $y \in B$ with

$$\inf A \leq x < \inf A + \frac{\varepsilon}{2} \qquad \text{and} \qquad \inf B \leq y < \inf B + \frac{\varepsilon}{2}.$$

Since $x + y \in A + B$ we must have

$$\inf(A + B) \leq x + y \leq \inf A + \frac{\varepsilon}{2} + \inf B + \frac{\varepsilon}{2} = \inf A + \inf B + \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we must have $\inf(A + B) \leq \inf A + \inf B$.

We have inequality in both directions, so $\inf(A+B) = \inf A + \inf B$. ♦

◇ **1.3-5.** Let $S \subset [0, 1]$ consist of all infinite decimal expansions $x = 0.a_1a_2a_3\cdots$ where all but finitely many digits are 5 or 6. Find $\sup S$.

Answer. $\sup(S) = 1$. ◇

Solution. The numbers $x_n = 0.99999\ldots9995555555\ldots$ consisting of n 9's followed by infinitely many 5's are all in S. Since these come as close to 1 as we want, we must have $\sup S = 1$. ♦

♦

◇ **1.5-3.** Let x_n be a sequence with $\limsup x_n = b \in \mathbb{R}$ and $\liminf x_n = a \in \mathbb{R}$. Show that x_n has subsequences u_n and l_n with $u_n \to b$ and $l_n \to a$.

Sketch. Use Proposition 1.5.5 to show that there are points $x_{N(n)}$ within $1/n$ of a (or b). Make sure you end up with a subsequence. ◇

Solution. If b is a finite real number and $b = \limsup x_n$, then for each N and for each $\varepsilon > 0$ there is an index n with $n > N$ and $b - \varepsilon < x_n \leq b + \varepsilon$. Use this repeatedly to generate the desired subsequence.

Step One: There is an index $n(1)$ such that $|b - x_{n(1)}| < 1$.

Step Two: There is an index $n(2)$ such that $n(2) > n(1)$ and $|b - x_{n(2)}| < 1/2$.

... and so forth.

Induction Step: Having selected indices

$$n(1) < n(2) < n(3) < \cdots < n(k)$$

with $|b - x_{n(j)}| < 1/j$, for $1 \leq j \leq k$, there is an index $n(k + 1)$ with $n(k + 1) > n(k)$ and $|b - x_{n(k+1)}| < 1/(k + 1)$

By induction, this gives a subsequence converging to b.

The inductive definition of a subsequence converging to the limit inferior is similar. Notice that the tricky part is to make the selection of indices dependent on the earlier ones so that the indices used are increasing. This is what makes the selection of terms a subsequence. They remain in the same order as they appeared in the original sequence. ♦

◇ **1E-6.** Let A and B be two nonempty sets of real numbers with the property that $x \leq y$ for all $x \in A$, $y \in B$. Show that there exists a number $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in A$, $y \in B$. Give a counterexample to this statement for rational numbers (it is, in fact, equivalent to the completeness axiom and is the basis for another way of formulating the completeness axiom known as ***Dedekind cuts***).

Solution. Let y_0 be in B. Then $x \leq y_0$ for every x in A. So A is bounded above. Since we are given that it is not empty, $r = \sup A$ exists in $\mathbb{R}$. Since r is an upper bound for A, we have $x \leq r$ for every x in A. If $y \in B$, then $y \geq x$ for every x in A, so y is an upper bound for A, and $r \leq y$. Thus we have $x \leq r \leq y$ for every x in A and every y in B.

In $\mathbb{Q}$ this fails. Let $A = \{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\}$ and $B = \{y \in \mathbb{Q} \mid y > 0 \text{ and } y^2 > 0\}$. The $x \leq y$ for every x in A and every y in B. But there is no rational number r with $r^2 = 2$. So $A \cap B = \emptyset$, $A \cup B = \mathbb{Q}$, and there is no r in $\mathbb{Q}$ with $x \leq r \leq y$ for every x in A and y in B. ♦

◇ **1E-8.** For nonempty sets $A, B \subset \mathbb{R}$, determine which of the following statements are true. Prove the true statements and give a counterexample for those that are false:

(a) $\sup(A \cap B) \leq \inf\{\sup(A), \sup(B)\}$.
(b) $\sup(A \cap B) = \inf\{\sup(A), \sup(B)\}$.
(c) $\sup(A \cup B) \geq \sup\{\sup(A), \sup(B)\}$.
(d) $\sup(A \cup B) = \sup\{\sup(A), \sup(B)\}$.

Solution. (a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.

This is true if the intersection is not empty.

If $x \in A \cap B$, then $x \in A$, so $x \leq \sup A$. Also, $x \in B$, so $x \leq \sup B$. Thus x is no larger than the smaller of the two numbers $\sup A$, and $\sup B$. That is, $x \leq \inf\{\sup A, \sup B\}$ for every x in $A \cap B$. Thus $\inf\{\sup A, \sup B\}$ is an upper bound for $A \cap B$. So $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.

Here is another argument using Proposition 1.3.3. We notice that $\sup(A \cap B) \subseteq A$, so by Proposition 1.3.3 we should have $\sup(A \cap B) \leq \sup A$. Similarly, $\sup(A \cap B) \leq \sup B$. So $\sup(A \cap B)$ is no larger than the smaller of the two numbers $\sup A$ and $\sup B$. That is

$$\sup(A \cap B) \leq \inf\{\sup A, \sup B\}.$$

There is a problem if the intersection is empty. We have defined $\sup(\emptyset)$ to be $+\infty$, and this is likely to be larger than $\inf\{\sup A, \sup B\}$.

One can make a reasonable argument for defining $\sup(\emptyset) = -\infty$. Since any real number is an upper bound for the empty set, and the supremum is to be smaller than any other upper bound, we should take $\sup(\emptyset) = -\infty$. (Also, since any real number is a lower bound for $\emptyset$, and the infimum is to be larger than any other lower bound, we could set $\inf(\emptyset) = +\infty$). If we did this then the inequality would be true even if the intersection were empty.

(b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$
This one need not be true even if the intersection is not empty. Consider the two element sets $A = \{1, 2\}$ and $B = \{1, 3\}$. Then $A \cap B = \{1\}$. So $\sup(A \cap B) = 1$. But $\sup A = 2$ and $\sup B = 3$. So $\inf\{\sup A, \sup B\} = 2$ We do have $1 \leq 2$, but they are certainly not equal.
We know that $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$. Can we get the opposite inequality?

(c) $\sup(A \cup B) \geq \sup\{\sup A, \sup B\}$

(d) $\sup(A \cup B) = \sup\{\sup A, \sup B\}$
Neither A nor B nor the union is empty, so we will not be troubled with problems in the definition of the supremum of the empty set.
If x is in $a \cup B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \leq \sup A$. If $x \in B$, then $x \leq \sup B$. In either case it is no larger than the larger of the two numbers $\sup A$ and $\sup B$. So $x \leq \sup\{\sup A, \sup B\}$ for every x in the union. Thus $\sup\{\sup A, \sup B\}$ is an upper bound for $A \cup$. So

$$\sup(A \cup B) \leq \sup\{\sup A, \sup B\}$$

In the opposite direction, we note that $A \subseteq A \cup B$, so by Proposition 1.3.3, we have $\sup A \leq \sup(A \cup B)$. Similarly $\sup B \leq \sup(A \cup B)$. So $\sup(A \cup B)$ is at least as large as the larger of the two numbers $\sup A$ and $\sup B$. That is

$$\sup(A \cup B) \geq \sup\{\sup A, \sup B\}$$

We have inequality in both directions, so in fact equality must hold.

♦

◇ **1E-23.** Let $P \subset \mathbb{R}$ be a set such that $x \geq 0$ for all $x \in P$ and for each integer k there is an $x_k \in P$ such that $kx_k \leq 1$. Prove that $0 = \inf(P)$.

Sketch. Since 0 is a lower bound for P, $0 \leq \inf(P)$. Use the given condition to rule out $0 < \inf(P)$. ◇

Solution. With $k = 1$ the hypothesis gives an element x_1 of the set P with $x_1 \leq 1$. In particular, P is not empty. We have assumed that $x \geq 0$ for every x in P. So 0 is a lower bound for P. We conclude that $\inf P$ exists as a finite real number and that $0 \leq \inf P$.

Now let $\varepsilon > 0$. By the Archimedean Principle there is an integer k with $0 < 1/k < \varepsilon$. By hypothesis, there is an element x_k of P with $kx_k \leq 1$. So $0 \leq x_k \leq 1/k < \varepsilon$. So ε is not an lower bound for P and $\inf P \leq \varepsilon$. This holds for every $\varepsilon > 0$, so $\inf P \leq 0$. We have inequality in both directions, so $\inf P = 0$ as claimed. ♦

◇ **1E-26.** Assume that $A = \{a_{m,n} \mid m = 1, 2, 3, \ldots \text{ and } n = 1, 2, 3, \ldots\}$ is a bounded set and that $a_{m,n} \geq a_{p,q}$ whenever $m \geq p$ and $n \geq q$. Show that

$$\lim_{n \to \infty} a_{n,n} = \sup A.$$

Solution. Since A is a bounded, nonempty subset of $\mathbb{R}$, we know that $c = \sup A$ exists as a finite real number and that $a_{j,k} \leq c$ for all j and k. For convenience, let $b_n = a_{n,n}$. If $n \leq k$, we have $b_n = a_{n,n} \leq a_{k,k} = b_k$. So the sequence $(b_n)_1^\infty$ is increasing and bounded above by c. So $b = \lim_{n \to \infty} b_n$ exists and $b \leq c$. Let $d < c$, then there is a $a_{k,j}$ in A with $d < a_{k,j} \leq c$. If $n \geq \max(k, j)$, then

$$d < a_{k,j} \leq a_{n,n} = b_n \leq b_{n+1} \leq \ldots.$$

So $b = \lim b_n > d$. This is true for every $d < c$, so $b \geq c$. We have inequality in both directions, so

$$\lim_{n \to \infty} a_{n,n} = \lim_{n \to \infty} b_n = b = c = \sup A$$

as claimed. ♦