◇ **1.1-4.** Prove that in an ordered field, if $\sqrt{2}$ is a positive number whose square is 2, then $\sqrt{2} < 3/2$. (Do this without using a numerical approximation for $\sqrt{2}$.)

Solution. It is convenient to set up a lemma first. This lemma corresponds to the fact that the squaring and square root functions are monotone increasing on the positive real numbers.

Lemma. *If $a > 0$ and $b > 0$, then $a^2 \le b^2 \iff a \le b$.*

Proof: Suppose $0 \le a \le b$. If we use Property 1.1.2 xi twice we find $a^2 \le ab$ and $ab \le b^2$. Transitivity of inequality gives $a^2 \le b^2$.

In the other direction, if $a^2 \le b^2$, then

$$0 \le b^2 - a^2 = (b-a)(b+a).$$

If $b + a = 0$, then $a = b = 0$ since both are non-negative. If it is not 0, then it is positive and so is $(b+a)^{-1}$. So

$$\begin{aligned} 0 \cdot (b+a)^{-1} &\le ((b-a)(b+a))(b+a)^{-1} = (b-a)((b+a)(b+a)^{-1}) \\ 0 &\le b - a \\ a &\le b \end{aligned}$$

We have implication in both directions as claimed.

Using this we can get our result. If $\sqrt{2} \ge 3/2$, we would have $2 \ge 9/4$ and so $8 \ge 9$. Subtracting 8 from both sides would give $0 \ge 1$, but we know this is false. Thus $\sqrt{2} \ge 3/2$ has led to a contradiction and cannot be true. We must have $\sqrt{2} < 3/2$ as claimed. ♦

◇ **1.1-5.** Give an example of a field with only three elements. Prove that it cannot be made into an ordered field.

Sketch. Let $\mathbb{F} = \{0, 1, 2\}$ with arithmetic mod 3. For example, $2 \cdot 2 = 1$ and $1 + 2 = 0$. To show it cannot be ordered, get a contradiction from (for example) $1 > 0$ so $1 + 1 = 2 > 0$ so $1 + 2 = 0 > 0$. ◇

Solution. Let $\mathbb{F} = \{0, 1, 2\}$ with arithmetic mod 3. For example, $2 \cdot 2 = 1$ and $1 + 2 = 0$. The commutative associative and distributive properties work for modular arithmetic with any base. When the base is a prime, the result is a field. In particular in arithmetic modulo 3 we have

$$1 \cdot 1 = 1 \quad \text{and} \quad 2 \cdot 2 = 1.$$

Thus 1 and 2 are their own reciprocals. Since they are the only two nonzero elements, we have a field.

To show it cannot be ordered, get a contradiction from (for example) $1 > 0$ so $1+1 = 2 > 0$ so $1+2 = 0 > 0$. We know that $0 \le 1$ in any ordered field. So $1 \le 1 + 1 = 2$ by order axiom 15. Transitivity gives $0 \le 2$. So far there is no problem. But, if we add 1 to the inequality $1 \le 2$ obtained above, we find $2 = 1 + 1 \le 2 + 1 = 0$. So $2 \le 0$. Thus $0 = 2$. If we multiply by 2, we get $0 = 0 \cdot 2 = 2 \cdot 2 = 1$. So $0 = 1$. But we know this is false. ♦